Title: Conformal numerical method for self force applications in the time domain

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Abstract: "In 2034 LISA is due to be launched, which will provide the opportunity to extract physics from stellar objects and systems that would not otherwise be possible, among which are EMRIs. Unlike previous sources detected at LIGO, these sources can be simulated using an accurate computation of the gravitational self-force, resulting from the gravitational effects of the compact object orbiting around the massive BH. Whereas the field has seen outstanding progress in the frequency domain, metric reconstruction and self-force calculations are still an open challenge in the time domain. Such computations would not only further corroborate frequency domain calculations/models but also allow for full self-consistent evolution of the orbit under the effect of the self-force . Given we have a priori information about the local structure of the discontinuity at the particle, we will show how we can construct discontinuous spatial and temporal discretizations  by operating on discontinuous Lagrange and Hermite interpolation formulae and hence recover higher order accuracy. We will show how this technique in conjunction with well-suited conformal (hyperboloidal slicing) and numerical (discontinuous time symmetric ) methods can provide a relatively simple method of lines numerical recipe approach to the problem.  We will show, in particular, how this method can be applied to solve the Regge-Wheeler and Zerilli equations with a moving particle source in the time domain.

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Note to organizers: if both my talk and my supervisor, Charalampos Markakis, are selected could this please be after his ? Thank you for your consideration

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# *Conformal Discontinuous Numerical method for gravitational self-force applications in the time domain*

*Towards a numerical recipe: From BHPT through Regge-Wheeler-Zerilli Formalism towards GSF computation*

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07/06/2021

**Lidia Gomes Da Silva**

# Motivation and previous work

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Self-force calculations are due in the TD, RWZ and Teukolsky equations with a source are typically solved as an ODE in the frequency domain, this needs corroboration.

- Approximate δ as a Gaussian pulse (Harms, Bernuzzi, Brugmann, Zenginoglu)
- Construct finite difference representation of δ (Hughes et al)
- Domain decomposition + time dependent grid + jump conditions across the particle (Canizares, Sopuerta, Field, Hesthaven, Heffernan, Ottewill, Diener, Warburton, Wardell et all)

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# Numerical recipe towards computing the GSF

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$$[-\partial_t^2 + \partial_x^2 - V^{RW/Z}{}_l(x)]\phi_{lm}{}^{RW/Z} = G_{lm}{}^{RW/Z}(t,x)\delta\big(x - \xi(t)\big) + F_{lm}{}^{RW/Z}(t,x)\delta'\big(x - \xi(t)\big)$$

- Decompose in spin weighted spherical harmonics, obtain 1+1D RWZ equation
- Use hyperboloidal grid covering this 2D Lorentzian manifold from the horizon to $\mathcal{J}^+$
- Evolve in time using the Method of Lines recipe
  - Discontinuous collocation methods for spatial discretization
  - Discontinuous symmetric methods for time integration

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# *Method of lines recipe*

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We take our equation:

$$[-\partial_t^2 + \partial_x^2 - V_l(x)]\phi_{lm} = G_{lm}(t,x)\delta\big(x - \xi(t)\big) + F_{lm}(t,x)\delta'\big(x - \xi(t)\big)$$

Reduce it to a first order system

$$\frac{dU}{d\tau} = \boldsymbol{L}\,\boldsymbol{U} + \boldsymbol{S}, \qquad \boldsymbol{U} = \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix}$$

Discretize ***U,L*** in space using finite difference, pseudospectral or Fourier collocation nodes

Integrate in time using time symmetric methods as a coupled ODE system, example Hermite integration ***(Haris's talk)***

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# Discontinuous collocation method - Lagrange interpolation

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N$^{th}$ order polynomial

$$p(x) = \sum_{j=0}^{N} c_j \pi_j$$

Collocation conditions

$$p(x) = f_i, \qquad i = 0,1, \dots N$$

Solution: Lagrange interpolating polynomial

$$p(x) = \sum_{j=0}^{N} f_j(x)\pi_j(x)\,, \pi_j = \prod_{k=0,k\neq j}^{N} \frac{x - x_k}{x_j - x_k}$$

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# Discontinuous CM: Discontinuous Lagrange Interpolation

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Nth order piecewise polynomial

$$p(x) = \sum_{j=0}^{N} \left[\theta(x-\xi)c_j^{+} + \theta(\xi - x)c_j^{-}\right]x^{j}$$

Collocation conditions

$$p(x) = f_i, \qquad i = 0,1, \ldots, N$$

Jump conditions

$$p^{(k)}(\xi^{+}) - p^{(k)}(\xi^{-}) = J_k, \qquad k = 0,1,2, \ldots$$

Solution: Interpolating piecewise polynomial

$$p(x) = \sum_{j=0}^{N}\left[f_j(x) + \Delta\left(x_j - \xi; x - \xi\right)\right]\pi_j(x)$$

