Title: The Stabilizer Subtheory Has a Unique Noncontextual Model

Speakers: David Schmid

Series: Quantum Foundations

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Abstract: We give a complete characterization of the (non)classicality of all stabilizer subtheories. First, we prove that there is a unique nonnegative and diagram-preserving quasiprobability representation of the stabilizer subtheory in all odd dimensions, namely Grossâ€™s discrete Wigner function. This representation is equivalent to Spekkensâ€™ epistemically restricted toy theory, which is consequently singled out as the unique noncontextual ontological model for the stabilizer subtheory. Strikingly, the principle of noncontextuality is powerful enough (at least in this setting) to single out one particular classical realist interpretation. Our result explains the practical utility of Grossâ€™s representation, e.g. why (in the setting of the stabilizer subtheory) negativity in this particular representation implies generalized contextuality, and hence sheds light on why negativity of this particular representation is a necessary resource for universal quantum computation in the state injection model. This last fact, together with our result, implies that generalized contextuality is also a necessary resource for universal quantum computation in this model. In all even dimensions, we prove that there does not exist any nonnegative and diagram-preserving quasiprobability representation of the stabilizer subtheory, and, hence, that the stabilizer subtheory is contextual in all even dimensions.

The stabilizer subtheory has a unique noncontextual model

David Schmid
Haoxing Du
John Selby
Matt Pusey

Ravi Kunjwal
Elie Wolfe
Rob Spekkens

Why study nonclassicality?

- -intrinsic foundational interest
- -nonclassicality powers technology
- -guides us in interpreting QT
- -which guides us in extending QT

Classicality = generalized noncontextuality

-equivalent to quantum optics notion of classicality

-equivalent to GPT notion of classicality

-emerges in quantum Darwinism limit

-motivated by Leibniz's principle

-generalized contextuality subsumes KS contextuality, Bell nonlocality, and anomalous WVs

-gen. contextuality is a resource for quantum computation, quantum communication, state discrim., cloning, metrology

Results covered this talk

Relationships between various representations of operational theories and of generalized probabilistic theories (GPTs)

Such representations have a very simple mathematical structure

For any stabilizer subtheory in **odd** dimensions, the unique noncontextual model is given by Gross's discrete Wigner function (or equivalently, Spekkens' toy model).

Every stabilizer subtheory in **even** dimensions is contextual.

Generalized contextuality is a necessary resource for quantum computation (in the state injection model)

Gross's discrete Wigner representation has been very useful in studying quantum computation

-extending the Gottesman-Knill simulations to all processes represented nonnegatively in Gross's repn

-every state useful for magic state distillation has negativity in Gross's repn

-every state that promotes the stabilizer subtheory to universal quantum computation via magic state distillation must be Kochen-Specker contextual

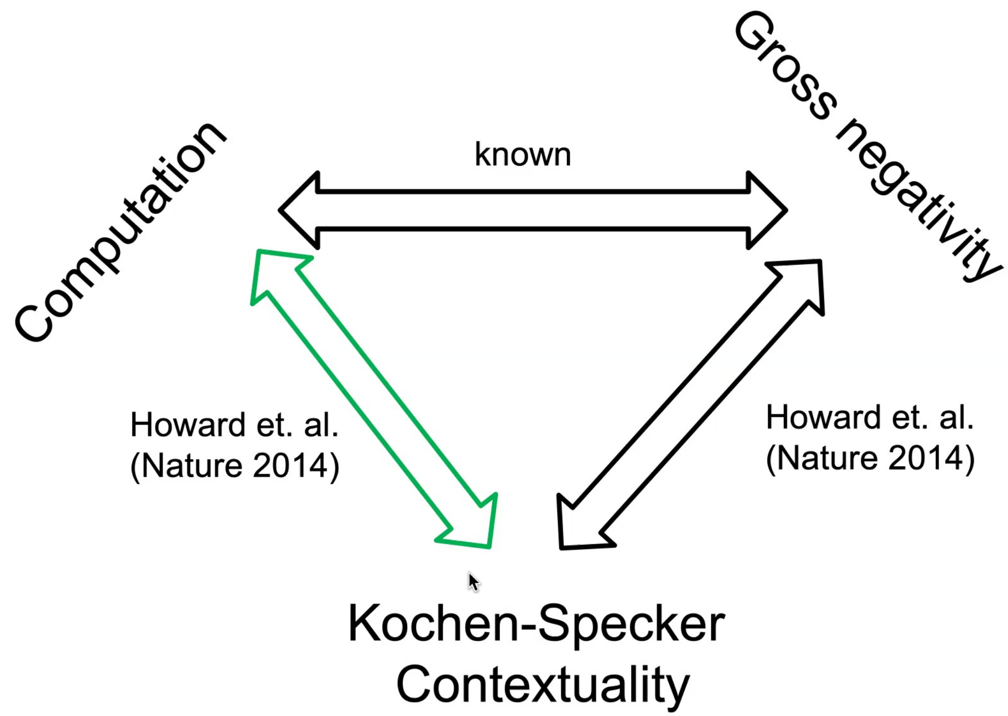
-negativity in Gross's repn is a necessary resource for computation

Emerson, Wallman, et. al.

But negativity in one particular quasiprobability repn
is *not* generally sufficient to establish nonclassicality

So why would negativity in one particular repn (Gross's) be
associated to a strong form of nonclassicality (UQC)?

Because Gross's is the unique classical
(noncontextual) representation!



“Contextuality supplies the magic for quantum computation”

Preliminaries

Process Theory G

collection of processes (on some systems) which is closed under composition

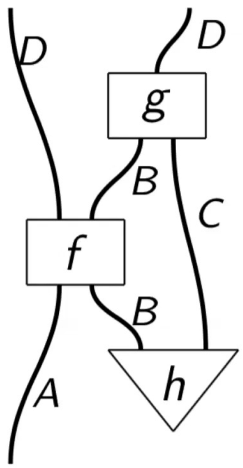
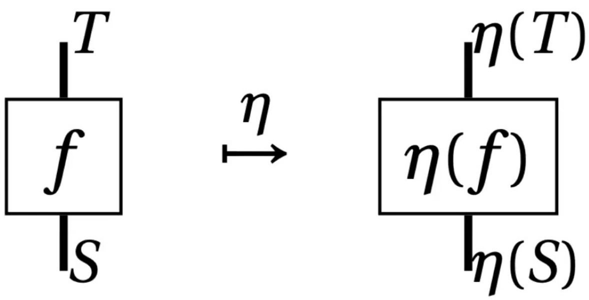


Diagram Preserving map

$\eta : G \to G'$

takes processes from one theory to processes of another



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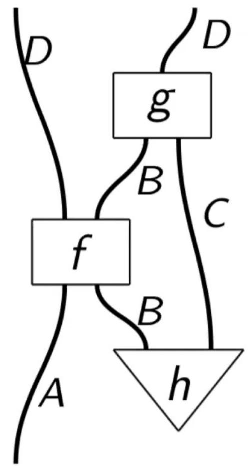

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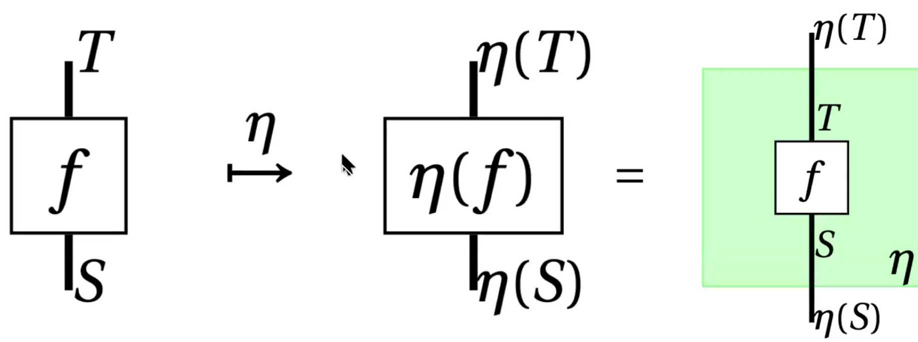

