Title: Kerr self-force via elliptic PDEs: Background and theory (part 1)

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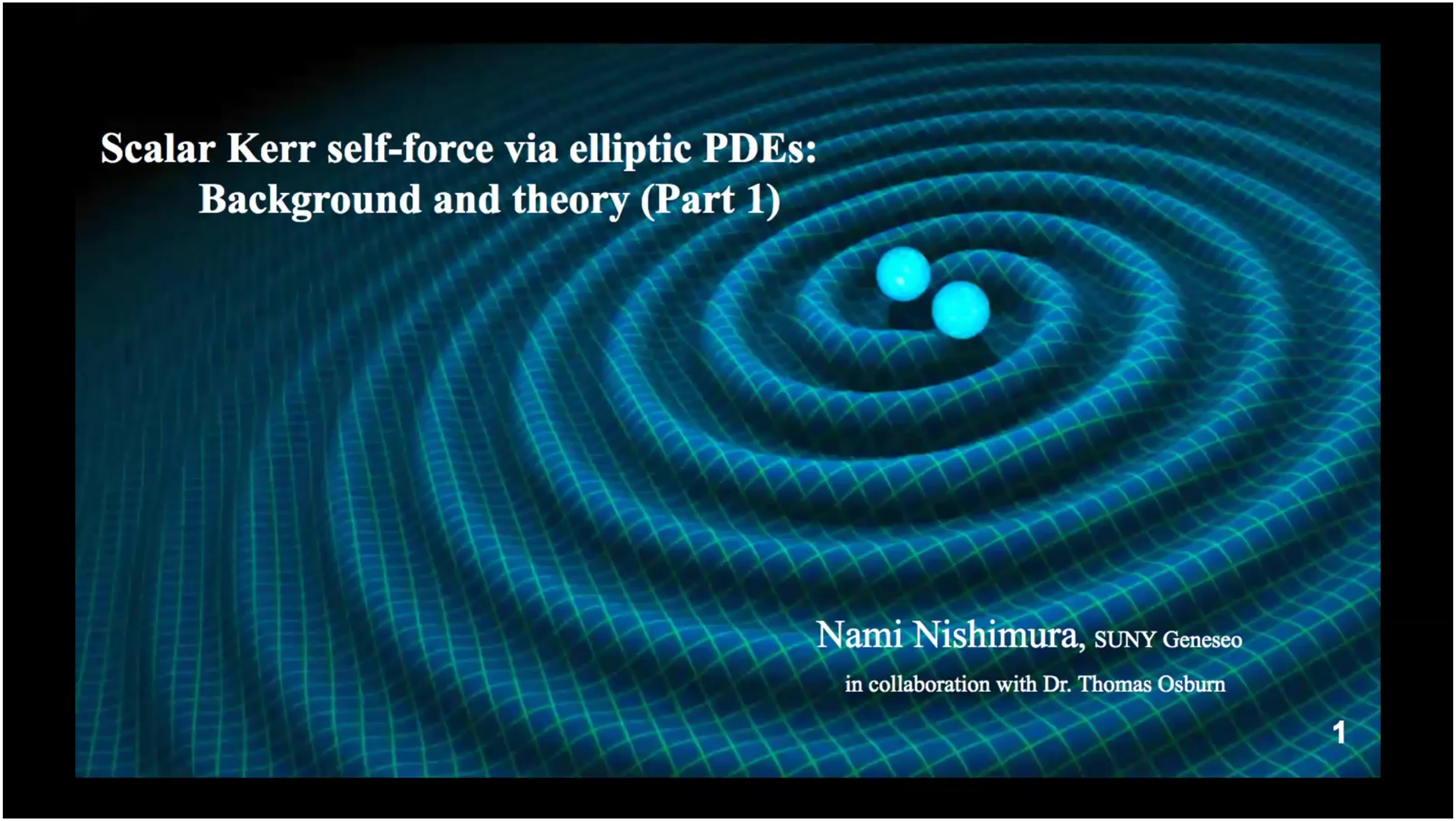
Abstract: Our long-term goal is to calculate the Lorenz gauge gravitational self-force for an extreme mass-ratio binary system in Kerr spacetime. Past work in the time-domain has encountered time instabilities for the two lowest modes m=0 and m=1. In order to overcome this problem, we enter the frequency-domain, which introduces elliptic PDEs. To develop an appropriate scheme, we first investigate the scalar self-force in Kerr spacetime by separating the Ï¦ and t variables. To calculate the self-force, we use the effective source method. This presentation discusses the background and theory, which will be followed by a discussion of numerical methods in a later presentation.


