Title: Kerr self-force via elliptic PDEs: Numerical methods (part 2)

Speakers: Thomas Osburn

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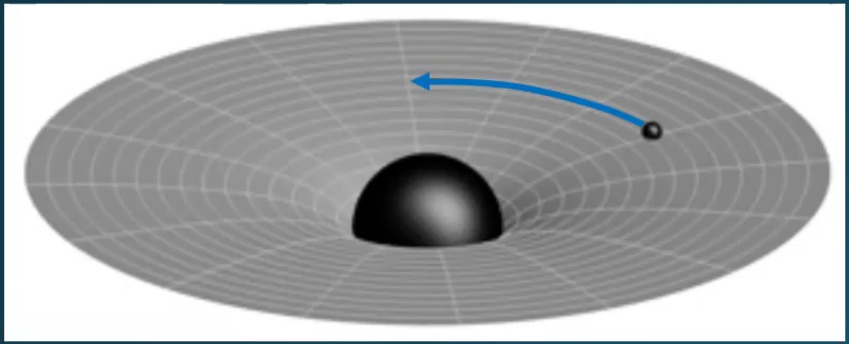
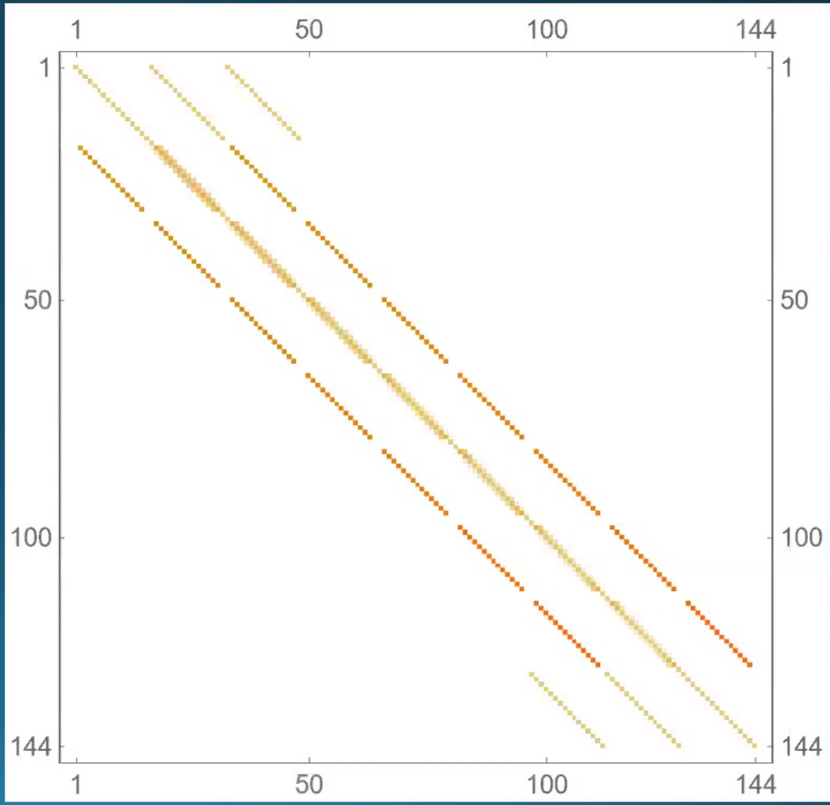
Abstract: I will discuss the numerical methods we use to calculate the self-force on a scalar charge orbiting a Kerr black hole. We apply a 2nd-order finite difference scheme on a rectangular grid in the r*-Î¸ plane. By working in the frequency domain and separating the Ï• variable (but not Î¸) we encounter elliptic PDEs, which present certain numerical challenges. One challenge is that every grid point is coupled to every other grid point so that a simultaneous solution requires solving a large linear system. Another related challenge involves how imperfect boundary conditions can introduce errors that would inevitably pollute the entire domain. We have applied various techniques to overcome these challenges, such as analyzing the boundary behavior to impose more sophisticated boundary conditions with improved accuracy (for a fixed outer boundary position). Various preliminary self-force results will be presented and discussed.

Kerr self-force via Elliptic PDEs: Numerical methods (part 2)





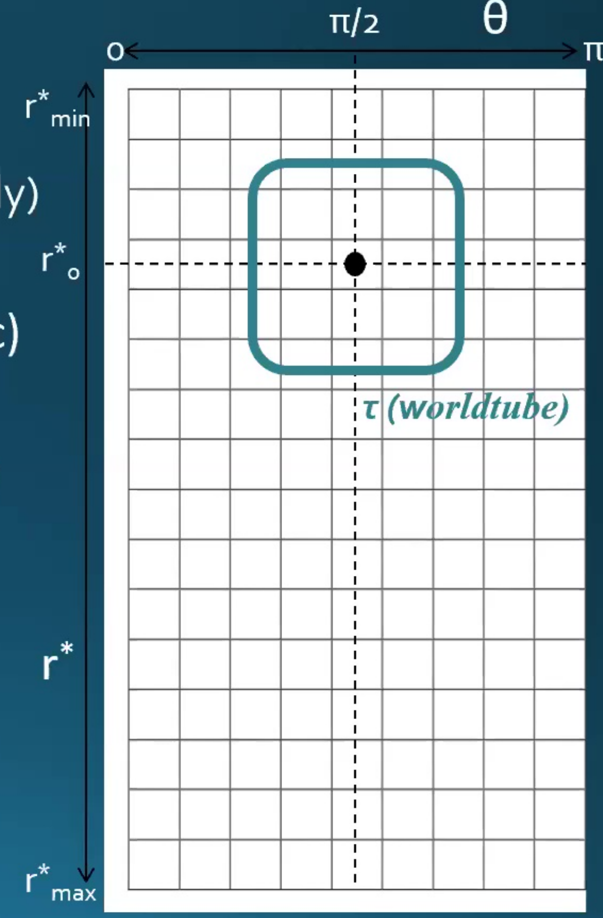
Tommy Osburn, tosburn@geneseo.edu
SUNY Geneseo
In collaboration with **Nami Nishimura**

Review/overview

- Lorenz gauge Kerr self-force calculations have encountered prior challenges:
  - non-separable in θ (so far)
  - numerical instabilities in time domain