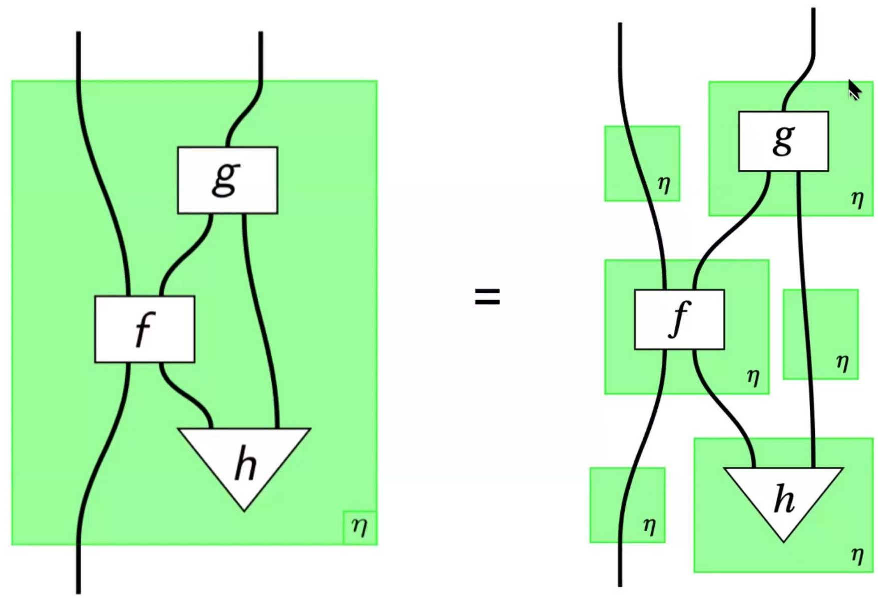
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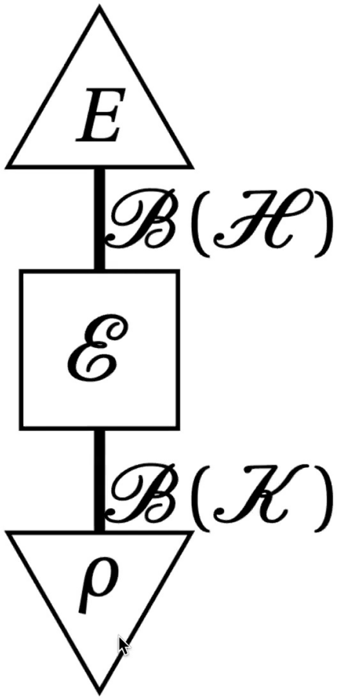
Quantum theory as a process theory

systems are vector spaces of Hermitian operators on Hilbert space

processes are channels

—processes with no inputs are density operators

—processes with no outputs are effects



Quantum theory as a process theory

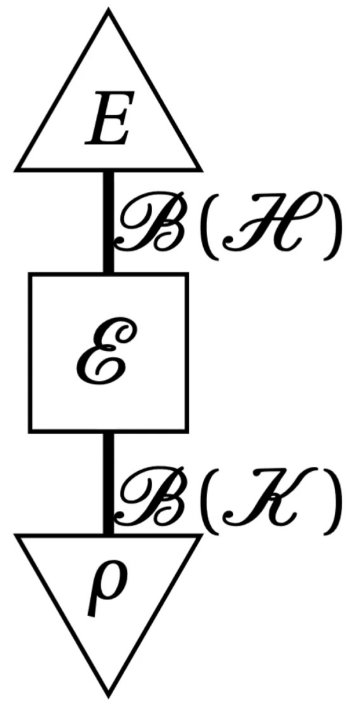
systems are vector spaces of Hermitian operators on Hilbert space

processes are channels

—processes with no inputs are density operators

—processes with no outputs are effects

$$\mathrm{Tr}[E\mathcal{E}(\rho)] =$$



Substochastic matrices as a process theory

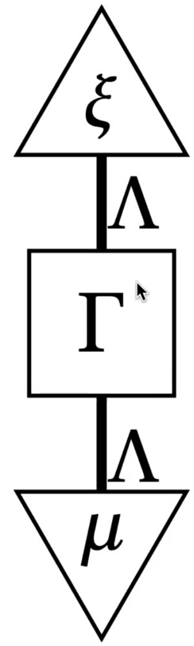
systems are sets

processes are substochastic maps

—processes with no inputs are sub-norm prob distributions

—processes with no outputs are response functions

$$\sum_{\lambda,\lambda'} \xi(\lambda')\Gamma(\lambda'|\lambda)\mu(\lambda) \quad = $$



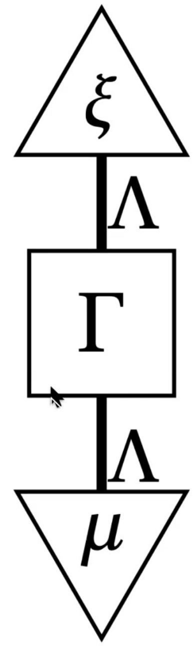
e.g. Louivillian mechanics

QuasiSubstochastic matrices as a process theory

systems are sets

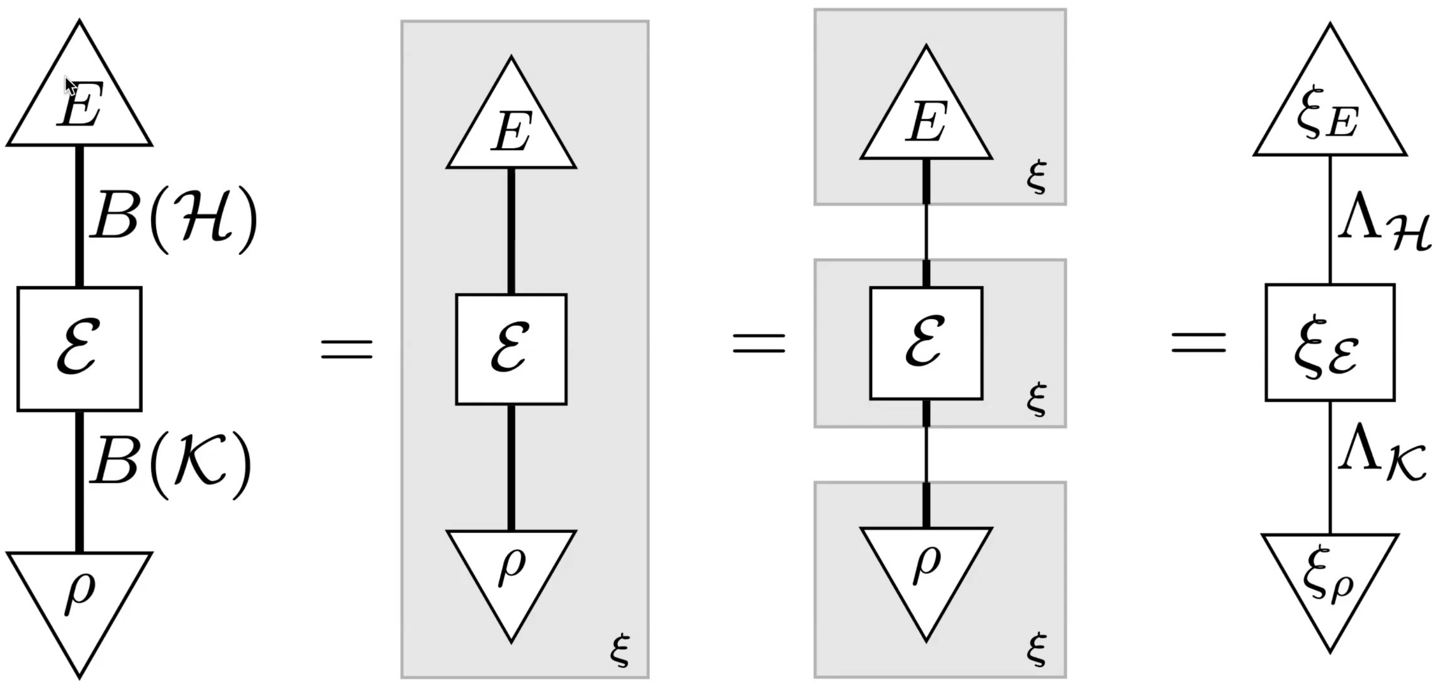
processes are quasisubstochastic maps

$$\sum_{\lambda,\lambda'} \xi(\lambda')\Gamma(\lambda'|\lambda)\mu(\lambda) \quad = $$



can go negative!