Scalar Kerr self-force via elliptic PDEs:
Background and theory (Part 1)
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in collaboration with Dr. Thomas Osburn
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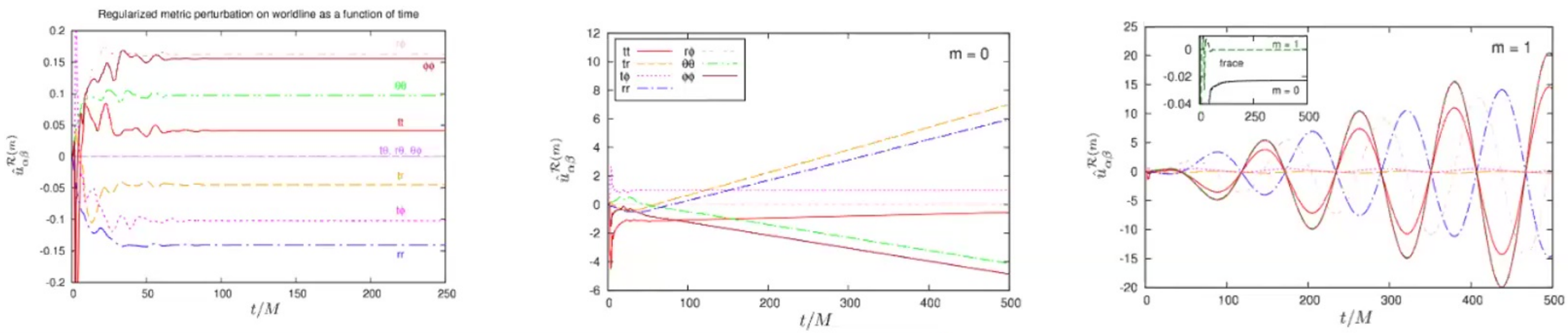
Past works and our long-term goal

Gravitational self-force in EMRIs (Extreme Mass Ratio Inspirals) has been calculated….

1. **In time-domain in Lorenz gauge** → Instabilities for m=0, 1 (Monopole and dipole cannot be evolved stably using time-domain method)



Regularized metric perturbation (MP) for m=0 and m=1 does not converge

L.Barack and S.R.Dolan.Phys.Rev.D87,084066(2013)

Past works and our long-term goal

Gravitational self-force in EMRIs (Extreme Mass Ratio Inspirals) has been calculated….

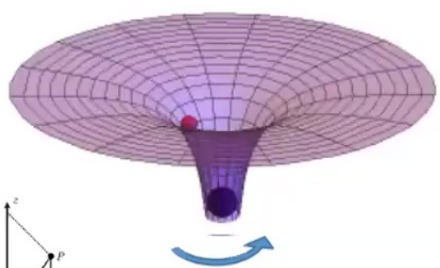
1. **In time-domain in Lorenz gauge** → Instabilities for m=0, 1 (Monopole and dipole cannot be evolved stably using time-domain method)

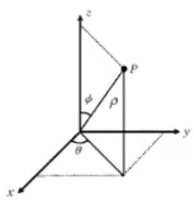
Difference in domain	
Time domain	Frequency domain
Involves a direct numerical time-integration of the Hyperbolic PDEs Can accommodate arbitrary particle orbits **Catastrophic instability in Lorenz gauge for m=0 and m=1**	Involves numerical computations of Elliptic PDEs with Fourier transform in time Accurate for circular or near-circular particle orbits, but not for high eccentricity orbit

Our Long-term goal and theoretical model

Our long-term goal : calculate the Lorenz gauge gravitational self-force in Kerr spacetime for an extreme mass-ratio binary system in frequency domain.