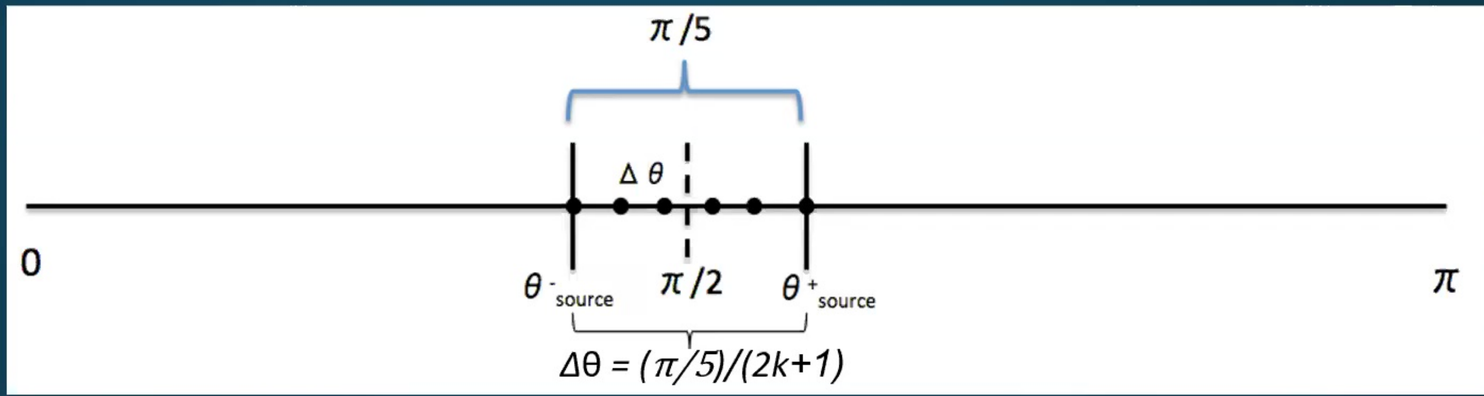
(Dolan, Barack 2013)
(Thornburg, Dolan currently)

- This work: r-θ PDEs in frequency domain ($\frac{\partial}{\partial t} = -i\omega \longrightarrow$ Elliptic)
- Scalar self-force - Mathematica development by Nami:
  - Rectangular grid in r_*- θ plane (m-modes ⟶ separated φ)
  - 2^{nd} order finite difference ⟶ sparse linear system
  - Effective source – Barry Wardell's code
  - Boundary conditions are approximate ⟶ error?
  - Convergence tests
- Future directions
  - Gravitational case
  - 2^{nd} order gravitational self-force?
  - Eccentric/inclined orbits?

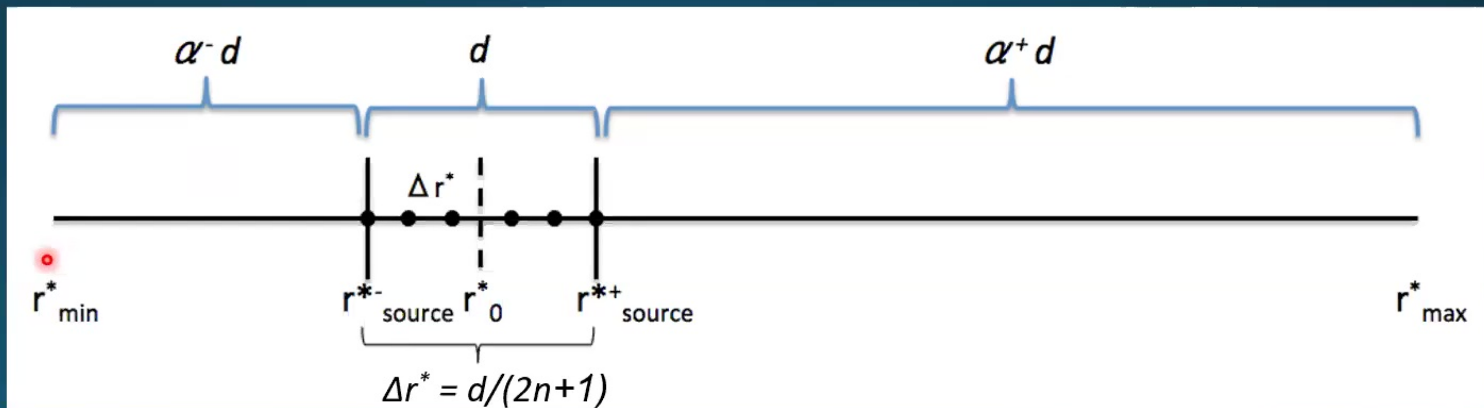


Numerical grid

const. r_* slice

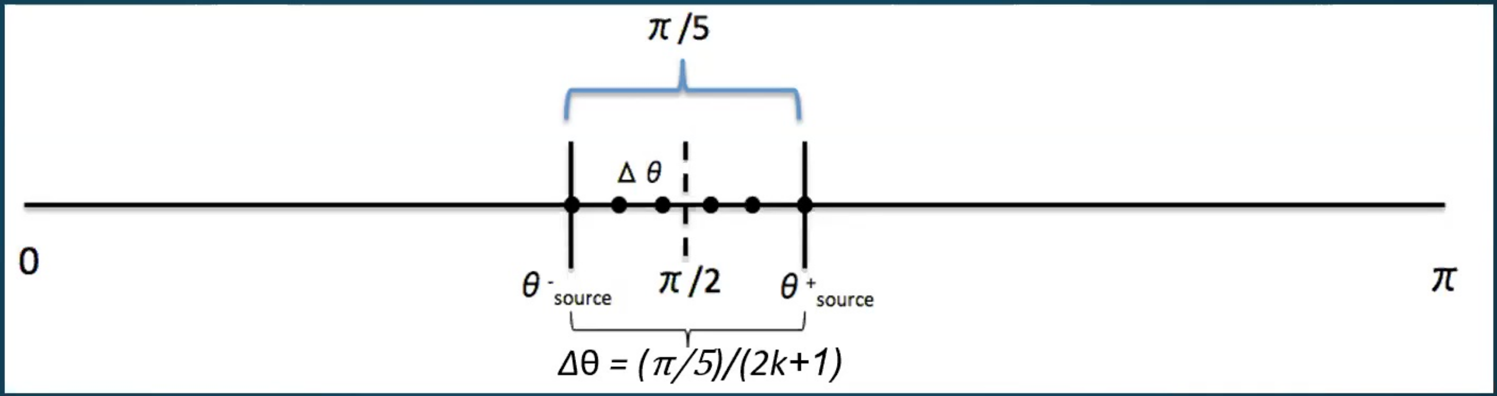


const. θ slice

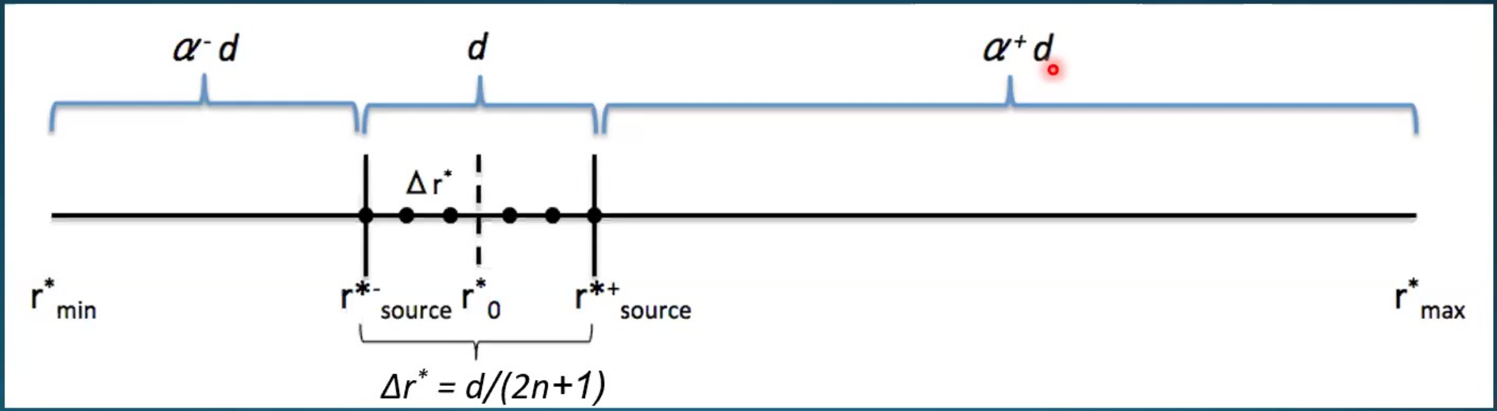


Numerical grid

const. r_* slice



const. θ slice



boundary pos. measured in multiples of ***d*** (worldtube diameter)