Intuition:



E
$B(\mathcal{H})$
$\mathcal{E}$
$B(\mathcal{K})$
ρ
$=$
E
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$=$
ξ_E
$\Lambda_{\mathcal{H}}$
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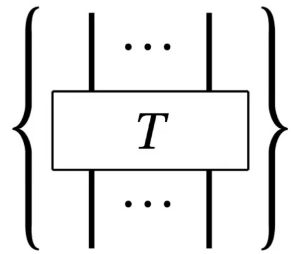
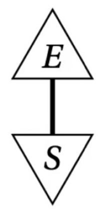
$\Lambda_{\mathcal{K}}$
ξ_ρ

Operational theories

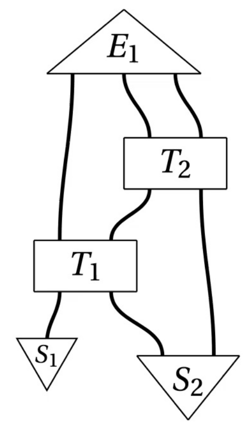
Operational Theories

-systems 

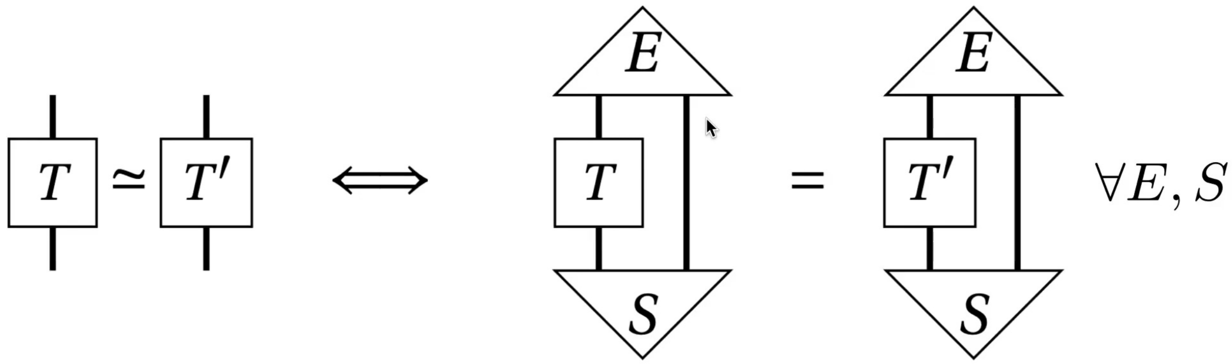
-laboratory procedures applied to systems 