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# *Discontinuous CM: Examples*

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Interpolation

$$p(x) = \sum_{j=0}^{N} [f_j(x) + \Delta(x_j - \xi; x_i - \xi)], \qquad D_{ij}^{n} = \pi_j^{(n)}(x_i)$$

Differentiation

$$p^{(n)}(x_i) = \sum_{j=0}^{N} D_{ij}^{n} [f_j(x) + \Delta(x_j - \xi; x_i - \xi)]$$

$$\Delta(x_j - \xi; x_i - \xi) = [\theta(x_i - \xi) - \theta(x_j - \xi)] \sum_k \frac{J_k}{k!} (x_j - \xi)^k$$

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# *Discontinuous flat wave equation*

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We use the distributional forced 1 + 1D wave equation as our toy model:

$$-\partial_t^2 \psi + \partial_x^2 \psi = G(t)\delta(x - vt)$$

Approximate $\psi(t, x)$ as a piecewise polynomial:

$$\psi(t,x) \approx \sum_{j=0}^{N} \left[\psi_j(t) + \Delta\left(x_j - \xi(t); x - \xi(t)\right)\right] \pi_j(x)$$

$\Delta$ is piecewise constant wrt *x* so not differentiated when approx. spatial derivatives

$$\partial_x^n \psi(t,x) \approx \sum_{j=0}^{N} D_{ij}^n \left[\psi_j + \Delta\left(x_j - \xi(t); x_i - \xi(t)\right)\right]$$

No summation over i, so rightmost term reduces to just a vector of length N+1, i.e

$$s_i^n(t) = \sum_{j=0}^{N} D_{ij}^n \left[\Delta\left(x_j - \xi(t); x_i - \xi(t)\right)\right]$$

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# *Discontinuous flat wave equation*

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We can express $\psi_J$ as a linear combination of

$$\psi_J(x,t) = \sum_{i=0}^{\infty} J_i(t)\phi_i\big(x;\xi(t)\big)$$

The jumps $J_i$ are obtained by the generalization of the Frobenius method and given by the recurrence relation

$$J_0 = \gamma^2 G(t)$$

$$J_1 = \gamma^2\big(F(t) - J_0\ddot{\xi} - 2\dot{J}_0\dot{\xi}\big)$$

$$J_{n+2} = \gamma^2\big(\ddot{J}_n - 2\dot{J}_{n+1}\dot{\xi} - J_{n+1}\ddot{\xi}\big)$$

For any other terms in a PDE such as a potential or hyperboloidal coefficients we use

$$f(x)\psi^J\big(x,t;\xi(t)\big) = \left(\sum_{m=0}^{\infty}\psi_m\sum_{r=0}^{m}\binom{m}{r}\right) f^{(r)}\big(\xi(t)\big)J_{m-r}(t)$$

and update $J_{n+2}$ accordingly.

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# Results

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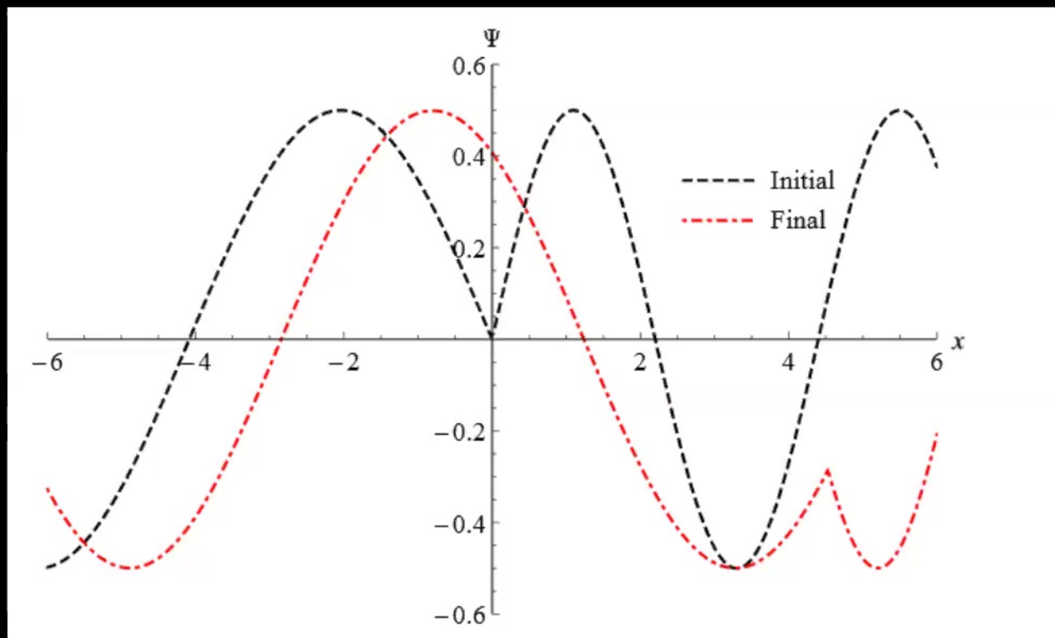


