Title: A Solution to the Stable Marriage Problem

Speakers: Emily Riehl

Collection: Perimeter Public Lectures

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Abstract: In her Perimeter Institute public lecture, premiering May 12, mathematician Emily Riehl will invite viewers to consider what might be called the "matchmaker's dilemma."

Imagine a matchmaker who wishes to arrange opposite-sex marriages in a dating pool of single men and single women (there's a mathematical reason for the heteronormative framework, which will be explained).

The matchmaker's goal is to pair every man and woman off into couples that will form happy, stable marriages - so perfectly matched that nobody would rather run off with someone from a different pairing.

In the real world, things don't work out so nicely. But could they work out like that if the matchmaker had a computer algorithm to calculate every single factor of compatibility?

In her talk, recorded as part of the Perimeter Institute Public Lecture Series, Riehl will examine that question, its sexist implications, an algorithmic solution, and real-world applications.

An associate professor of mathematics at Johns Hopkins University, Riehl has published more than 20 papers and two books on higher category theory and homotopy theory. She studied at Harvard and Cambridge and earned her PhD at the University of Chicago.

In addition to her research, Riehl is active in promoting access to the world of mathematics. She is a co-founder of Spectra: the Association for LGBT Mathematicians, and has presented on mathematical proof and queer epistemology as part of several conferences and lecture series.

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Emily Riehl

A SOLUTION TO THE STABLE MARRIAGE PROBLEM

Perimeter Institute Public Lecture Series

May 12, 2021

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COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

— DAVID GALE + LLOYD SHAPLEY, 1962

"...it may be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical. For this purpose we recommend the statement and proof of our Theorem 1. The argument is carried out not in mathematical symbols but in ordinary English; there are no obscure or technical terms. Knowledge of calculus is not presupposed. In fact, one hardly needs to know how to count. Yet any mathematician will immediately recognize the argument as mathematical..."

THE MATCHMAKER'S DILEMMA:

Imagine a heterosexual dating pool of single men and women.

Is it possible to arrange marriages so that no unmatched couple is tempted to elope?

To answer the QUESTION it helps to introduce some DEFINITIONS

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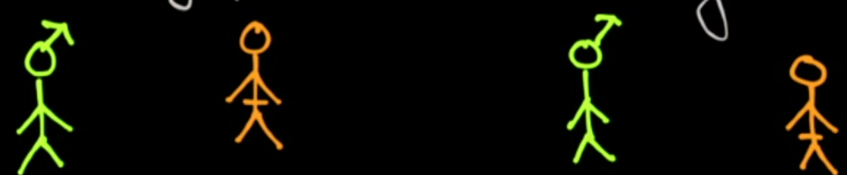
A DATING POOL is a collection of equal numbers of men and women, each of whom has a fixed PREFERENCE LIST ranking each of the members of the opposite sex.

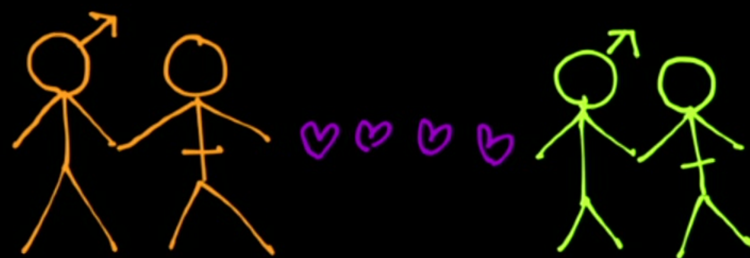
	Helena	Hermia		Demetrius	Lysander
#1	Demetrius	Lysander	#1	Hermia	Helena
#2	Lysander	Demetrius	#2	Helena	Hermia

A MARRIAGE ARRANGEMENT is a heterosexual matching: a partitioning of the dating pool into pairs consisting of one man + one woman



Lysander + Hermia   Demetrius + Helena

A marriage arrangement is UNSTABLE if there is an unmatched couple who each prefer each other to their assigned spouses.



A marriage arrangement is STABLE if there are no instabilities.

QUESTION: Is it always possible to arrange stable marriages?

THE "STABLE ROOMMATES" PROBLEM

Consider a SAME SEX DATING POOL with the following preference lists:

	Ada	Emmy	Marie	Vera
#1	Emmy	Marie	Ada	—
#2	—	—	—	—
#3	Vera	Vera	Vera	—

For this dating pool any marriage arrangement has an instability:

Without the heteronormative framework, the stable marriage problem has no solution!

THE "STABLE ROOMMATES" PROBLEM

Consider a SAME SEX DATING POOL with the following preference lists:

	Ada	Emmy	Marie	Vera
#1	Emmy	Marie	Ada	—
#2	—	—	—	—
#3	Vera	Vera	Vera	—

For this dating pool any marriage arrangement has an instability:

Emmy + Marie   Ada + Vera

Without the heteronormative framework, the stable marriage problem has no solution!

