Title: Self-consistent adiabatic inspiral and transition motion

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Abstract: We describe the transition to plunge of a point particle around the last stable orbit of Kerr at leading order in the transition-timescale expansion. Taking systematically into account all self-force effects, we prove that the transition motion is still described by the PainlevÃ© transcendent equation of the first kind. Using an asymptotically matched expansions scheme, we consistently match the quasi-circular adiabatic inspiral with the transition motion. The matching requires to take into account the secular change of angular velocity due to radiation-reaction during the adiabatic inspiral, which consistently leads to a leading-order radial self-force in the slow timescale expansion.

Self-consistent adiabatic inspiral and transition motion

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Based on arXiv 2102.12747 (with Geoffrey Compère at ULB)

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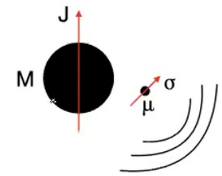
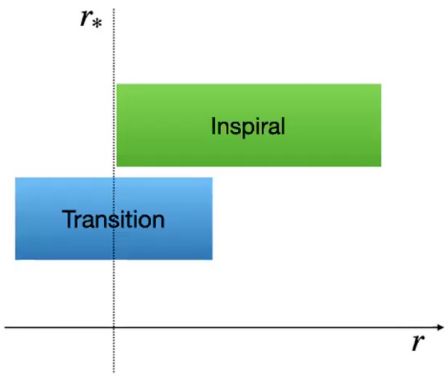
Introduction

- An EMRI within GR
- Kerr black hole (M, J) + compact object (μ, σ), $\eta = \mu/M$
- Neglect structure and spin of the secondary object



- Inspiral on quasi-circular and equatorial orbits
- At the ISCO the slow-timescale expansion of the inspiral breaks down
- A different treatment is needed which matches smoothly onto the slow-timescale expansions



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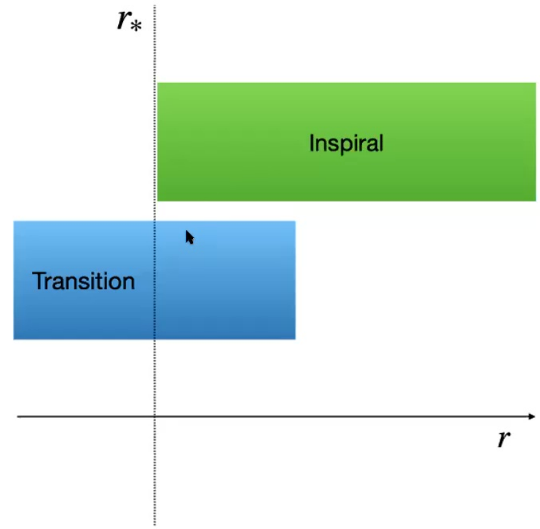
Matched asymptotic expansions

Treatment of the transition in small mass ratio regime [Ori & Thorne, 2000] and [Kesden, 2011]

- Leading-order transition neglecting self-force
- Quasi-circularity is inconsistent with normalisation of the 4-velocity

The plan:

- Leading-order (adiabatic) inspiral including all self-force effects
- Leading-order transition including all self-force effects
- Matched asymptotic expansions in the overlapping region



Forced equatorial geodesics

- Trajectory $z^\mu = (t, r, \pi/2, \phi)$ angular velocity $\Omega = d\phi/dt$ and redshift $U = dt/d\tau = -g^{tt}e + g^{t\phi}\ell$
- The normalization of the 4-velocity and the radial geodesic equation

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - V, \quad \frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{\partial V}{\partial r} + f^r, \quad V \equiv 1 - \frac{2}{r} + \frac{\ell^2 + a^2(1-e^2)}{r^2} + \frac{2(\ell - ae)}{r^3}$$