***Figure 5:*** Numerical solution using the discontinuous collocation methods with an order 2 Hermite integration time stepper. We selected initial function to be the exact solution provided in https://arxiv.org/abs/0902.1287 .

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# Results

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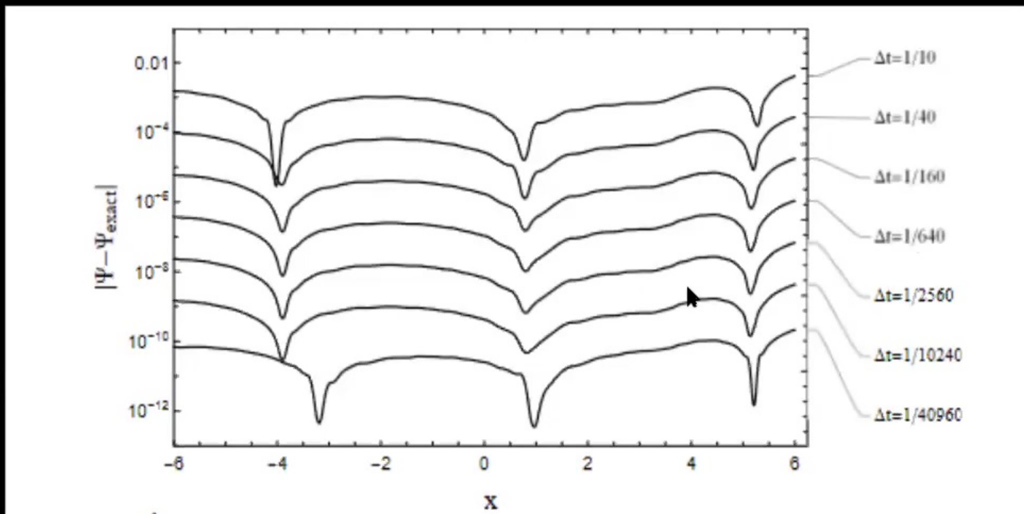


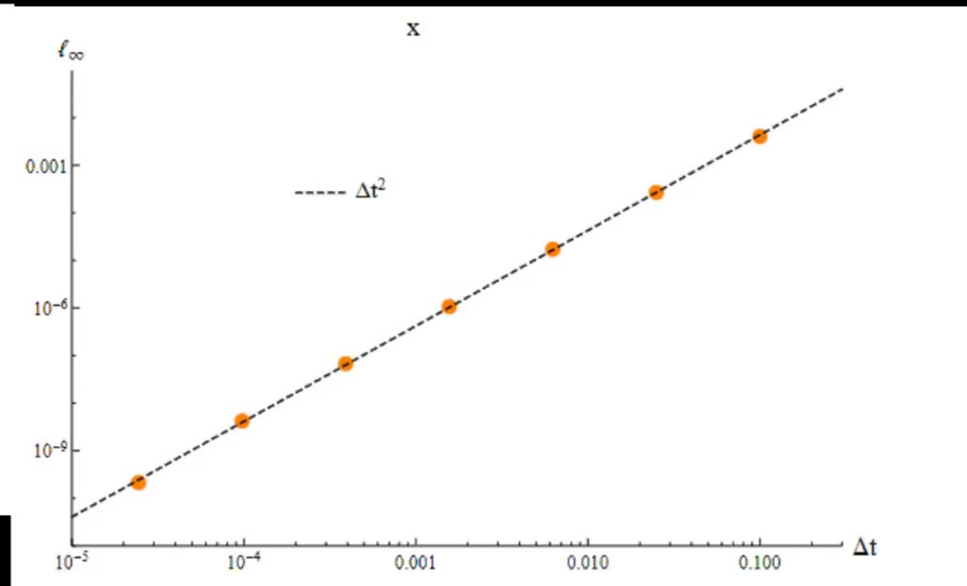


***Figure 6:*** The figure on the left shows the difference between the numerical solution and the exact solution for decreasing time steps. The figure on the right demonstrates that the l∞ error norm scales as Δt2, as expected of our method.