Finite difference equations

- Blue positions below: finite difference equation

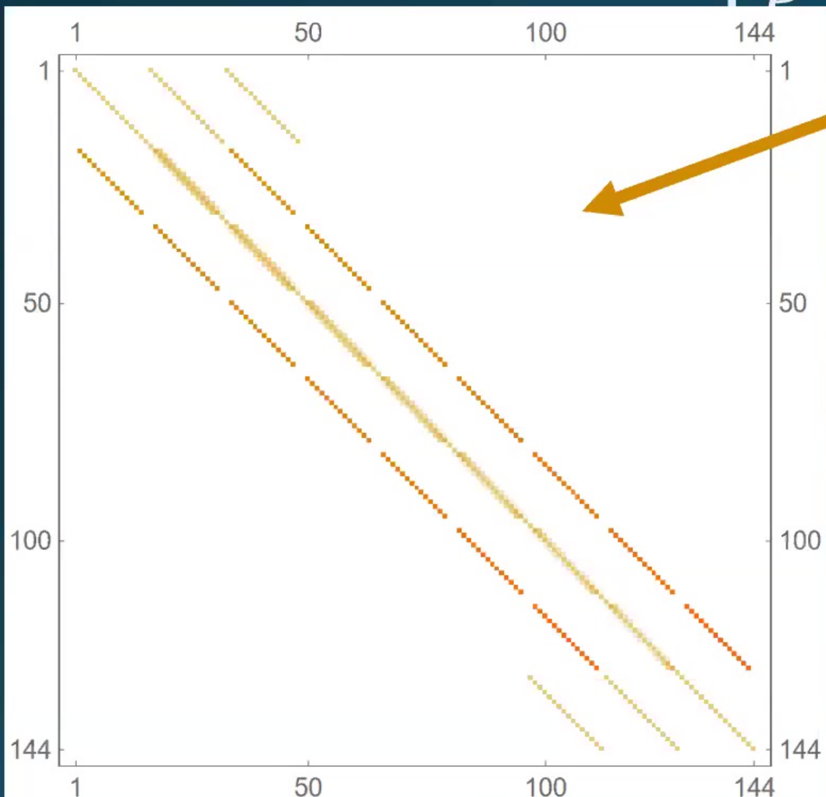
$$A\,\frac{\Psi^m_{21}-2\Psi^m_{31}+\Psi^m_{41}}{\Delta r_*^2}+B\,\frac{\Psi^m_{30}-2\Psi^m_{31}+\Psi^m_{32}}{\Delta\theta^2}+\cdots=0$$

- One algebraic equation and unknown for each grid point

- Vector of unknowns: $\begin{bmatrix}\Psi^m_{00}\\ \Psi^m_{01}\\ \Psi^m_{02}\\ \vdots\\ \Psi^m_{10}\\ \Psi^m_{11}\\ \Psi^m_{12}\\ \vdots\end{bmatrix}$

$$\begin{array}{cc} & \theta \longrightarrow \\ r_* \downarrow & \begin{pmatrix} \Psi^m_{00} & \Psi^m_{01} & \Psi^m_{02} & \Psi^m_{03} & \Psi^m_{04} & \Psi^m_{05} \\ \Psi^m_{10} & \Psi^m_{11} & \Psi^m_{12} & \Psi^m_{13} & \Psi^m_{14} & \Psi^m_{15} \\ \Psi^m_{20} & \Psi^m_{21} & \Psi^m_{22} & \Psi^m_{R23} & \Psi^m_{R24} & \Psi^m_{R25} \\ \Psi^m_{30} & \Psi^m_{31} & \Psi^m_{32} & \Psi^m_{R33} & \Psi^m_{R34} & \Psi^m_{R35} \\ \Psi^m_{40} & \Psi^m_{41} & \Psi^m_{42} & \Psi^m_{R43} & \Psi^m_{R44} & \Psi^m_{R45} \end{pmatrix} \end{array}$$

Sparse linear system



$$\begin{bmatrix} a & b & c & d & \cdots \\ e & f & g & h & \cdots \\ & j & k & l & \cdots \\ & n & o & p & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \Psi^m_{00} \\ \Psi^m_{01} \\ \vdots \\ \Psi^m_{10} \\ \Psi^m_{11} \end{bmatrix} = \begin{bmatrix} S^m_{00} \\ S^m_{01} \\ \vdots \\ S^m_{10} \\ S^m_{11} \end{bmatrix}$$

$$\begin{pmatrix} \Psi^m_{00} & \Psi^m_{01} & \Psi^m_{02} & \Psi^m_{03} & \Psi^m_{04} & \Psi^m_{05} \\ \Psi^m_{10} & \Psi^m_{11} & \Psi^m_{12} & \Psi^m_{13} & \Psi^m_{14} & \Psi^m_{15} \\ \Psi^m_{20} & \Psi^m_{21} & \Psi^m_{22} & \Psi^m_{R23} & \Psi^m_{R24} & \Psi^m_{R25} \\ \Psi^m_{30} & \Psi^m_{31} & \Psi^m_{32} & \Psi^m_{R33} & \Psi^m_{R34} & \Psi^m_{R35} \\ \Psi^m_{40} & \Psi^m_{41} & \Psi^m_{42} & \Psi^m_{R43} & \Psi^m_{R44} & \Psi^m_{R45} \end{pmatrix}$$

Worldtube and effective source

- Green positions below: finite difference equation

$$A\,\frac{\Psi^m_{R24}-2\Psi^m_{R34}+\Psi^m_{R44}}{\Delta r_*^2}+B\,\frac{\Psi^m_{R33}-2\Psi^m_{R34}+\Psi^m_{R35}}{\Delta\theta^2}+\cdots=S^m_{eff}$$