-probability rule 

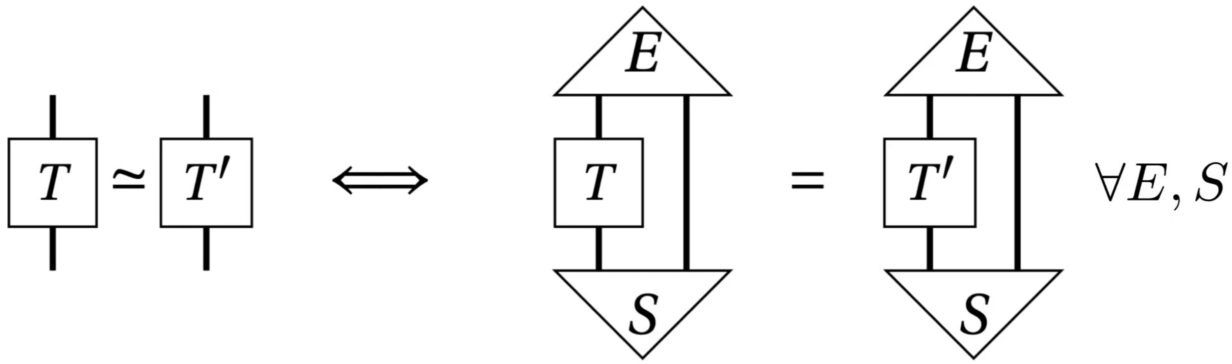
$= \quad \Pr(E|S)$

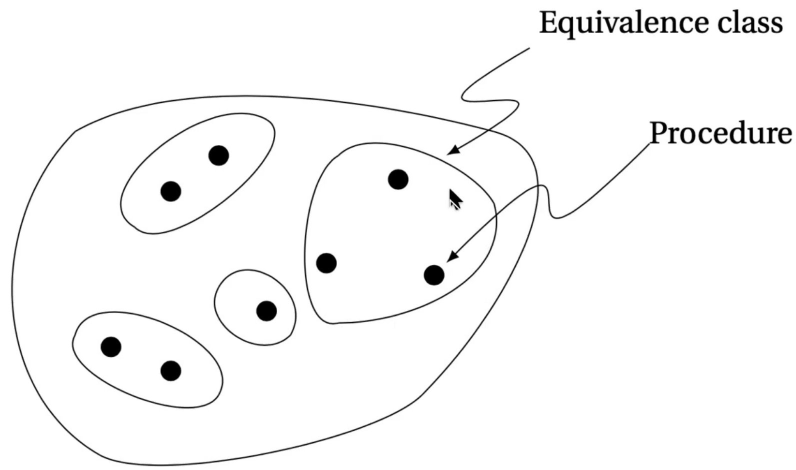


Operational Equivalence of Two Procedures

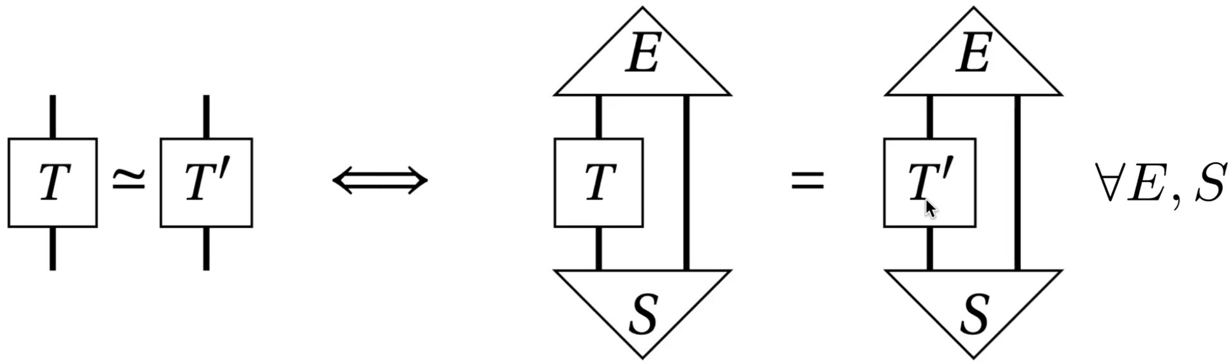


Operational Equivalence of Two Procedures



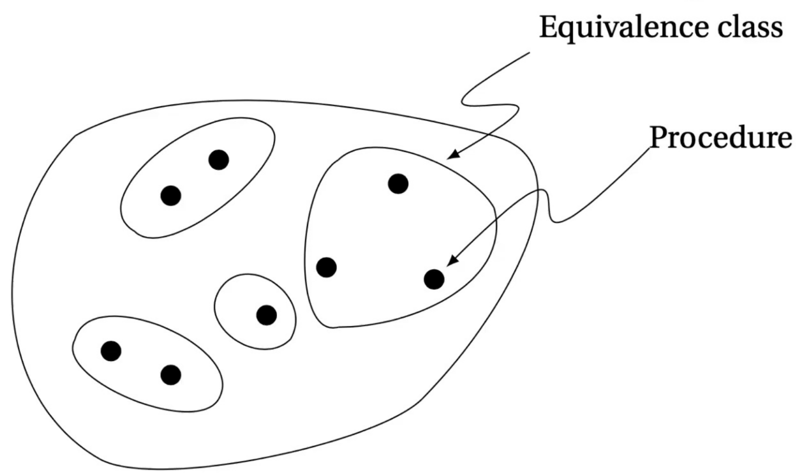


Operational Equivalence of Two Procedures

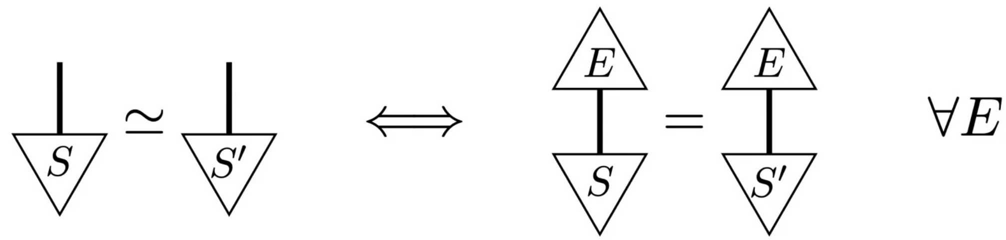


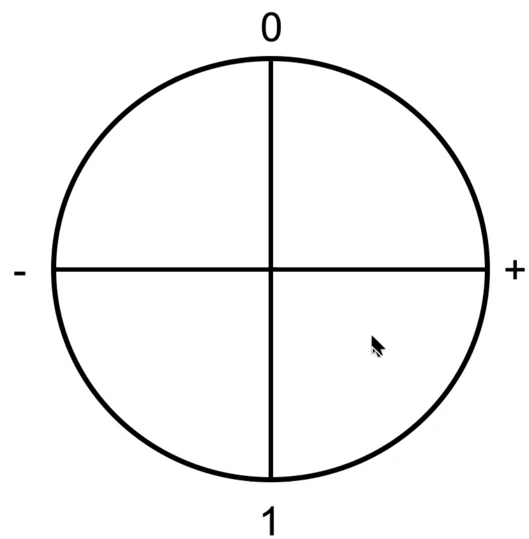
$$(\tilde{T}, c_T)$$

Equivalence Class Context



e.g. for states:



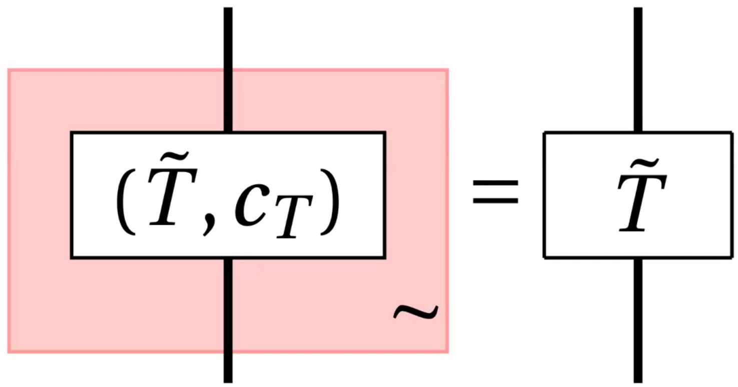


Quotiented operational theories

see Pavia group’s OPTs

Quotienting with respect to Operational Equivalence

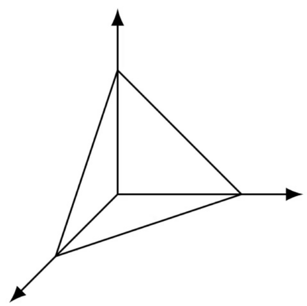
define a quotienting map $\sim :: (\tilde{T}, c_T) \mapsto \tilde{T}$



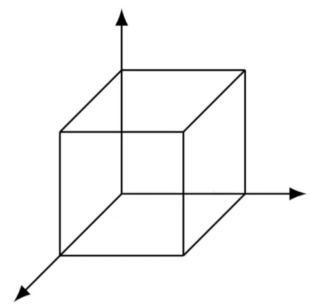
The quotiented theory is the generalized probabilistic theory (GPT) associated with the operational theory

Simple view of GPTs

state space



effect space



inner products give Pr(effect|state)

-observable statistics (and hence the theory) are given by the geometry

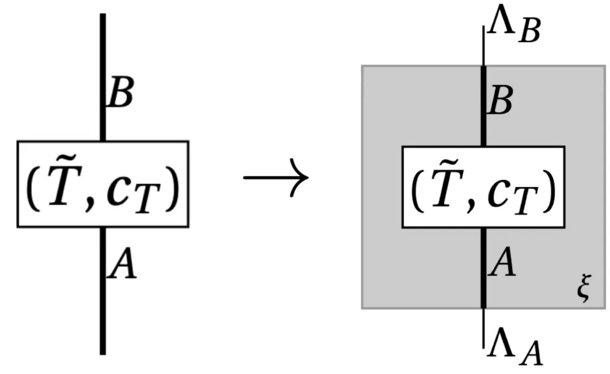
-states/effects are distinct IFF they give distinct probabilities

Representations

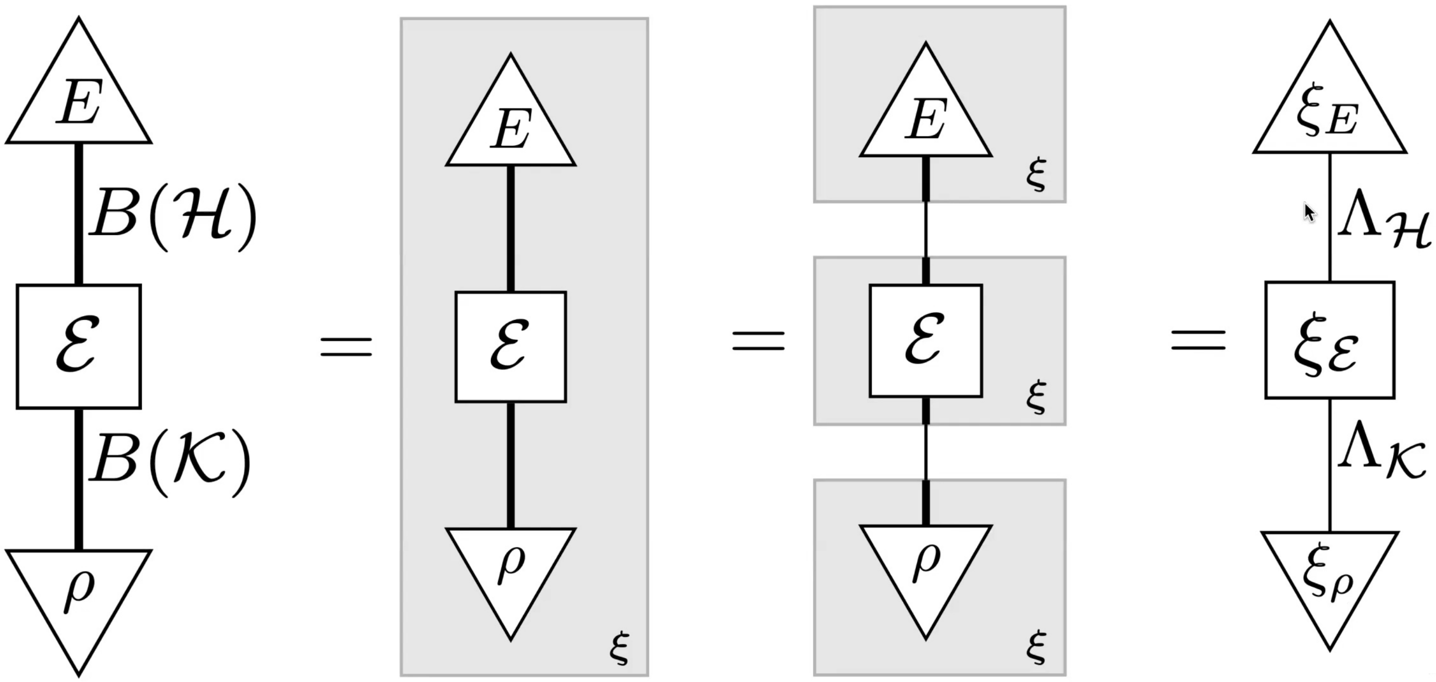
Ontological model of an operational theory