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# Conformal methods for BHPT – Minimal Gauge Slicing

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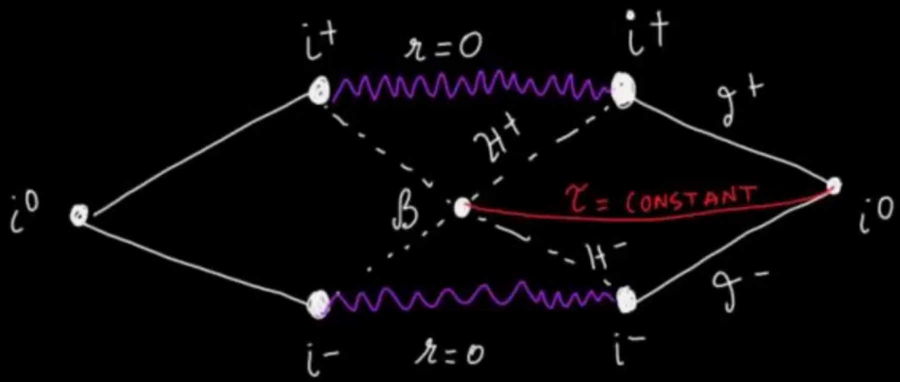
Implement following transformation

$$\begin{cases} t = \tau - h(\sigma) \\ x = g(\sigma) \end{cases} \quad \Leftrightarrow \quad \begin{cases} t = \tau - \frac{1}{2}\left(\ln\sigma + \ln(1-\sigma) - \frac{1}{\sigma}\right) \\ x = \frac{1}{2}\left(\frac{1}{\sigma} + \ln(1-\sigma) - \ln\sigma\right) \end{cases}$$

where here we follow the minimal gauge, slicing found in https://arxiv.org/abs/2004.06434




$$H = \frac{1}{2}\left(-g^{\tau\tau}\partial_\tau\phi^2(\tau,\sigma) + g^{\sigma\sigma}\partial_\sigma\phi^2(\tau,\sigma) + V(\sigma)\psi^2(\tau,\sigma)\right)$$

***More details in Rodrigo's talk today at 11.45***

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# *Discontinuous hyperboloidal  RWZ equation*

For any other terms in a PDE such as a potential or hyperboloidal coefficients we use

$$f(x)\psi^J\big(x,t;\xi(t)\big) = \left(\sum_{m=0}^{\infty}\psi_m\sum_{r=0}^{m}\binom{m}{r}\right)f^{(r)}\big(\xi(t)\big)J_{m-r}(t)$$

and update $J_{n+2}$ accordingly.

We simply update our recurrence relation accordingly to the now hyperboloidal RWZ equation, for minimal gauge update is as follows:

$$J_0 = \gamma h^2 G(u(\tau))$$

$$J_1 = \gamma h^2\big(F(u(\tau)) - J_0\ddot{\xi} - 2J_0\dot{\xi} + C(\xi)\dot{J}_0 + B(\xi)J_0\dot{\xi}\big)$$

$$J_{n+2} = \gamma h^2\big(\ddot{J}_n - 2\dot{J}_{n+1}\dot{\xi} - J_{n+1}\ddot{\xi}\big) + \gamma h^2\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}\left(C^r(\xi)J_{n-r+1} - B^r(\xi)\big(\dot{J}_{n-r} - \dot{\xi}J_{n-r+1}\big) + H^r(\xi)J_{n-r}\right)\right)$$
$$+ \gamma h^2\textstyle\sum_{n=1}^{\infty}\left(\sum_{r=1}^{n}\binom{n}{r}\left(E^{(r)}(\xi)\dot{\xi}J_{n+2-r} - A^{(r)}(\xi)J_{n+2-r}\right)\right)$$

# Conclusions

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***Pros:***

- Fixed grid
- Particle can be anywhere
- Discontinuities handled by adding jump terms to the Lagrange or Hermite interpolation formulae
- Recursive jump relations inform a priori our discontinuous numerical method

***Cons:***

- Need large number of jumps to reach machine precision (~10-30)
- Complexity of jump relations increase in hyperboloidal slicing

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# *Future steps*

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- Use $\phi_{lm}$ to reconstruct metric perturbations in the time domain
- Gravitational wave extraction
- Compute hyperbolic scattering angles (for comparison with PN/PM/EOB)
- Self-consistent evolution under the influence of GSF in TD
- Apply the waveform model in Kerr and perform equivalent calculations for the Teukolsky equation with source (Michael O'Boyle and Nelson Eiro)

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