- S^m_{eff} from Barry Wardell's C code at barrywardell.net

- Because worldline between grid points, no issue numerically evaluating S^m_{eff} directly

$$\begin{pmatrix}
\Psi^m_{00} & \Psi^m_{01} & \Psi^m_{02} & \Psi^m_{03} & \Psi^m_{04} & \Psi^m_{05} \\
\Psi^m_{10} & \Psi^m_{11} & \Psi^m_{12} & \Psi^m_{13} & \Psi^m_{14} & \Psi^m_{15} \\
\Psi^m_{20} & \Psi^m_{21} & \Psi^m_{22} & \Psi^m_{R23} & \Psi^m_{R24} & \Psi^m_{R25} \\
\Psi^m_{30} & \Psi^m_{31} & \Psi^m_{32} & \Psi^m_{R33} & \Psi^m_{R34} & \Psi^m_{R35} \\
\Psi^m_{40} & \Psi^m_{41} & \Psi^m_{42} & \Psi^m_{R43} & \Psi^m_{R44} & \Psi^m_{R45}
\end{pmatrix}$$

worldtube

Crossing the worldtube? Two field types

- Purple positions below: finite difference equation

$$A\,\frac{\Psi^m_{04}-2\Psi^m_{14}+\Psi^m_{24}}{\Delta r_*^2}+B\,\frac{\Psi^m_{13}-2\Psi^m_{14}+\Psi^m_{15}}{\Delta\theta^2}+\cdots=0$$

does not appear in the vector of unknowns!

$$\begin{pmatrix}
\Psi^m_{00} & \Psi^m_{01} & \Psi^m_{02} & \Psi^m_{03} & \Psi^m_{04} & \Psi^m_{05}\\
\Psi^m_{10} & \Psi^m_{11} & \Psi^m_{12} & \Psi^m_{13} & \Psi^m_{14} & \Psi^m_{15}\\
\Psi^m_{20} & \Psi^m_{21} & \Psi^m_{22} & \Psi^m_{R23} & \Psi^m_{R24} & \Psi^m_{R25}\\
\Psi^m_{30} & \Psi^m_{31} & \Psi^m_{32} & \Psi^m_{R33} & \Psi^m_{R34} & \Psi^m_{R35}\\
\Psi^m_{40} & \Psi^m_{41} & \Psi^m_{42} & \Psi^m_{R43} & \Psi^m_{R44} & \Psi^m_{R45}
\end{pmatrix}$$

worldtube

Crossing the worldtube? Two field types

- Purple positions below: finite difference equation

$$A\,\frac{\Psi^m_{04} - 2\Psi^m_{14} + \Psi^m_{R24} + \Psi^m_{P24}}{\Delta r_*^2} + B\,\frac{\Psi^m_{13} - 2\Psi^m_{14} + \Psi^m_{15}}{\Delta\theta^2} + \cdots = \overset{\text{known}}{0}$$

- Use singular puncture field to transform appropriately

$$\begin{array}{c}\begin{pmatrix}
\Psi^m_{00} & \Psi^m_{01} & \Psi^m_{02} & \Psi^m_{03} & \Psi^m_{04} & \Psi^m_{05} \\
\Psi^m_{10} & \Psi^m_{11} & \Psi^m_{12} & \Psi^m_{13} & \Psi^m_{14} & \Psi^m_{15} \\
\Psi^m_{20} & \Psi^m_{21} & \Psi^m_{22} & \Psi^m_{R23} & \Psi^m_{R24} & \Psi^m_{R25} \\
\Psi^m_{30} & \Psi^m_{31} & \Psi^m_{32} & \Psi^m_{R33} & \Psi^m_{R34} & \Psi^m_{R35} \\
\Psi^m_{40} & \Psi^m_{41} & \Psi^m_{42} & \Psi^m_{R43} & \Psi^m_{R44} & \Psi^m_{R45}
\end{pmatrix} & \text{worldtube}\end{array}$$

Crossing the worldtube? Two field types

- Purple positions below: finite difference equation

$$A\frac{\Psi^m_{04}-2\Psi^m_{14}+\boxed{\Psi^m_{R24}}}{\Delta r_*^2}+B\frac{\Psi^m_{13}-2\Psi^m_{14}+\Psi^m_{15}}{\Delta\theta^2}+\cdots=\boxed{-\frac{A}{\Delta r_*^2}\Psi^m_{P24}}$$

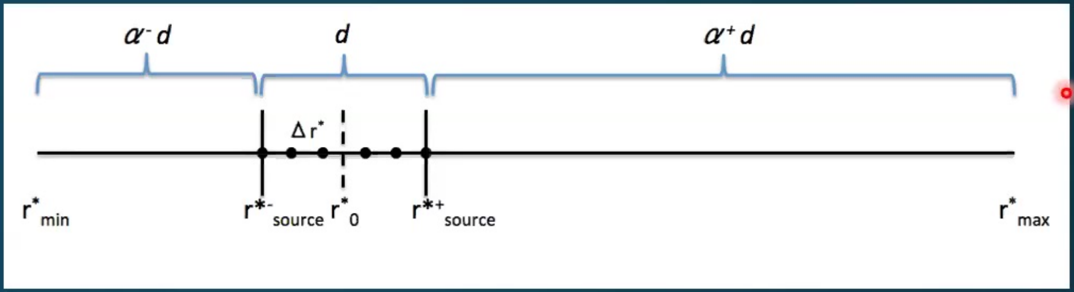
- Use singular puncture field to transform appropriately
- Jumps from crossing worldtube are accounted for through source vector
- Coupling matrix NOT affected

$$\begin{pmatrix}
\Psi^m_{00} & \Psi^m_{01} & \Psi^m_{02} & \Psi^m_{03} & \Psi^m_{04} & \Psi^m_{05} \\
\Psi^m_{10} & \Psi^m_{11} & \Psi^m_{12} & \Psi^m_{13} & \Psi^m_{14} & \Psi^m_{15} \\
\Psi^m_{20} & \Psi^m_{21} & \Psi^m_{22} & \Psi^m_{R23} & \Psi^m_{R24} & \Psi^m_{R25} \\
\Psi^m_{30} & \Psi^m_{31} & \Psi^m_{32} & \Psi^m_{R33} & \Psi^m_{R34} & \Psi^m_{R35} \\
\Psi^m_{40} & \Psi^m_{41} & \Psi^m_{42} & \Psi^m_{R43} & \Psi^m_{R44} & \Psi^m_{R45}
\end{pmatrix}$$

worldtube

Numerical boundary conditions

- θ BCs are straightforward
- r^* BCs involve subtleties

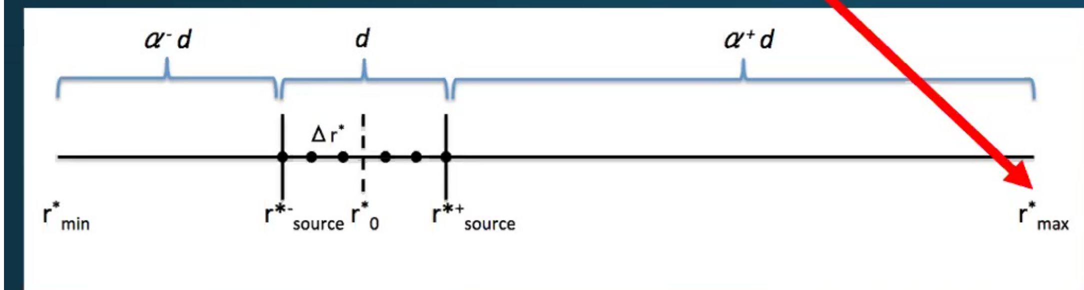


@ $r^* = \pm\infty$: $\Psi^m e^{-i\omega t} = A(\theta)e^{-i\omega(t\mp r^*)} \longrightarrow \boxed{\frac{\partial \Psi^m}{\partial r^*} \mp i\omega\Psi^m = 0}$

