Title: Crossing Symmetry in the Planar Limit

Speakers: Sebastian Mizera

Series: Quantum Fields and Strings

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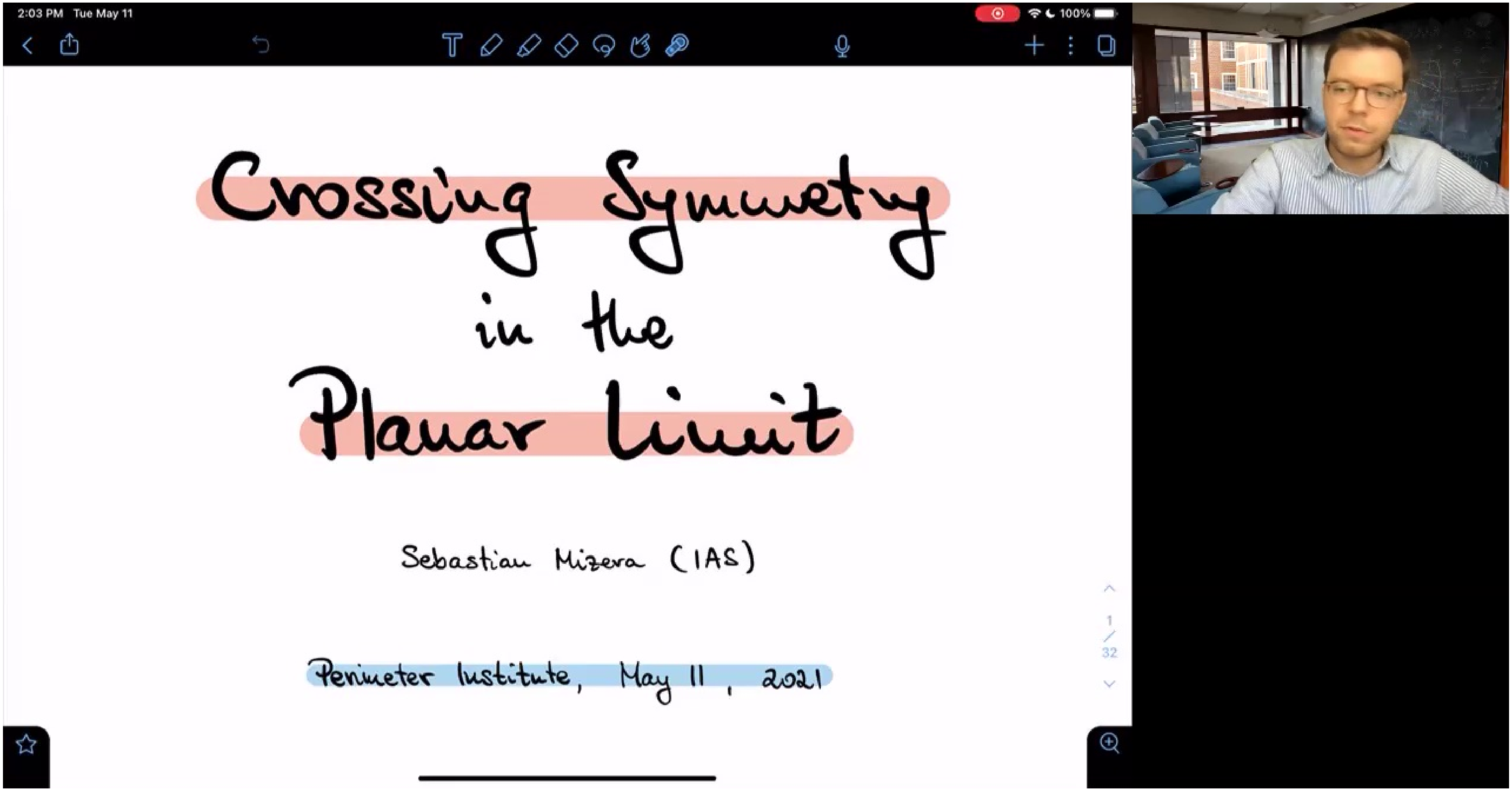
Abstract: Crossing symmetry asserts that particles are indistinguishable from anti-particles traveling back in time. In quantum field theory, this statement translates to the long-standing conjecture that probabilities for observing the two scenarios in a scattering experiment are described by one and the same function. Why could we expect it to be true? In this talk we examine this question in a simplified setup and take steps towards illuminating a possible physical interpretation of crossing symmetry. To be more concrete, we consider planar scattering amplitudes involving any number of particles with arbitrary spins and masses to all loop orders in perturbation theory. We show that by deformations of the external momenta, one can smoothly interpolate between the future and the past lightcones without encountering any singularities.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Crossing Symmetry
in the
Planar Limit
Sebastian Mizera (IAS)
Perimeter Institute, May 11, 2021
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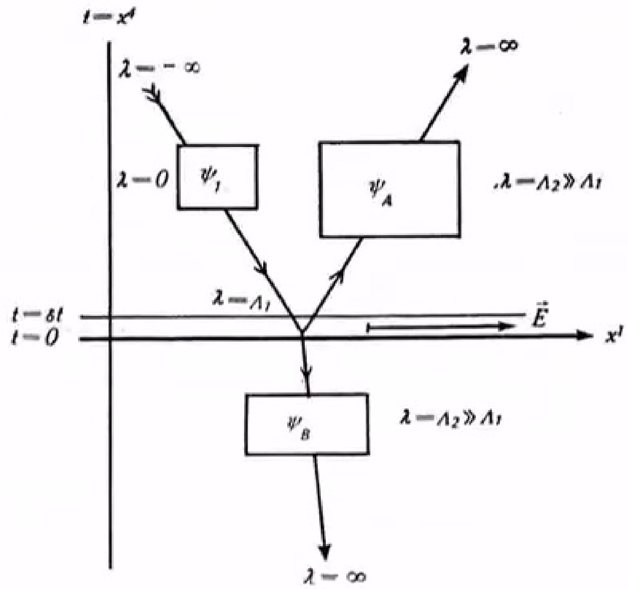
80th anniversary of the first Feynman diagram!