We first investigate the scalar self-force





Our Theoretical Model

- A point particle with a scalar charge q moving in a circular orbit around a Kerr black hole
- Extreme mass ratio binary system
- Kerr background shows azimuthal symmetry
- Separate t and ϕ variables

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The Scalar-Field Equation

- Boyer-Lindquist coordinate ϕ is pathological at the event horizon

New coordinate $\Rightarrow$ $d\varphi = d\phi + \frac{a}{\Delta} dr \qquad \varphi(\phi, r) = \phi + \underbrace{\frac{a}{r_+ - r_-} \ln\left|\frac{r - r_+}{r - r_-}\right|}_{\Delta\phi(r)}$

$a = J/M \quad |a| > 0$
$a > 0$ for the prograde
$a < 0$ for the retrograde

- We decompose the field equation into azimuthal modes to reduce the dimensionality

$$\boxed{\Phi(t, r, \theta, \phi) = e^{-i\omega t} \sum_{m=-\infty}^{\infty} \Phi^m e^{im\varphi}} \quad \overset{\text{Fourier Series}}{\Longleftrightarrow} \quad \boxed{\Phi^m(t, r, \theta) = \frac{e^{-i\omega t}}{2\pi} \int_{-\pi}^{\pi} e^{-im\varphi} \Phi(t, r, \theta, \varphi)\, d\varphi}$$

- Kerr tortoise coordinate r^*

$$\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta} \qquad r_* = r + \frac{2M}{r_+ - r_-}\left(r_+ \ln\left|\frac{r - r_+}{2M}\right| - r_- \ln\left|\frac{r - r_-}{2M}\right|\right)$$

… r* maps the interval (r_+, ∞) to the interval $(-\infty, \infty)$, where r_+ is the event horizon.

The Scalar-Field Equation

The scalar field is governed by Klein-Gordon equation

$$\boxed{\Box\Phi = \frac{1}{\sqrt{-g}}\delta\left(\sqrt{-g}\, g^{\mu\gamma}\delta_\gamma\Phi\right) = S}$$

Kerr metric determinant: $g = -\rho^2 \sin^2\theta$

$\Psi^m = \mathrm{r}\,\Phi^m$ We introduce the radial-factored field Ψ^m

m-mode scalar wave operator

$$\boxed{\Box_\Psi^m \Psi^m = S_\Psi^m}$$

$$\Box_\Psi^m = -\omega^2 + \frac{4amMr}{\Sigma^2}\omega - \frac{(r^2+a^2)}{\Sigma^2}\frac{\partial^2}{\partial r_*^2} - \left[\frac{2iamr(r^2+a^2) - 2a^2\Delta}{r\Sigma^2}\right]\frac{\partial}{\partial r_*}$$
$$-\frac{\Delta}{\Sigma^2}\left[\frac{\partial^2}{\partial\theta^2} + cot\theta\frac{\partial}{\partial\theta} - \frac{m^2}{sin^2\theta} - \frac{2M}{r}\left(1 - \frac{a^2}{Mr}\right) - \frac{2iam}{r}\right]$$

$$\Sigma^2 = (r^2+a^2)^2 - a^2\Delta sin^2\theta$$

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Effective Source

We introduce a "puncture" scalar field Φ_p, the local approximation of the divergent singular field Φ_s near the particle's worldline.

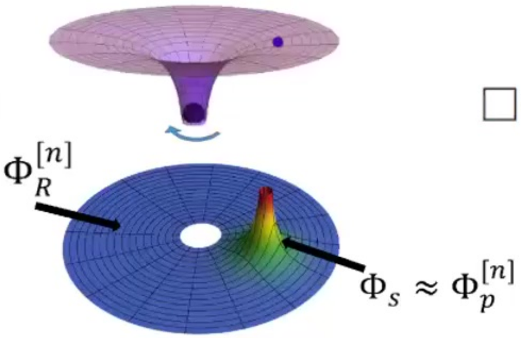


Image: B.Wardell, I.Vega, J.Thornburg and P.Diener(2012)