- Super-radiance near horizon? Accounted for (horizon BC no issues)
- But, truncating r^*_{max} with the above BC introduces significant error

Large r* analysis:

$$\Psi^m = e^{i\omega r^*}\left(A(\theta) + \frac{B(\theta)}{r} + \frac{C(\theta)}{r^2} + \vartheta(\frac{1}{r^3})\right)$$



"naïve" condition -

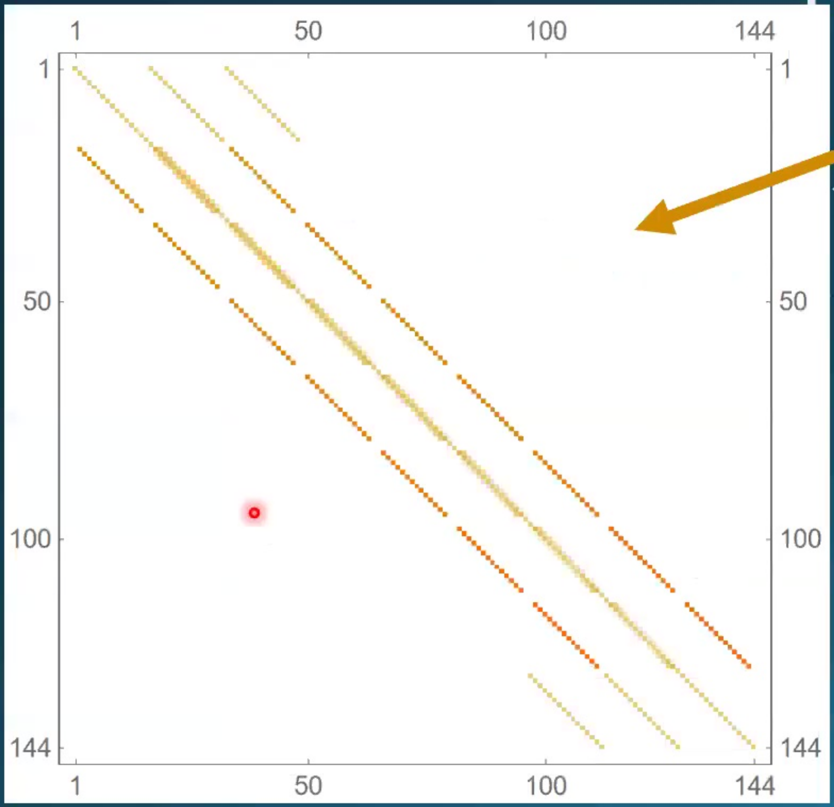
$$\boxed{\text{BC1: } \frac{\partial \Psi^m}{\partial r^*} - i\omega\Psi^m = \vartheta(\frac{1}{r_{max}^2})}$$

Less naïve - $\boxed{\text{BC2: } \frac{\partial^2 \Psi^m}{\partial {r^*}^2} - 2i\omega\frac{\partial \Psi^m}{\partial r^*} - i\omega\Psi^m = \vartheta(\frac{1}{r_{max}^3})}$

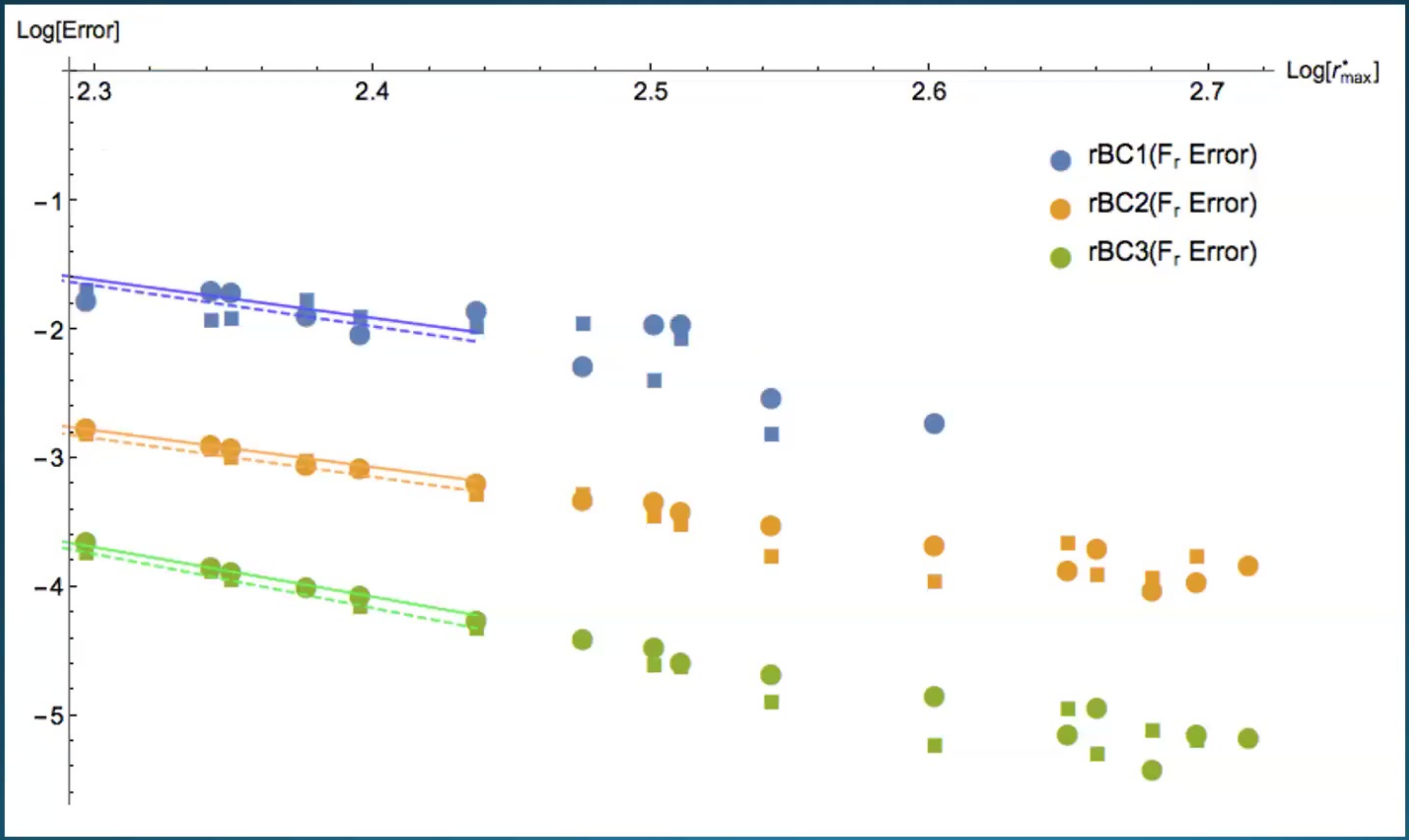
Sophisticated? - $\boxed{\text{BC3: } \frac{\partial^3 \Psi^m}{\partial {r^*}^3} - 3i\omega\frac{\partial^2 \Psi^m}{\partial {r^*}^2} - 3\omega^2\frac{\partial \Psi^m}{\partial r^*} + i\omega^3\Psi^m = \vartheta(\frac{1}{r_{max}^4})}$