- The energy and angular momentum flux-balance equations

$$\frac{d\ell}{d\tau} = f_\phi, \qquad \frac{de}{d\tau} = -f_t$$

- The orthogonality of the force with the velocity $f_\mu u^\mu = 0$

$$\Omega^{-1}\frac{de}{d\tau} - \frac{d\ell}{d\tau} = f_r \frac{dr}{d\tau}(\Omega U)^{-1}$$

- Angular velocity and deviation from circular geodesic angular velocity

$$\Omega^{geo} = \frac{1}{r^{3/2} + a}, \quad \delta = (\Omega^{-1} - \Omega^{-1}_{geo}), \quad \rightarrow \quad \Omega = \frac{1}{r^{3/2} + a + \delta}, \qquad \Omega = \frac{2(ae - \ell) + r\ell}{r(r^2 + a^2)e + 2a(ae - \ell)}$$

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The adiabatic inspiral

- Slow-timescale expansion for $X = (r, a, \delta, \Omega, U, e, \ell)$ where $\tilde{\tau} = \eta\tau$ is the slow proper time

$$X = X_{(0)}(\tilde{\tau}) + \eta X_{(1)}(\tilde{\tau}) + O_{\tilde{\tau}}(\eta^2)$$

- Consistently with the flux-balance equations and the orthogonality condition

$$f^a = \eta f^a_{(1)}(\tilde{\tau}) + O_{\tilde{\tau}}(\eta^2), \quad a = t, \phi$$
$$f^r = f^r_{(0)}(\tilde{\tau}) + \eta f^r_{(1)}(\tilde{\tau}) + O_{\tilde{\tau}}(\eta^2)$$

- Equations at lowest order: algebraic equations

$$\Omega = \frac{2(ae-\ell)+r\ell}{r(r^2+a^2)e+2a(ae-\ell)}$$

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - V^{geo}, \qquad \frac{d^2r}{d\tau^2} + \frac{1}{2}\frac{\partial V^{geo}}{\partial r} = f^r \quad \longrightarrow$$

$$U = dt/d\tau = -g^{tt}e + g^{t\phi}\ell$$

$$U_{(0)},\ e_{(0)},\ \ell_{(0)}$$

$$\Omega_{(0)} = \frac{1}{r^{3/2}_{(0)} + a_{(0)} + \delta_{(0)}}$$

$$f^r_{(0)} = \Delta U^2_{(0)}\Omega^2_{(0)} r^{-4}_{(0)} \delta_{(0)}(\delta_{(0)} + 2r^{3/2}_{(0)})$$

- Note on quasi-circularity: the inspiral is not described by a geodesic circular angular velocity

$$\frac{d\log r}{d\phi} = \frac{dr/d\tilde{t}}{rd\phi/d\tilde{t}} = \frac{\eta}{r\Omega U}\frac{dr}{d\tilde{\tau}} = O_{\tilde{\tau}}(\eta) \quad \longrightarrow$$

$\delta_{(0)}$ is small throughout the inspiral

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Solution in the ISCO limit

- The equations of motion:

$$\Omega^{-1}\frac{de}{d\tau} - \frac{d\ell}{d\tau} = f_r \frac{dr}{d\tau}(\Omega U)^{-1} \longrightarrow \frac{da_{(0)}}{d\tilde{\tau}} = 0 \longrightarrow \text{the spin is constant at adiabatic order}$$

$$\frac{d\ell}{d\tau} = f_\phi, \qquad \frac{de}{d\tau} = -f_t$$

$$\frac{dr_{(0)}}{d\tilde{\tau}} = \frac{r_{(0)}^2 A_\delta}{\delta_{(0)}(\delta_{(0)} + 2r_{(0)}^{3/2})}\left(e_{(0)} f^t_{(1)} - \ell_{(0)} f^\phi_{(1)}\right)$$

$$\frac{d\delta_{(0)}}{d\tilde{\tau}} = -\frac{A_\delta^{3/2}}{B_\delta} f^t_{(1)} + \frac{\sqrt{r_{(0)}} D_\delta}{2\Delta B_\delta}\frac{dr_{(0)}}{d\tilde{\tau}}$$