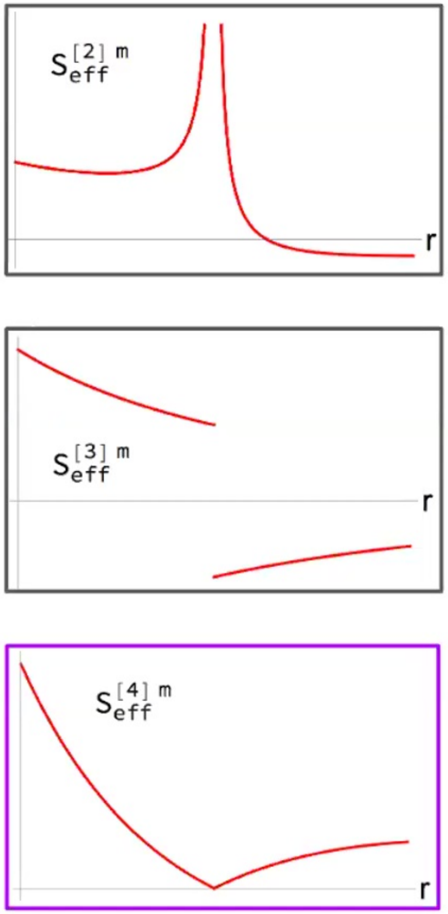
$$\Phi_R^{[n]} = \Phi - \Phi_P^{[n]} \qquad \Phi_R^{[n]} : \text{Residual field}$$

$$\Box\Phi = \Box\Phi_R^{[n]} + \Box\Phi_P^{[n]} \equiv S_{eff}^{[n]}$$

$$S_{eff}^{[n]m} = \Box\Phi_p^{[n]m}$$

Inverse Form of Fourier Expansions

$$S_{eff}^{[n]m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Box\Phi_p^{[n]} e^{-im\varphi} d\varphi$$



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Effective Source

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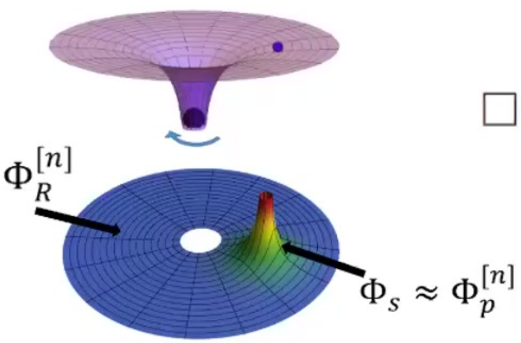

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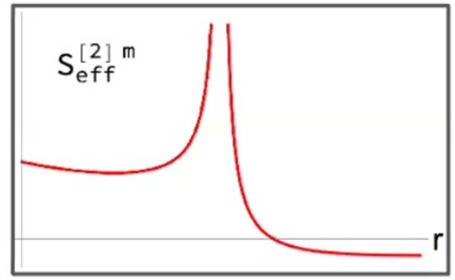
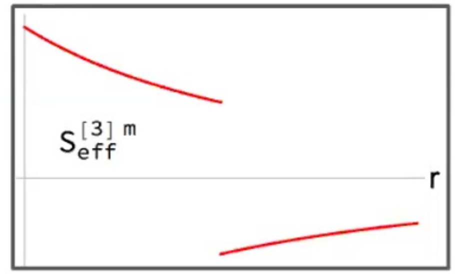
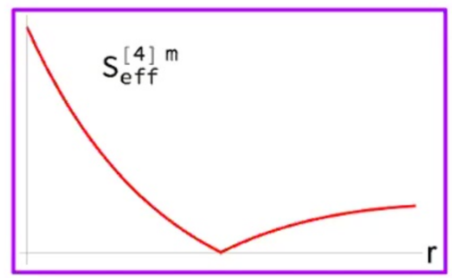
Inverse Form of Fourier Expansions

$$S_{eff}^{[n]m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Box\Phi_p^{[n]} e^{-im\varphi} d\varphi$$

...We expand the singular field up to 4th order by using Taylor Expansion to gain finite Effective Source (B.Wardell , "kerr-circular.c", https://github.com/barrywardell/EffectiveSource)