Solving the linear system

$$\begin{bmatrix} a & b & c & d & \cdots \\ e & f & g & h & \cdots \\ i & j & k & l & \cdots \\ m & n & o & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \Psi^m_{00} \\ \Psi^m_{01} \\ \vdots \\ \Psi^m_{10} \\ \Psi^m_{11} \\ \vdots \end{bmatrix} = \begin{bmatrix} S^m_{00} \\ S^m_{01} \\ \vdots \\ S^m_{10} \\ S^m_{11} \\ \vdots \end{bmatrix}$$

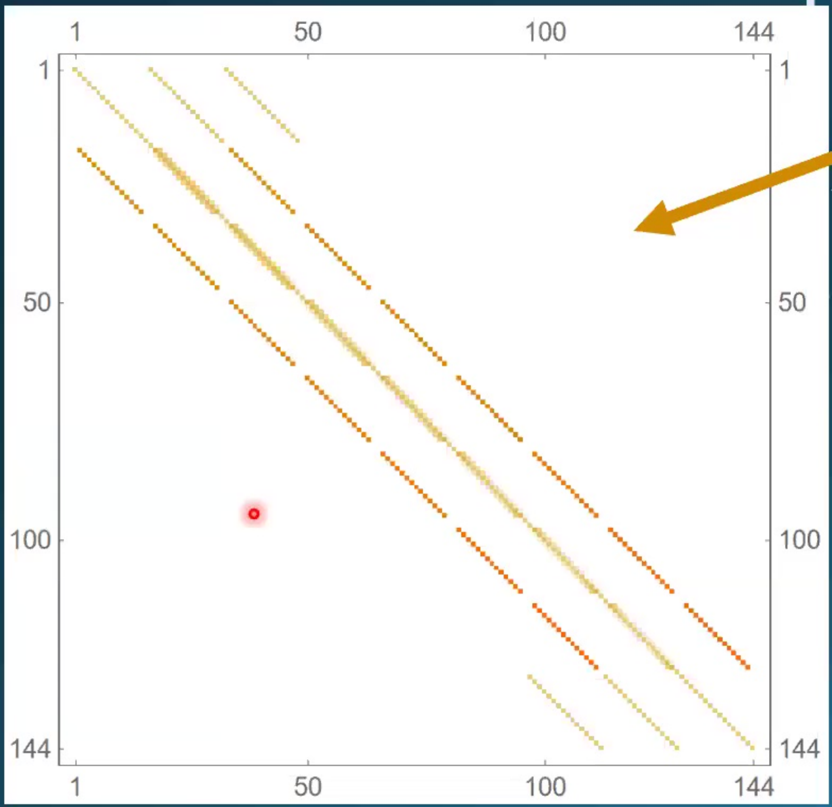


How far away must the outer boundary (r^*_{max}) be?



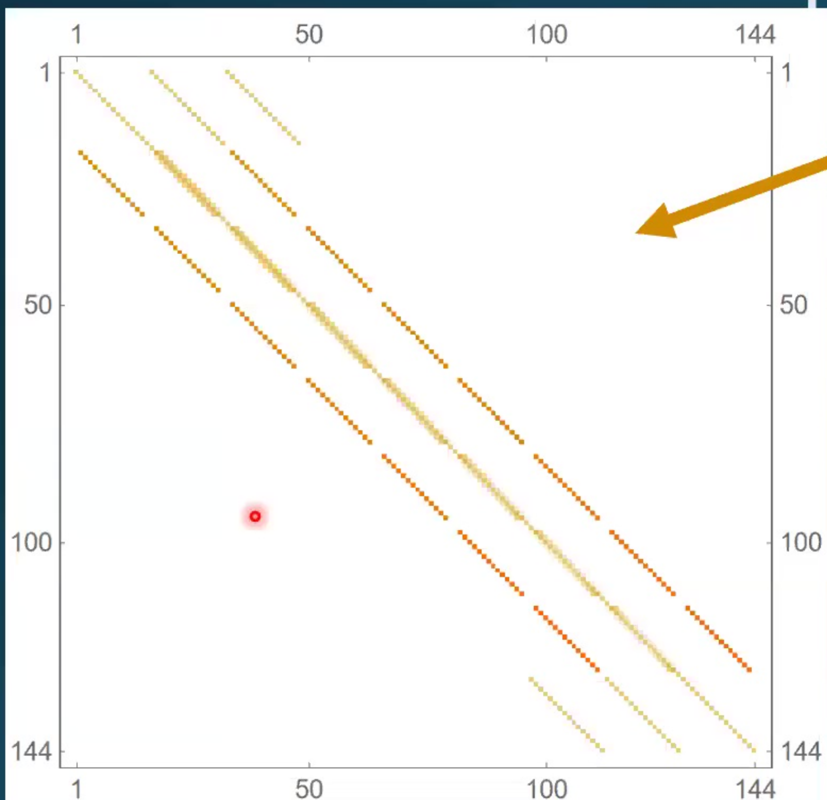
Solving the linear system

$$\begin{bmatrix} a & b & c & d & \cdots \\ e & f & g & h & \cdots \\ i & j & k & l & \cdots \\ m & n & o & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \Psi_{00}^{m} \\ \Psi_{01}^{m} \\ \vdots \\ \Psi_{10}^{m} \\ \Psi_{11}^{m} \\ \vdots \end{bmatrix} = \begin{bmatrix} S_{00}^{m} \\ S_{01}^{m} \\ \vdots \\ S_{10}^{m} \\ S_{11}^{m} \\ \vdots \end{bmatrix}$$



Solving the linear system

$$\begin{bmatrix} a & b & c & d & \cdots \\ e & f & g & h & \cdots \\ i & j & k & l & \cdots \\ m & n & o & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \Psi^m_{00} \\ \Psi^m_{01} \\ \vdots \\ \Psi^m_{10} \\ \Psi^m_{11} \\ \vdots \end{bmatrix} = \begin{bmatrix} S^m_{00} \\ S^m_{01} \\ \vdots \\ S^m_{10} \\ S^m_{11} \\ \vdots \end{bmatrix}$$

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LinearSolve[mat, b, Method → "Krylov"]
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VS.