- As $\tilde{\tau} \to \tilde{\tau}_*$: solution in half-integer powers of $(\tilde{\tau}_* - \tilde{\tau})$. There are 3 free parameters to fix by matching with the transition

$$\ell_{(0)} - \ell_{(0)*} = \kappa^*_{(0),2}(\tilde{\tau}_* - \tilde{\tau}) + \kappa^*_{(0),3}(\tilde{\tau}_* - \tilde{\tau})^{3/2} + O(\tilde{\tau}_* - \tilde{\tau})^2$$

$$e_{(0)} - e_{(0)*} = \Omega_{(0)*}\kappa^*_{(0),2}(\tilde{\tau}_* - \tilde{\tau}) + e^*_{(0),3}(\tilde{\tau}_* - \tilde{\tau})^{3/2} + O(\tilde{\tau}_* - \tilde{\tau})^2$$

$$r_{(0)} - r_{(0)*} = r^*_{(0),1}(\tilde{\tau}_* - \tilde{\tau})^{1/2} + O(\tilde{\tau}_* - \tilde{\tau})$$

$$\delta_{(0)} = \delta^*_{(0),2}(\tilde{\tau}_* - \tilde{\tau}) + O(\tilde{\tau}_* - \tilde{\tau})^{3/2} \longrightarrow f^r_{(0)} = \frac{8}{3} r_{(0)*}^{-7/2}\delta_{(0)} + O(\tilde{\tau}_* - \tilde{\tau})^{3/2}$$

$$f^r_{(0)} = \Delta U_{(0)}^2 \Omega_{(0)}^2 r_{(0)}^{-4}\delta_{(0)}(\delta_{(0)} + 2r_{(0)}^{3/2})$$

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The transition to plunge

- Transition-timescale expansion where $s = \eta^{1/5}(\tau - \tau_*)$ is the transition proper time

$$r - r_{[0]*} = \eta^{2/5} R(\eta, s)$$
$$\ell - \ell_{[0]*} = \eta^{4/5} \xi(\eta, s)$$
$$e - e_{[0]*} = \Omega_{[0]*} \left[\eta^{6/5} Y(\eta, s) + \eta^{4/5} \xi(\eta, s)\right]$$
$$a = a_{[0]} + \sum_{i=1}^{\infty} \eta^{i/5} a_{[i]}(s)$$

$$R = \sum_{i=0}^{\infty} \eta^{i/5} R_{[i]}(s), \quad \xi = \sum_{i=0}^{\infty} \eta^{i/5} \xi_{[i]}(s), \quad Y = \sum_{i=0}^{\infty} \eta^{i/5} Y_{[i]}(s)$$

- The radial self-force and the deviation from geodesic circular angular velocity δ

$$f^r(s) = f^r_{[0]}(s)\eta^{4/5} + O_s(\eta), \quad f^r_{[0]}(s) = \epsilon_* R^2_{[0]}(s) - \zeta_* \xi_{[0]}(s)$$

$$\Omega = \frac{1}{r^{3/2} + a + \delta}, \quad \Omega = \frac{2(ae - \ell) + r\ell}{r(r^2 + a^2)e + 2a(ae - \ell)} \longrightarrow \delta = \delta_{[0]}\eta^{4/5} + O_s(\eta), \quad \delta_{[0]} = \frac{\pi_*}{\Delta_*^2} R^2_{[0]}(s) - \frac{A_*^{3/2}\Omega_{[0]*}}{\Delta_*} \xi_{[0]}(s)$$

Solution at early times

- The equations of motion:

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - V^{geo}$$

$$\frac{d^2r}{d\tau^2} + \frac{1}{2}\frac{\partial V^{geo}}{\partial r} = f^r$$

$$f^r = \left[\left(e - \frac{1}{2}\frac{\partial V^{geo}}{\partial e}\right)\frac{de}{d\tau} - \frac{1}{2}\frac{\partial V^{geo}}{\partial \ell}\frac{d\ell}{d\tau} - \frac{1}{2}\frac{\partial V^{geo}}{\partial a}\frac{da}{d\tau}\right]\left(\frac{dr}{d\tau}\right)^{-1}$$