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THE "DEFERRED-ACCEPTANCE" ALGORITHM

"To start, let each boy propose to his favorite girl. Each girl who receives more than one proposal rejects all but her favorite from among those who have proposed to her. However, she does not accept him yet, but keeps him on a string to allow for the possibility that someone better may come along later."

Day 1 { AM: each woman proposes to her top choice
PM: each man rejects all but his top suitor

Day n+1 { AM: each woman who was rejected on day n proposes to her next top choice
PM: each man rejects all but his top suitor

When everyone is engaged, the algorithm terminates.


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AN EXAMPLE
Day 1
Charlotte Elizabeth Jane Lydia
#1 Bingley Wickham Bingley Bingley
#2 Darcy Darcy Wickham Wickham
#3 Collins Bingley Darcy Darcey
#4 Wickham Collins Collins Collins
Day 2
Day 3
Bingley Collins Darcy Wickham
#1 Jane Jane Elizabeth Lydia
#2 Elizabeth Elizabeth Jane Jane
#3 Lydia Lydia Charlotte Elizabeth
#4 Charlotte Charlotte Lydia Charlotte
Day 4


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AN EXAMPLE
Charlotte Elizabeth Jane Lydia
#1 Bingley Wickham Bingley Bingley
#2 Darcy Darcy Wickham Wickham
#3 Collins Bingley Darcy Darcy
#4 Wickham Collins Collins Collins
Bingley Collins Darcy Wickham
#1 Jane Jane Elizabeth Lydia
#2 Elizabeth Elizabeth Jane Jane
#3 Lydia Lydia Charlotte Elizabeth
#4 Charlotte Charlotte Lydia Charlotte
Day 1 ~~Elizabeth-Wickham~~
Jane-Bingley
Day 2 Charlotte-Darcy
Day 3
Day 4

STABLE MARRIAGES

THEOREM 1 The deferred acceptance algorithm arranges stable marriages.

Proof: No woman can be part of an instability because every man she prefers to her assigned spouse rejected her in favor of someone he likes better. □

COROLLARY In any heterosexual dating pool, a stable matching exists!

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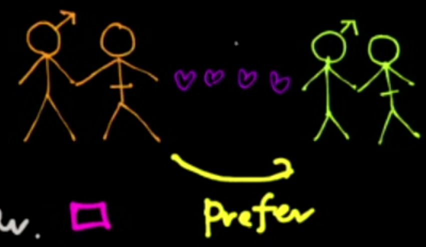

COROLLARY In any heterosexual dating pool, a stable matching exists!

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FINDING YOUR BEST POSSIBLE MATCH

Now that we know that a stable matching exists, it's interesting to think about how multiple stable marriage arrangements could be compared.

A woman's BEST POSSIBLE HUSBAND is her highest ranked man who she is matched with in any stable marriage arrangement.

A man's BEST POSSIBLE WIFE is his highest ranked woman who he is matched with in any stable marriage arrangement.

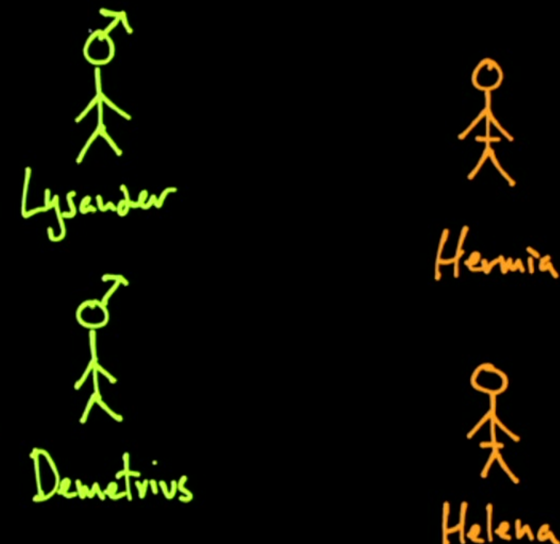
WHEN THE WOMEN PROPOSE, THE WOMEN WIN

THEOREM 2 The deferred acceptance algorithm, with the women proposing, assigns each and every woman her best possible husband.

Proof: We'll show that no woman is rejected by a "possible" husband.

For sake of contradiction:

- consider the first moment a woman Helena is rejected by a possible husband Demetrius
- this only happens if Demetrius receives a proposal from another woman, Hermia, who he prefers
- but if Helena + Demetrius are a possible match there must be another man, Lysander, who Hermia prefers (otherwise Hermia + Demetrius would elope)
- but then Hermia would never have proposed to Demetrius unless Lysander had previously rejected her — contradicting our assumption that Helena was the first woman rejected by a possible husband.