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LinearSolve[mat, b, Method → "Pardiso"]
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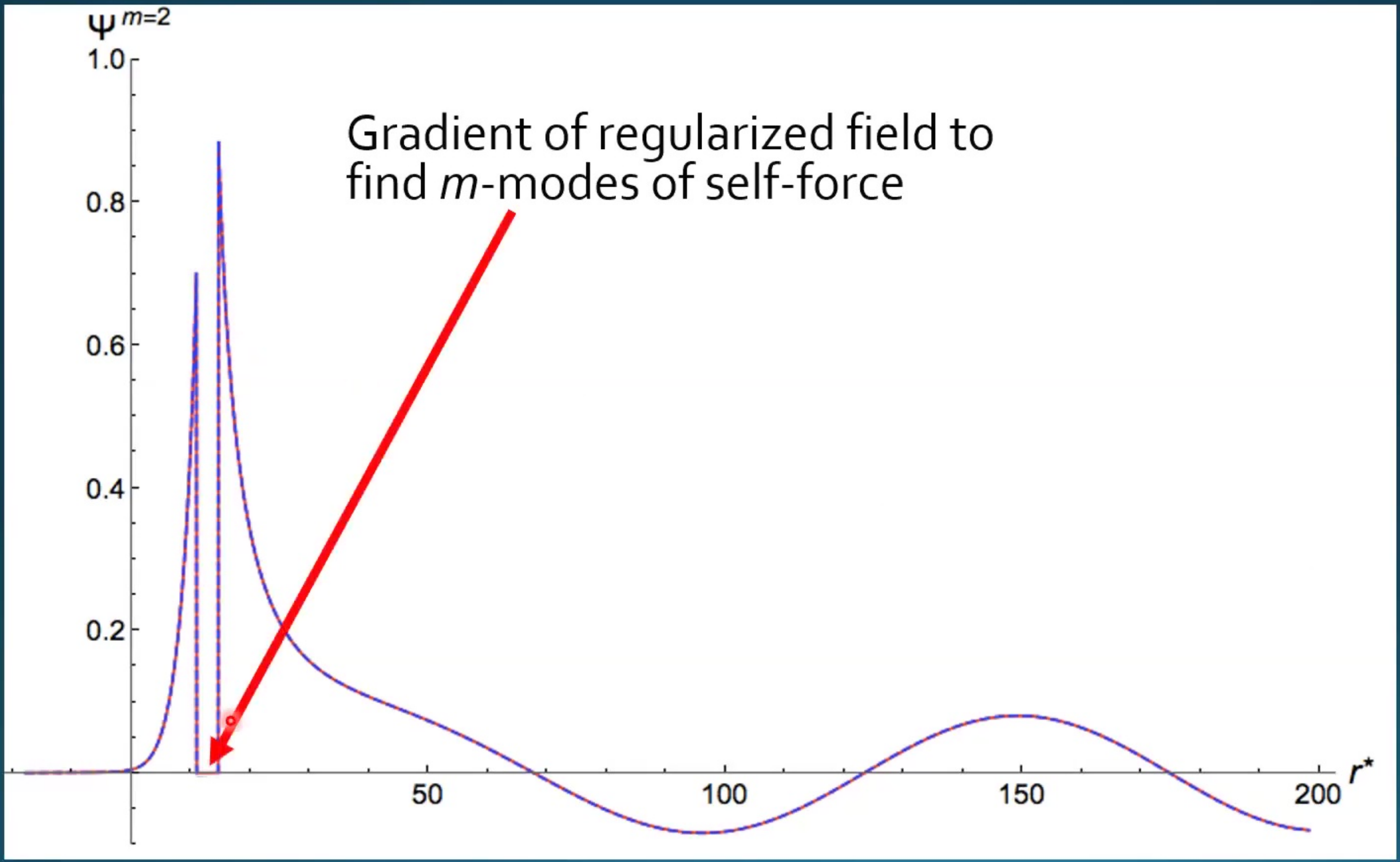
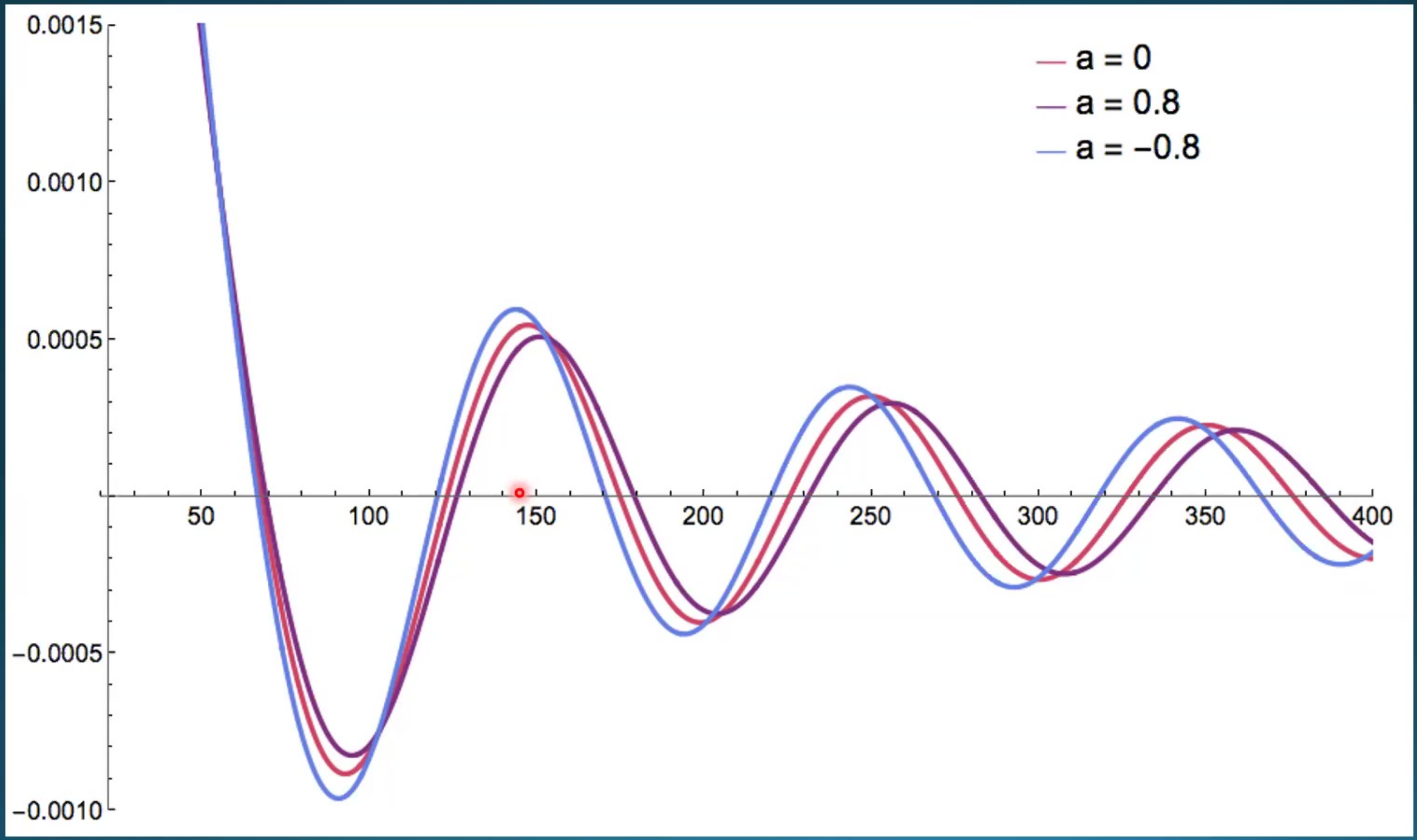
Scalar field results