diagram-preserving map

$$\xi : \mathbf{Op} \to \mathbf{SubStoch}$$



which: 1) reproduces the predictions 2) represents ignoring appropriately

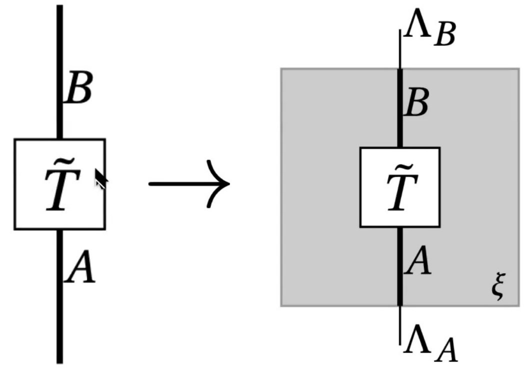


$$\mathrm{Tr}[E\mathcal{E}(\rho)] = \sum_{\lambda,\lambda'} \xi_E(\lambda')\xi_{\mathcal{E}}(\lambda'|\lambda)\xi_\rho(\lambda)$$

Ontological model of a GPT

diagram-preserving map

$$\xi : \mathbf{GPT} \rightarrow \mathbf{SubStoch}$$



which: 1) reproduces the predictions 2) represents ignoring appropriately

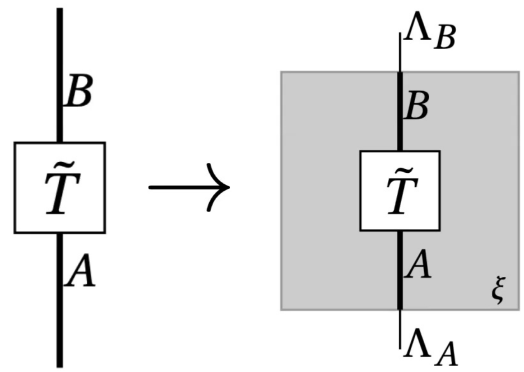
No possibility for map to depend on context!

Quasiprobability repn

~~Ontological model~~ of a GPT

diagram-preserving map

$$\xi : \mathbf{GPT} \to \sout{\mathbf{SubStoch}}\ \mathbf{QuasiSubStoch}$$



which: 1) reproduces the predictions 2) represents ignoring appropriately

Quasiprobability repn

~~Ontological model~~ of a GPT

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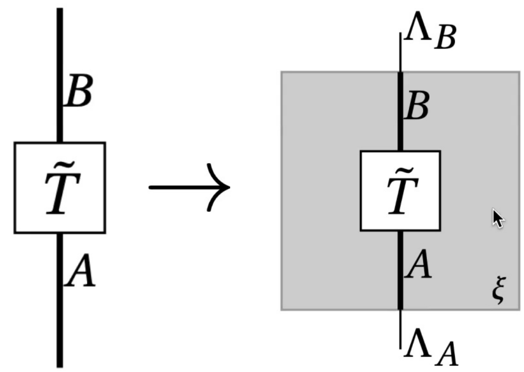
$$\xi : \mathbf{GPT} \to \cancel{\mathbf{SubStoch}}\ \mathbf{QuasiSubStoch}$$

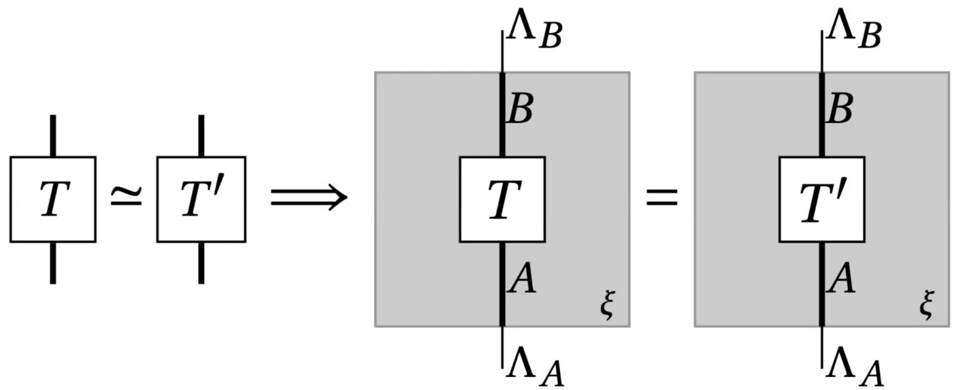


which: 1) reproduces the predictions 2) represents ignoring appropriately

Clearly, a *positive* quasiprobability representation is just an OM!

For operational theories, Classicality = Noncontextuality

An ontological model of an op thry is noncontextual iff it satisfies



$$(\widetilde{T}, c'_T)$$

$$(\widetilde{T}, c_T)$$

For OMs of GPTs, noncontextuality does not apply

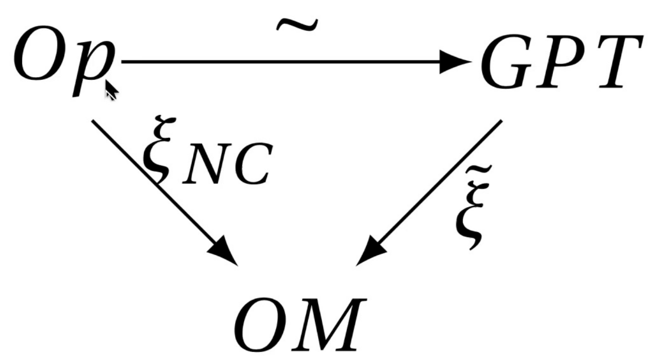
-there are no contexts in a GPT on which the representation could possibly depend:

<u>OM of operational theory</u>	<u>OM of GPT</u>
$(\widetilde{T}, c_T) \rightarrow \xi_{(\widetilde{T}, c_T)}(\lambda'\|\lambda)$	$\widetilde{T} \rightarrow \xi_{\widetilde{T}}(\lambda'\|\lambda)$

Consider an operational theory **Op** and the GPT **G** defined by quotienting it.

<u>Theorem:</u>

*There exists a NCOM of **Op** IFF there exists an OM of **G.***



PRX Quantum 2, 010331

So, if one's notion of classicality for operational theories is the existence of a NCOM, then one's notion of classicality for GPTs should be the *existence of an OM of one's GPT.*

PRX Quantum 2, 010331

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Two equivalent notions:
-existence of positive quasiprobability repn

PRX Quantum 2, 010331

So, if one’s notion of classicality for operational theories is the existence of a NCOM, then one’s notion of classicality for GPTs should be the *existence of an OM of one’s GPT.*

Two equivalent notions:
-existence of positive quasiprobability repn
-existence of a simplex embedding

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Relation to the traditional notion of classicality in GPTs?