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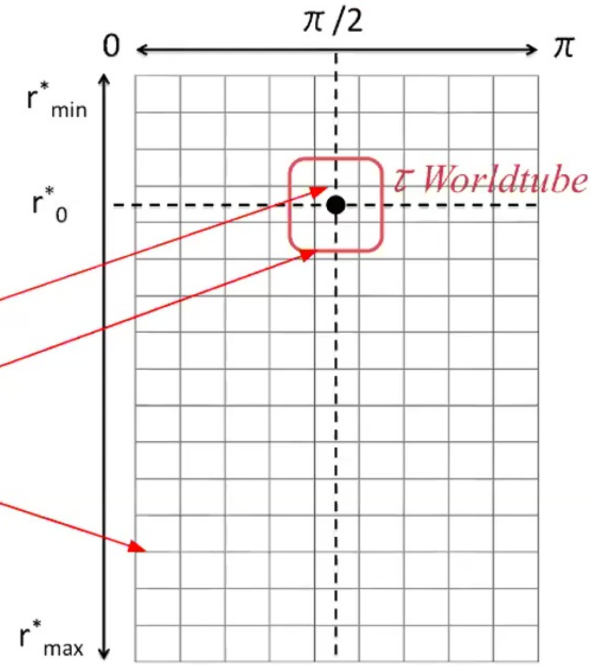
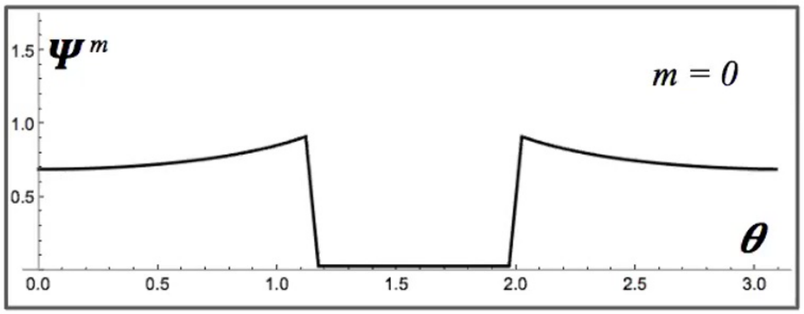
Worldtube Scheme

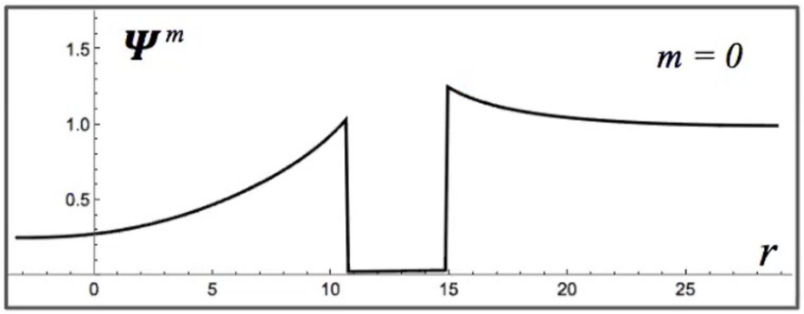
Boundary conditions only apply to the retarded field $\mathbf{\Psi}^{m}$

$$\begin{cases} \Box^{m}_{\Psi}\Psi^{m} = 0 & \textit{outside } \mathcal{T} \\ \Box^{m}_{\Psi}\Psi^{m}_{\mathrm{R}} = S^{m}_{\Psi eff} & \textit{inside } \mathcal{T} \\ \Psi^{m}_{\mathrm{R}} = \Psi^{m} - r\Phi^{m}_{\mathrm{P}} & \textit{across } \delta\mathcal{T} \end{cases}$$

where $\mathbf{\Psi}_{R}^{m}$ is the residual field mode
and $\mathbf{\Psi}^{m}$ is the full field mode







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Boundary Condition

$\boldsymbol{\theta}$ Boundary Conditions

$$\boxed{\frac{\partial \Psi^{m=0}}{\partial \theta}(\theta \in \{0,\pi\}) = 0 \qquad \Psi^{m\neq 0}(\theta \in \{0,\pi\}) = 0}$$

r* Boundary Conditions

… At r* = ± ∞, we assume that the waves are only propagating radially outward from the source

$$\Psi^{\pm} = f(\theta) e^{-i(\omega t \mp \omega r^*)}$$

$$\boxed{\frac{d\Psi^{+}_{r_*=\infty}}{dr_*} - i\,\omega\,\Psi^{+}_{r_*=\infty} = 0 \qquad \frac{d\Psi^{-}_{r_*=-\infty}}{dr_*} + i\,\omega\,\Psi^{-}_{r_*=-\infty} = 0}$$