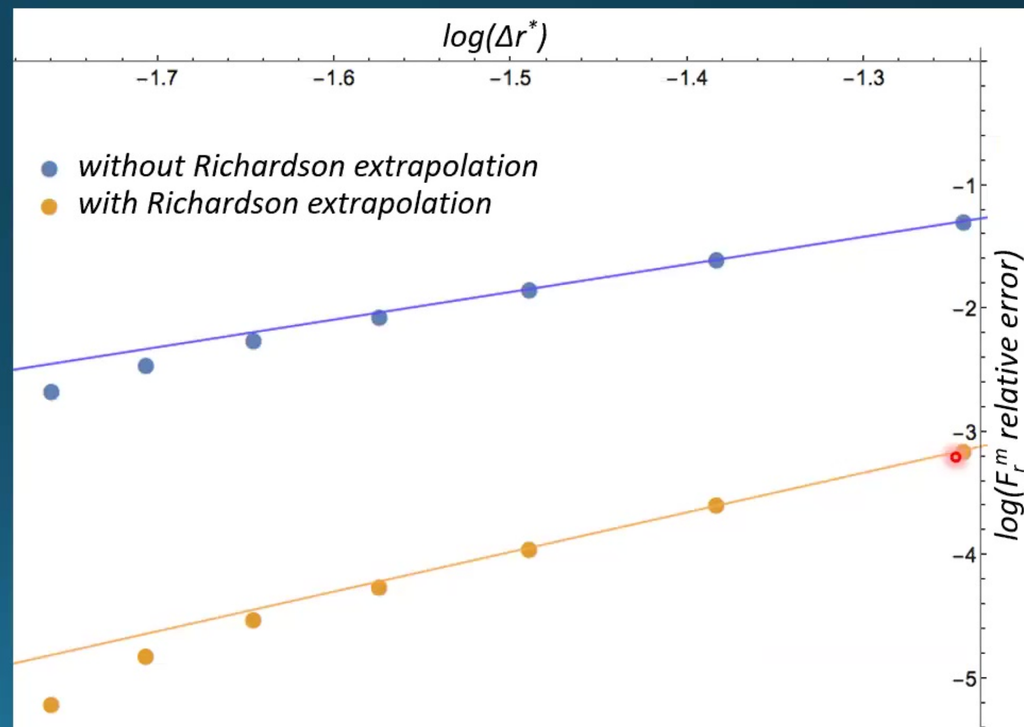


Scalar field results



Convergence test: decreasing Δr^*

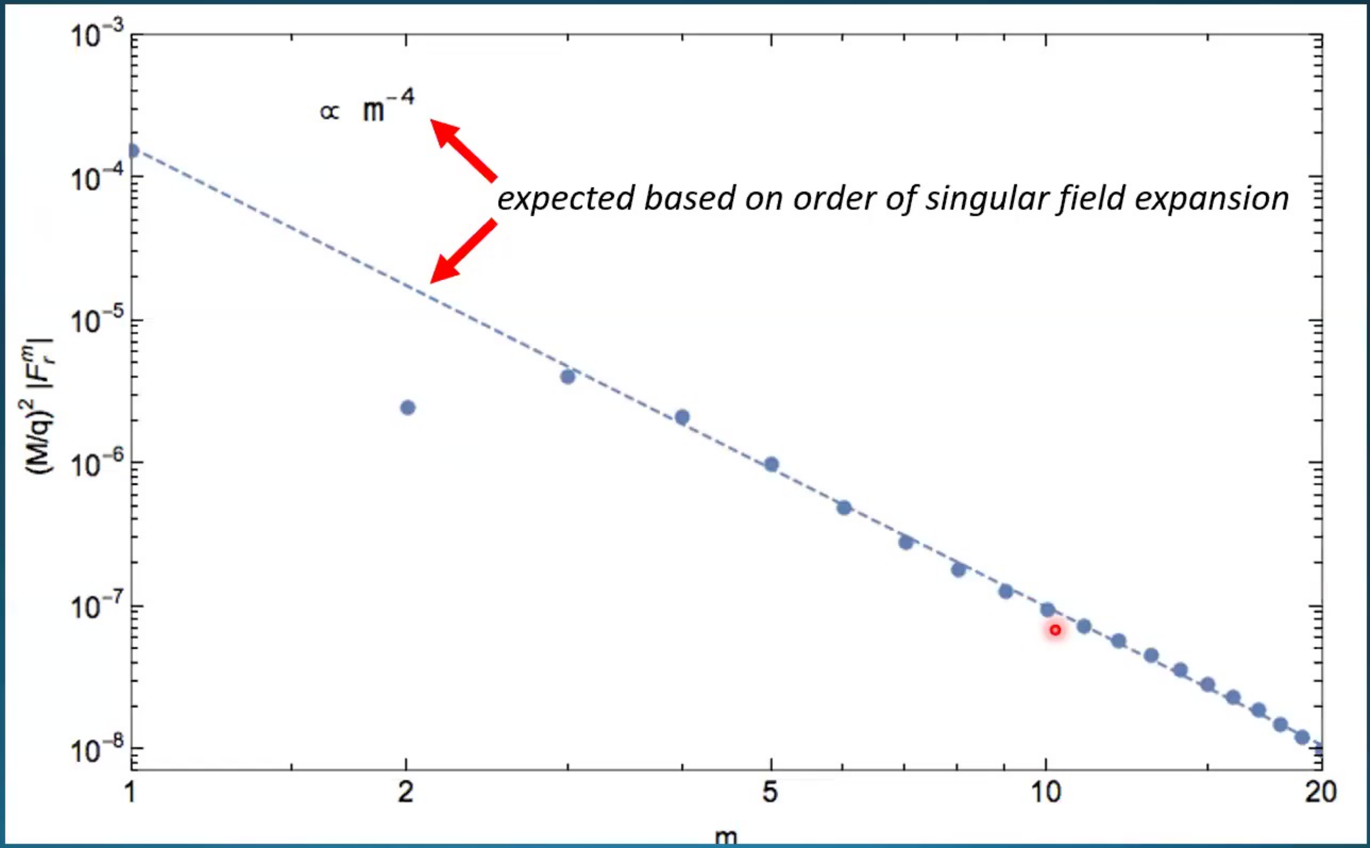
- Increase resolution until self-force converges (for each *m*-mode)
- 2[nd] order method converges rather slowly



Sum over m-modes: $F_r = \sum_m F_r^m$

- Maybe minor discrepancies?

- So far, self-force vs. past work agrees within ~**1%**
(Dolan, Barack, Wardell 2011)
(Thornburg, Wardell 2017)



Concluding notes and summary

- *Long-term goal*: calculate Lorenz gauge Kerr gravitational self-force

-*Progress so far*: calculate scalar Kerr self-force (to develop methods)

- Solved r- θ PDEs in frequency domain ($\frac{\partial}{\partial t} = -i\omega \longrightarrow$ Elliptic)
- Asymptotic analysis $\longrightarrow$ needed to develop sophisticated BCs
- Still a few minor issues to work out for optimized accuracy

- *Other method tried (failed)*: Hyperboloidal slicing and compactification

- *Future directions*:

- Use faster programming language (students: Balor Brennan, Luis Vasquez)
- Lorenz gauge Kerr gravitational self-force
- 2nd order self-force in Kerr spacetime

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  - 2^{nd} order self-force in Kerr spacetime
  - Eccentric/inclined orbits? I have some ideas, probably possible…