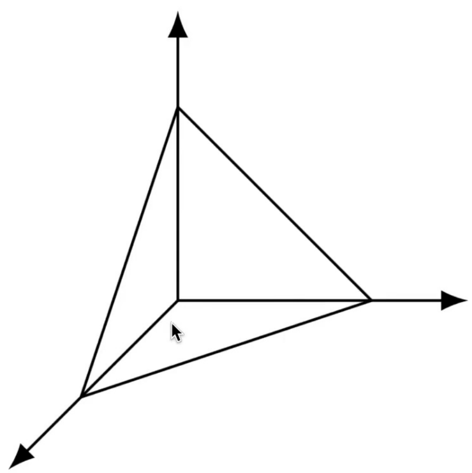
PRX Quantum 2, 010331

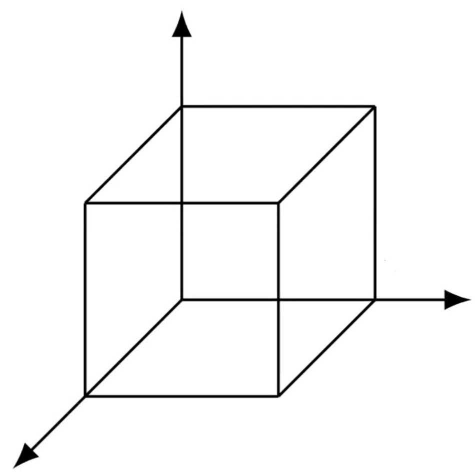
Traditionally, a GPT has been considered classical if it was *simplicial*:

state space: simplex

effect space: dual of simplex

d=3





PRX Quantum 2, 010331

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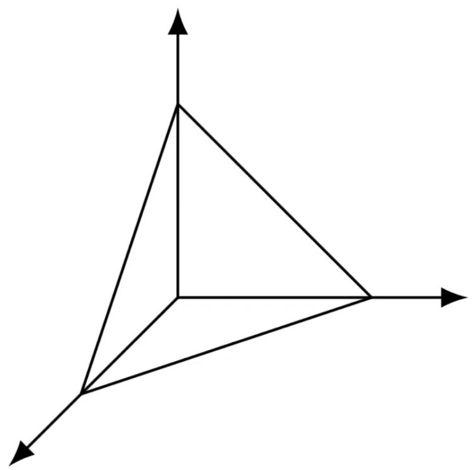
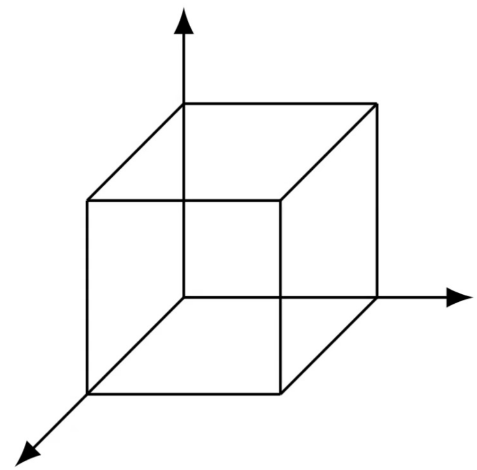
state space: simplex



effect space: dual of simplex



d=3

In a simplicial GPT, all measurements are compatible.

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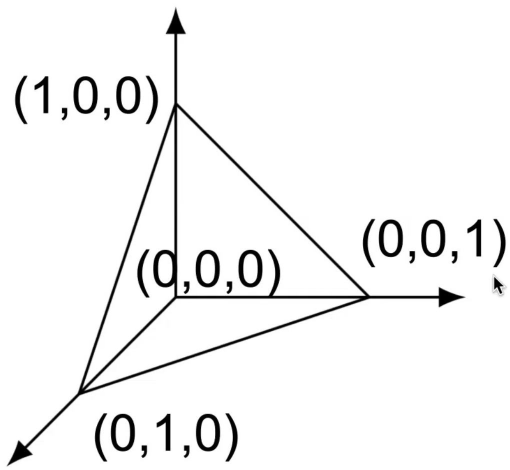

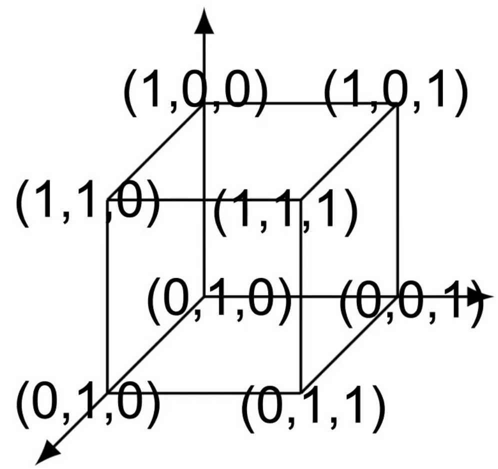

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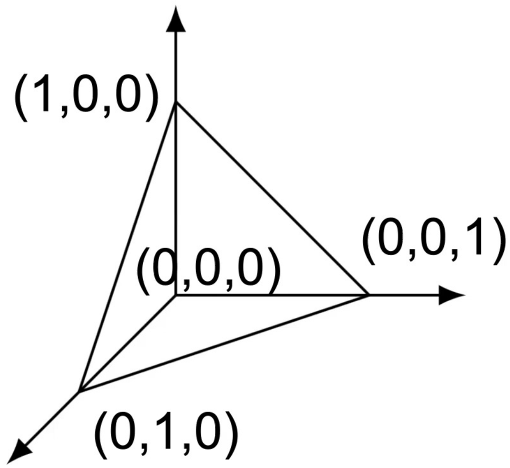
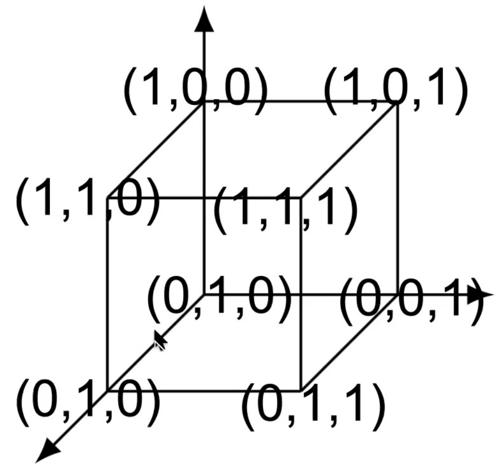
Theorem: A GPT admits of an OM if and only if it is embeddable in a simplicial GPT (of some dimension)





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At this point we have 4 equivalent conditions for a GPT to be classical:

1. the operational theory it came from is NC
2. it admits of an OM
3. it is simplex-embeddable
4. it admits of a positive quasiprobability representation

PRX Quantum 2, 010331

At this point we have 4 equivalent conditions for a GPT to be classical:

1. the operational theory it came from is NC
2. it admits of an OM
3. it is simplex-embeddable
4. it admits of a positive quasiprobability representation

New justifications for taking classicality = NC!

PRX Quantum 2, 010331

The structure of (diagram-preserving) quasiprobability representations

a quasiprobability representation of a GPT is "an ontological model whose probabilities can go negative"

usually only defined for representing QT

the most famous of these (the Wigner repn and Gross's repn) are diagram-preserving, but some (e.g. Q and P repns) are not

Structure theorem for DP quasiprobability repns of QT

pick a basis of trace-one Hermitian operators $\{F_\lambda\}_\lambda$

compute the unique dual basis satisfying $\mathrm{tr}[D_{\lambda'} F_\lambda] = \delta_{\lambda,\lambda'}$

states $$\xi_\rho(\lambda) = \mathrm{tr}[D_\lambda \rho]$$

effects $$\xi_E(\lambda) = \mathrm{tr}[F_\lambda E]$$

transformations $$\xi_{\mathcal{E}}(\lambda'|\lambda) = \mathrm{tr}[D_{\lambda'} \mathcal{E}(F_\lambda)]$$

(for tomographically complete scenarios with transformations) arXiv:2005.07161

Every OM of a tomographically local GPT is of analogous form, but where everything in the image of the map is positive

Every NCOM of a tomographically local op. theory is of this form, but where one quotients first

$\{F_\lambda\}_\lambda$ is a basis

$\Rightarrow$ number of ontic states = dimension of GPT

Powerful tool for studying noncontextuality

Excess baggage theorem ⇒ contextuality

Lillystone et. al.