$\longrightarrow$

$$\left(\frac{dR_{[0]}}{ds}\right)^2 = -\frac{2}{3}\alpha_* R_{[0]}^3 - 2\beta_*\kappa_* s R_{[0]} + \gamma_* Y_{[0]}$$

$$\frac{d^2R_{[0]}}{ds^2} = -(\alpha_* - \epsilon_*)R_{[0]}^2 - \kappa_*(\beta_* - \zeta_*)s$$

$$\frac{d}{ds}\left(Y_{[0]} - \frac{2\epsilon_*}{3\gamma_*}R_{[0]}^3 - \frac{2\kappa_*\zeta_*}{\gamma_*}R_{[0]}\right) = 2\kappa_*\frac{\beta_* - \zeta_*}{\gamma_*}R_{[0]}$$

$\longrightarrow$ Painlevé transcendent equation of the first kind

- At lowest order in η: algebraic equations that fix $e_{[0]*}$, $\ell_{[0]*}$ and $a_{[i]} = 0,\ i = 1,\ldots 6$
- As $s \to -\infty$ asymptotic solution to the Painlevé transcendent: series in half-integer powers of $(\tilde{\tau}_* - \tilde{\tau})$

$$r(s) = \sqrt{\frac{(\beta_* - \zeta_*)\kappa_*}{\alpha_* - \epsilon_*}}\sqrt{\tilde{\tau}_* - \tilde{\tau}} + r^*_{[0]} + \eta^{2/5}O(s^{-2}) + O_s(\eta^{3/5})$$

$$\ell(s) = \lambda_*(\tilde{\tau}_* - \tilde{\tau})^{3/2} + \kappa_*(\tilde{\tau}_* - \tilde{\tau}) + \ell^*_{[0]} + \eta^{6/5}O(s^{-1}) + O_s(\eta^{7/5})$$

$$e(s) = e^*_{[0],3}(\tilde{\tau}_* - \tilde{\tau})^{3/2} + \kappa_*\Omega_{[0]*}(\tilde{\tau}_* - \tilde{\tau}) + e_{[0]*} + \eta^{6/5}O(s^{-1}) + O_s(\eta^{7/5})$$

$$\delta(s) = \delta^*_{[0],2}(\tilde{\tau}_* - \tilde{\tau}) + \eta^{4/5}O(s^{-3/2}) + O_s(\eta)$$

$$f^r(s) = \frac{\kappa_*\beta_*}{2(\alpha_* - \epsilon_*)}\left.\frac{\partial^2 f^r}{\partial r^2}\right|_{[0]*}(\tilde{\tau}_* - \tilde{\tau}) + \eta^{4/5}O(s^{-3/2}) + O_s(\eta)$$

The match!

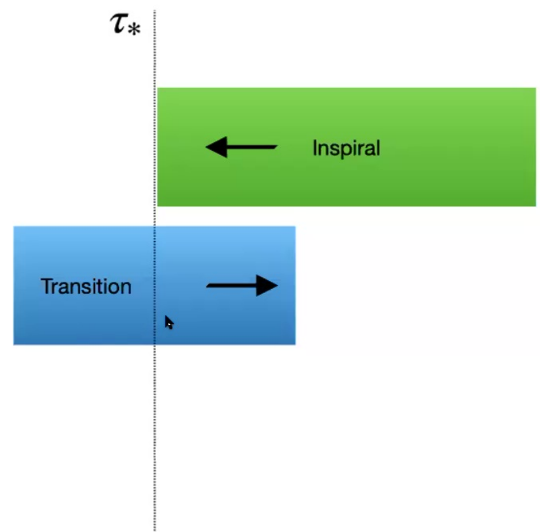
The inspiral

- is valid for all $\tau < \tau_*$
- reaches LSO as $\tau_* - \tau \ll \eta^{-1}$

The transition

- is valid for $-\eta^{-1/5} \ll \tau_* - \tau \ll \eta^{-1}$
- reaches LSO as $\tau_* - \tau \gg \eta^{-1/5}$

The overlapping region: $\eta^{-1/5} \ll \tau_* - \tau \ll \eta^{-1}$



The match!