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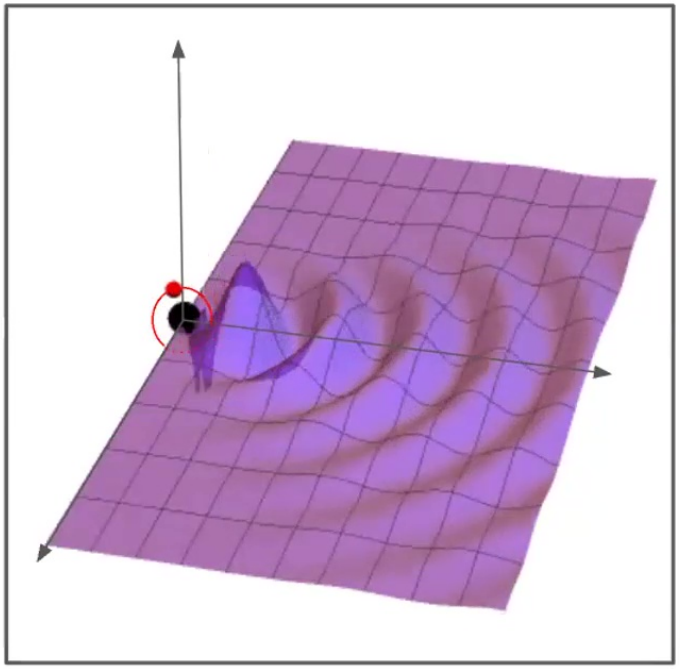
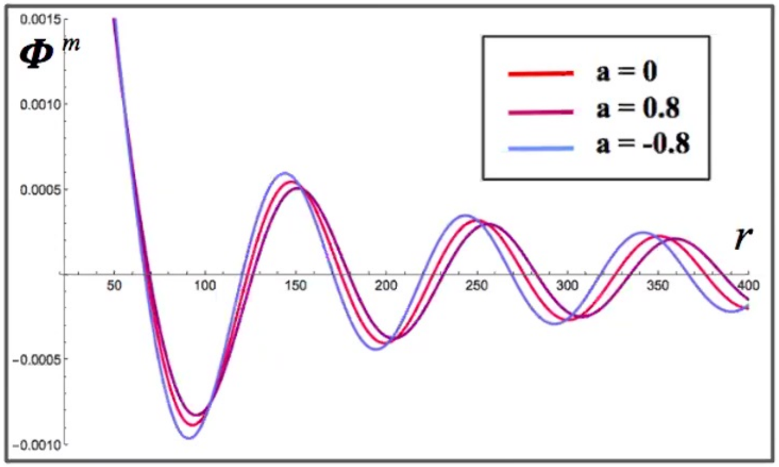
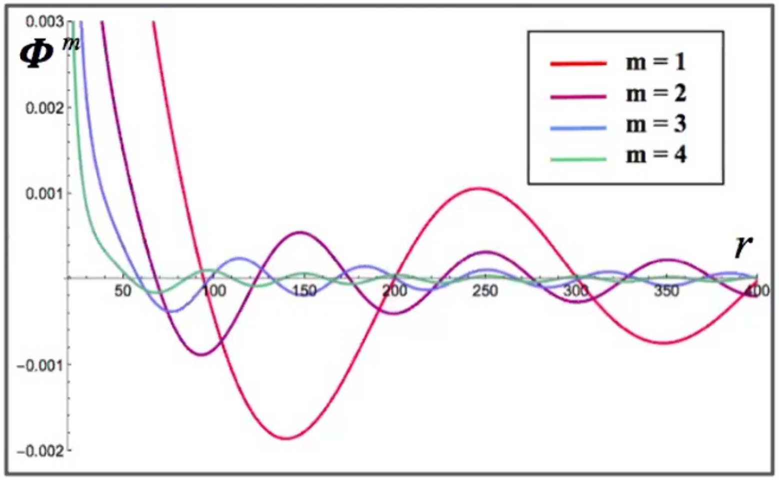
Results



Left: 3-D animation of the field
Upper right: m-mode contribution to the field, sliced at $\theta=\pi/2$
Lower right: a contribution to the field, sliced at $\theta=\pi/2$





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