8-state model (Wallman, Bartlett) for stabilizer qubits is contextual
-follows immediately from the fact that $8 > \text{GPTdim(qubit)} = 4$

Stabilizer subtheory

quantum error correction, information
processing, computation, foundations…

The stabilizer subtheory can be implemented fault-tolerantly.

Some nonstabilizer states promote it to universality.

⇒ State injection model for quantum computation.

$$\{|x\rangle\}_{1,2,\ldots,d} \qquad \omega = e^{\frac{2\pi i}{d}}$$

$$X|x\rangle = |x+1\rangle$$ position translator

$$Z|x\rangle = \omega^{x}|x\rangle$$ momentum translator

Weyl operators: $W_{p,q} := Z^{p}X^{q}$ $\qquad p, q \in \mathbb{Z}_d$

$\{|x\rangle\}_{1,2,\ldots,d}$ $\omega = e^{\frac{2\pi i}{d}}$

$X|x\rangle = |x+1\rangle$ position translator

$Z|x\rangle = \omega^x |x\rangle$ momentum translator

Weyl operators: $W_{p,q} := Z^p X^q$ $p, q \in \mathbb{Z}_d$

states: eigenstates of these
measurements: projective measurements in eigenbasis of these

allowed transformations are Clifford unitaries:

those that preserve Weyl operators

$$UW_{p,q}U^\dagger \propto W_{p',q'}$$

closed under composition and convex mixtures

for n systems of dim d, take tensor products of Weyl operators

Stabilizer subtheory on n qu*d*its is not the same as on one d^n-dimensional system closed under composition and convex mixtures

for n systems of dim d, take tensor products of Weyl operators

multi-particle view

Stabilizer subtheory on n qu*d*its is not the same as on one d^n-dimensional system

single-particle view

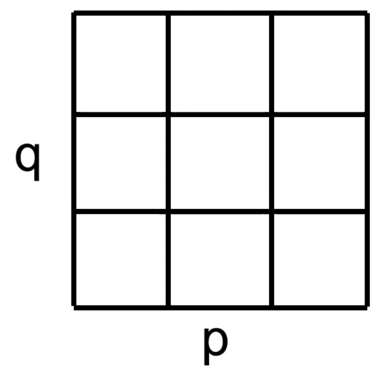
$$\{A_{p,q}\}_{p,q} := \left\{ \frac{1}{d} \sum_{p,q} \exp(pq - p'q' + i\pi/d) W_{p,q}{}^{\dagger} \right\}$$

Gross's Representation

$$\{A_{p,q}\}_{p,q} := \left\{ \frac{1}{d} \sum_{p,q} \exp(pq - p'q' + i\pi/d) W_{p,q}{}^{\dagger} \right\}$$

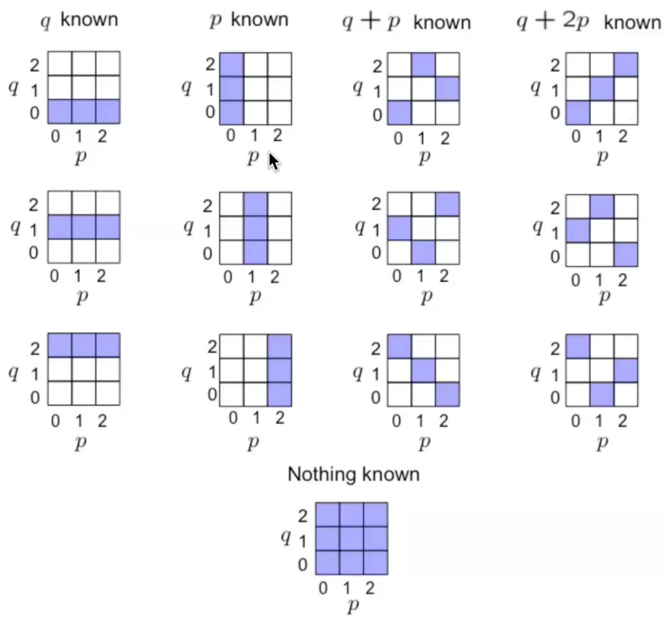
phase space point operators

qutrits– 3x3 phase space



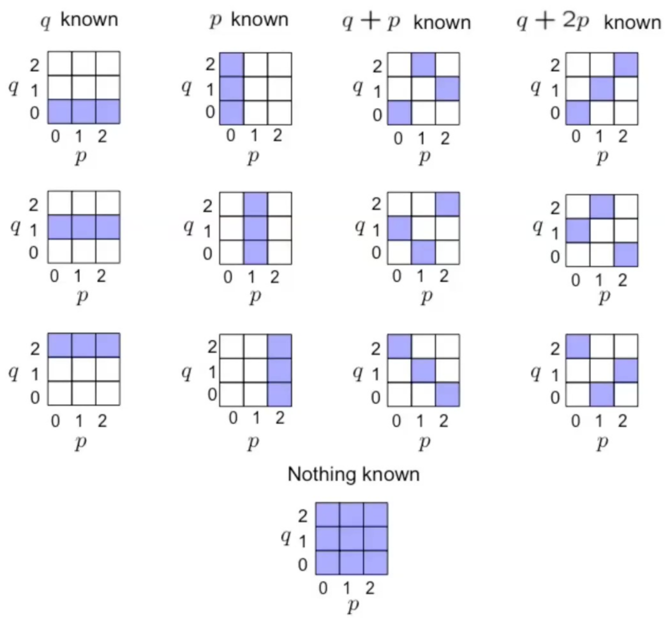
repns of stabilizer states

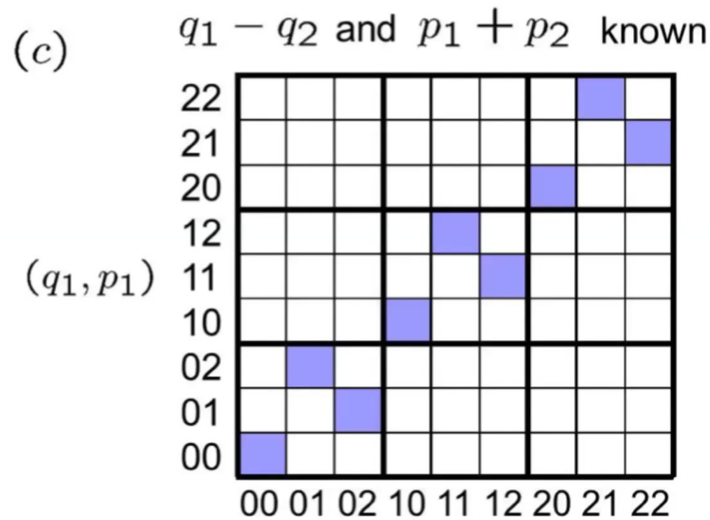
$$\xi_{\rho_{\text{stab}}}(p,q) = \text{tr}[A_{p,q}\rho_{\text{stab}}]$$