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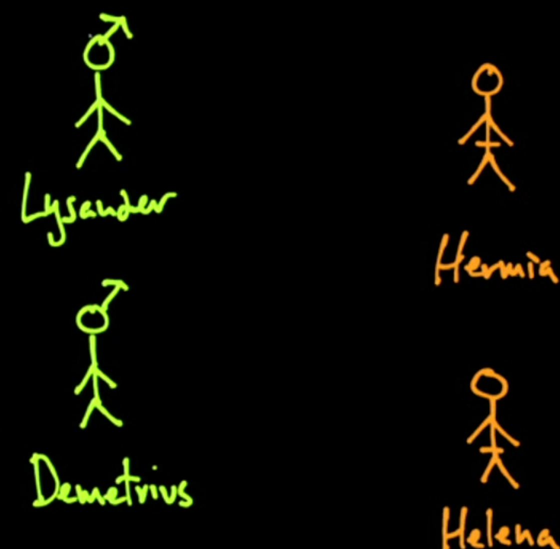

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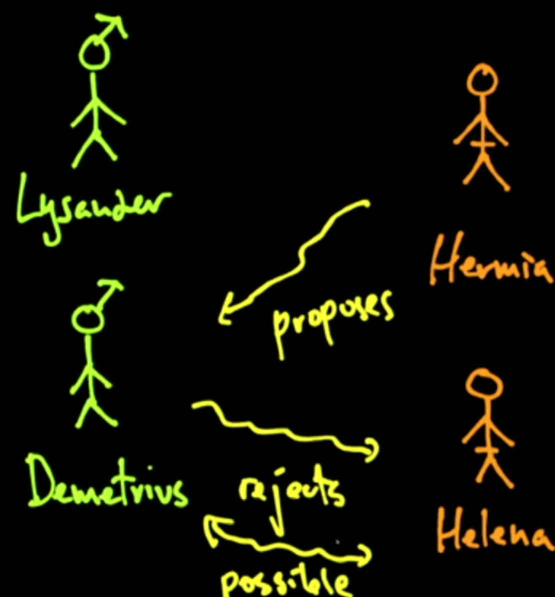

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WHEN THE MEN PROPOSE, THE MEN WIN

COROLLARY (by change of variables) The deferred acceptance algorithm, with the men proposing, assigns each and every man his best possible wife.

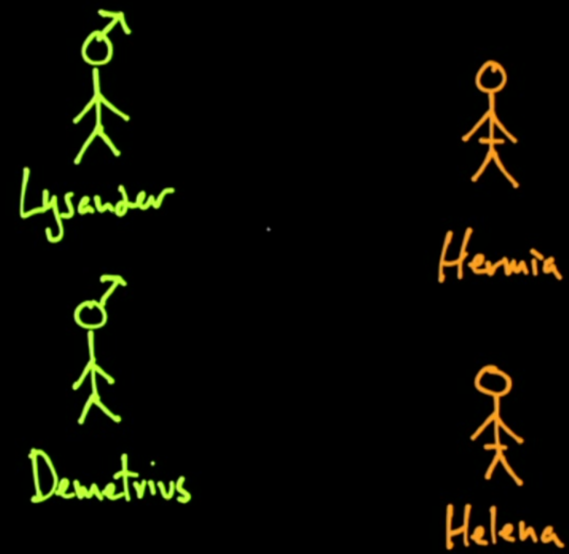
"It is clear that there is an entirely symmetrical procedure, with girls proposing to boys, which must also lead to a stable set of marriages ... when the boys propose, the result is optimal for boys, and when the girls propose it is optimal for girls."

WHEN THE WOMEN PROPOSE, THE MEN LOSE

THEOREM 3 The deferred acceptance algorithm, with the women proposing, assigns each and every man his worst possible wife.

Proof: We'll show that no man rejects a "possible" wife. For sake of contradiction:

- consider the first moment a man Demetrius rejects a possible wife Helena
- this only happens if Demetrius received a proposal from another woman, Hermia, who he prefers
- but if Helena + Demetrius are a possible match there must be another man, Lysander, who Hermia prefers (otherwise Hermia + Demetrius would elope)
- but then Hermia would never have proposed to Demetrius unless Lysander had previously rejected her — contradicting our assumption that Demetrius was the first man who rejects a possible wife. □



THE MEDICAL MATCH

Each year the CANADIAN RESIDENCY MATCHING PROGRAM uses an algorithm to match graduating medical students with residency programs at hospitals —

the DEFERRED-ACCEPTANCE ALGORITHM!!

THEOREM To obtain their best possible match, medical students should rank residencies in their true order of preference.

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A REAL WORLD COMPLICATION

Of course medical students are sometimes already married — to each other.

With the **COUPLES MATCH**, a pair of graduating medical students can elect to enter the medical match as a unit, submitting pairs of preferences rather than individual preference lists.

In practice this works fairly well, but it doesn't work so well in theory:

Eytan Ronn proved in 1990 that the couples match is NP-complete.

"NP-complete stable matching problems"

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THANK YOU

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