The inspiral

$$r = r_{(0)*} + r^{*}_{(0),1}(\tilde{\tau}_* - \tilde{\tau})^{1/2} + h.o.$$

$$\ell = \ell_{(0)*} + \kappa^{*}_{(0),2}(\tilde{\tau}_* - \tilde{\tau}) + \kappa^{*}_{(0),3}(\tilde{\tau}_* - \tilde{\tau})^{3/2} + h.o.$$

The transition

$$r = \sqrt{\frac{(\beta_* - \zeta_*)\kappa_*}{\alpha_* - \epsilon_*}}(\tilde{\tau}_* - \tilde{\tau})^{1/2} + r_{[0]*} + h.o.$$

$$\ell = \lambda_*(\tilde{\tau}_* - \tilde{\tau})^{3/2} + \kappa_*(\tilde{\tau}_* - \tilde{\tau}) + \ell_{[0]*} + h.o.$$

- The matching fixes the free coefficients of the inspiral $r^{*}_{(0),1} = \sqrt{\frac{(\beta_* - \zeta_*)\kappa_*}{\alpha_* - \epsilon_*}}, \quad \kappa^{*}_{(0),2} = \kappa_*, \quad \kappa^{*}_{(0),3} = \lambda_*$
- Also the energy e and the deviation from circular geodesic angular velocity $\delta_{(0)}$ match
- The leading-order radial self-force close to the LSO:

$$f^{r}_{(0)} = \frac{\kappa_* \beta_*}{2(\alpha_* - \epsilon_*)} \frac{\partial^2 f^r}{\partial r^2}\bigg|_{[0]*} (\tilde{\tau}_* - \tilde{\tau}) + O_\eta(\tilde{\tau}_* - \tilde{\tau})^{3/2}$$

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Summary

- Exact consistent match of the adiabatic quasi-circular inspiral with the transition motion at leading order in the small mass ratio expansion

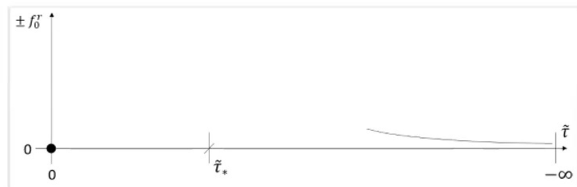
- The adiabatic inspiral with circular geodesic angular velocity does not match the solution in the transition regime

$$\Omega_{(0)} = \frac{1}{r_{(0)}^{3/2} + a_{(0)}} \rightarrow \Omega_{(0)} = \frac{1}{r_{(0)}^{3/2} + a_{(0)} + \delta_{(0)}}$$

- The adiabatic inspiral requires a zeroth-order (in the small mass ratio expansion) radial self-force $f^r_{(0)}$

$$f^r_{(0)} = \Delta U^2_{(0)} \Omega^2_{(0)} r^{-4}_{(0)} \delta_{(0)} (\delta_{(0)} + 2 r^{3/2}_{(0)})$$

$$f^r_{(0)} \sim (\tilde{\tau}_* - \tilde{\tau}) + O_\eta (\tilde{\tau}_* - \tilde{\tau})^{3/2}$$



Summary

- The Painlevé transcendent evolution in the transition remains true even in the presence of general self-force corrections
- It therefore controls the transition for any mass ratio (small mass ratio [Ori & Thorne, 2000], [Kesden, 2011], equal mass regime [Buonanno & Damour, 2000])

To do:

- Structure of the subleading match (and explicit match of the P1A inspiral)
- Understand $f^r_{(0)}$
- Include eccentric and non-equatorial inspirals