repns of stabilizer states

$$\xi_{\rho_{\mathrm{stab}}}(p,q) = \mathrm{tr}[A_{p,q}\rho_{\mathrm{stab}}]$$

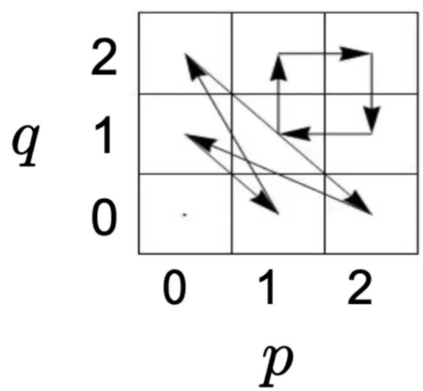




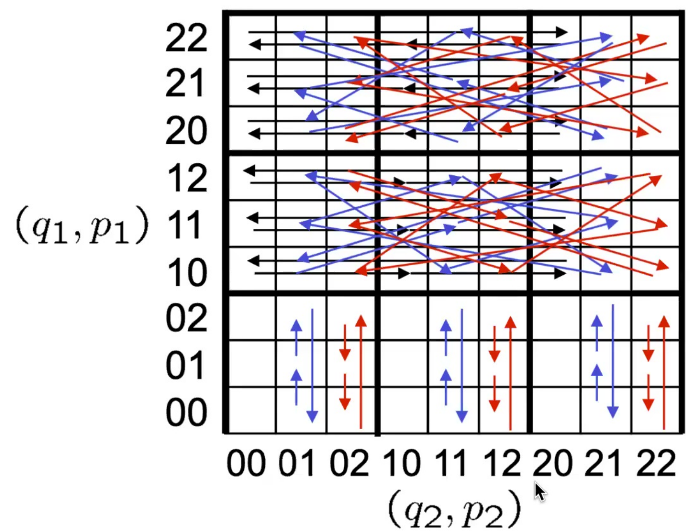
repns of stabilizer unitaries

$$\xi_{U_{\mathrm{stab}}}(p',q'|p,q) = \mathrm{tr}[A_{p',q'}U_{\mathrm{stab}}A_{p,q}U_{\mathrm{stab}}^{\dagger}]$$

$$q \mapsto p$$
$$p \mapsto -q$$



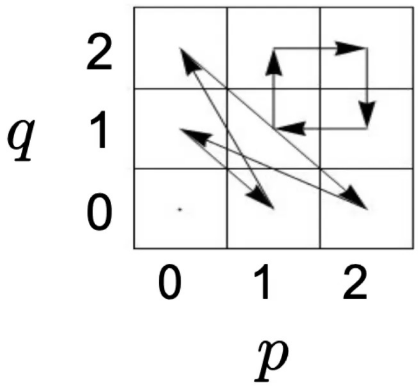
$$q_1 \mapsto q_1$$
$$p_1 \mapsto p_1 - p_2$$
$$q_2 \mapsto q_1 + q_2$$
$$p_2 \mapsto p_2$$



repns of stabilizer unitaries

$$\xi_{U_{\text{stab}}}(p', q' | p, q) = \text{tr}[A_{p',q'} U_{\text{stab}} A_{p,q} U^{\dagger}_{\text{stab}}]$$

$$\mathrm{q} \mapsto \mathrm{p}$$
$$\mathrm{p} \mapsto -\mathrm{q}$$



$$\mathrm{q}_1 \mapsto \mathrm{q}_1$$
$$\mathrm{p}_1 \mapsto \mathrm{p}_1 - \mathrm{p}_2$$
$$\mathrm{q}_2 \mapsto \mathrm{q}_1 + \mathrm{q}_2$$
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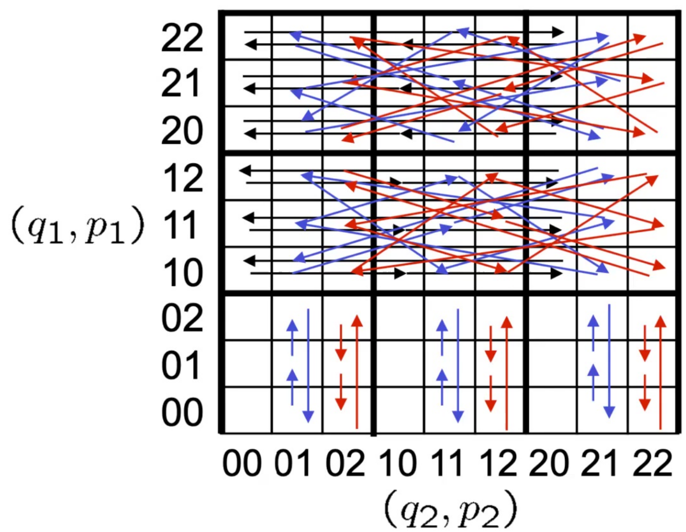


These are all valid stochastic processes

In **odd** dimensions, every stabilizer subtheory has a *unique* positive quasiprobability representation, namely Gross's. (Equivalently, there is a unique NCOM.)

Gross's repn ⇔ Spekkens' toy theory (odd dim)

In **odd** dimensions, every stabilizer subtheory has a *unique* positive quasiprobability representation, namely Gross's. (Equivalently, there is a unique NCOM.)

Gross's repn $\Leftrightarrow$ Spekkens' toy theory (odd dim)

In **even** dimensions, there is no positive quasiprobability representation for any stabilizer subtheory. (Equivalently, there is no NCOM.)

So there is a *unique* classical explanation of the stabilizer subtheory!

In the setting of the odd dimensional stabilizer subtheory, negativity *of Gross's repn* implies nonclassicality

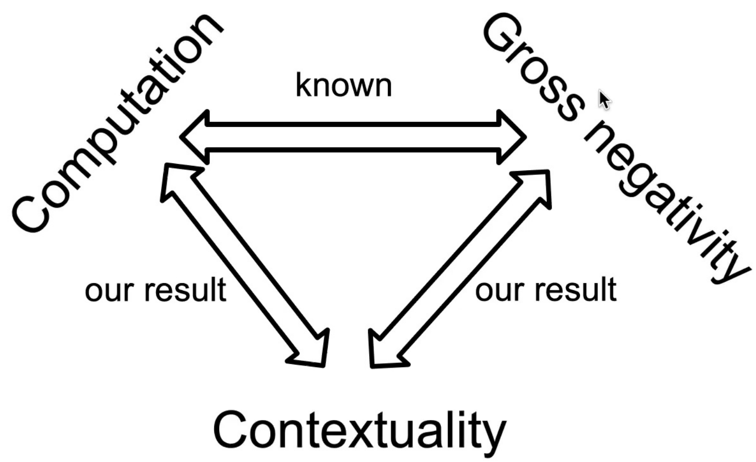
Gross proved that his repn was the only one among the family of "GHW repns" satisfying Clifford covariance

advantages of our uniqueness result over Gross's:

-generalized noncontextuality is a notion of classicality (covariance is not)

-Gross's result requires two ad hoc mathematical assumptions (d x d phase space, correct marginal probabilities)

This explains the demonstrated usefulness of Gross's repn in studying computation



Any state which promotes the stabilizer subtheory to UQC must have negativity in its Gross repn.

Negativity in Gross repn of a state ⇒ KS contextual

Howard et. al. (Nature 2014)

Negativity in Gross repn of a state ⇒ generalized contextuality

our result

Is generalized contextuality sufficient for computation?

Any state which promotes the stabilizer subtheory to UQC must have negativity in its Gross repn.

Negativity in Gross repn of a state ⇒ KS contextual

Howard et. al. (Nature 2014)

Negativity in Gross repn of a state ⇒ generalized contextuality

our result

So generalized contextuality (like KS contextuality) is necessary for UQC in this model

Is generalized contextuality sufficient for computation?

-not without caveats, at least
(even-dimensional stabilizer subtheory is efficiently simulable)

-every nonstabilizer pure state and unitary promotes to universality
-every nonstabilizer pure state and unitary has a negative Gross repn

$\Rightarrow$ for pure states and unitaries, Gross negativity
is a sufficient condition for promoting to UQC

Thank you!

The stabilizer subtheory has a unique noncontextual model

arXiv:2101.06263

A structure theorem for generalized-noncontextual ontological models

arXiv:2005.07161