[Stückelberg '41]

In the same paper, Stückelberg made a revolutionary (at the time) observation that free electrons are indistinguishable from free positrons moving back in time.

$p^\mu = (E, \vec{p})$ $\qquad$ $-p^\mu = (-E, -\vec{p})$

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$p^\mu = (E, \vec{p}) \qquad\qquad -p^\mu = (-E, -\vec{p})$

In quantum field theory, this statement needs to be phrased at the level of observables:

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independently of the way we observe it (any number and type of the remaining particles)

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Crossing symmetry:

Are the two scattering amplitudes boundary values of the same function?

[Gell-Mann, Goldberger, Thirring '54]

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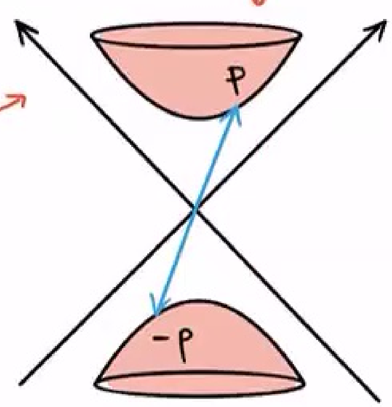
How are we supposed to think about crossing symmetry?

mass shell $p^2 = M^2$

future lightcone



past lightcone

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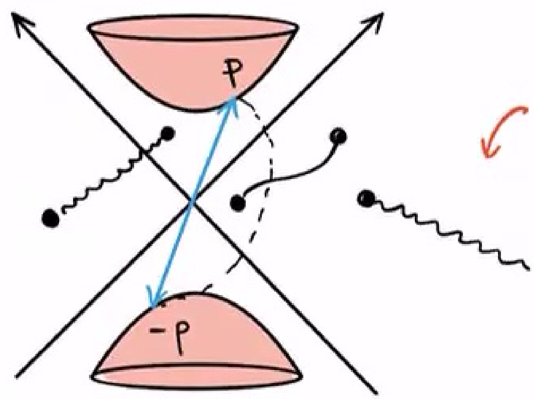
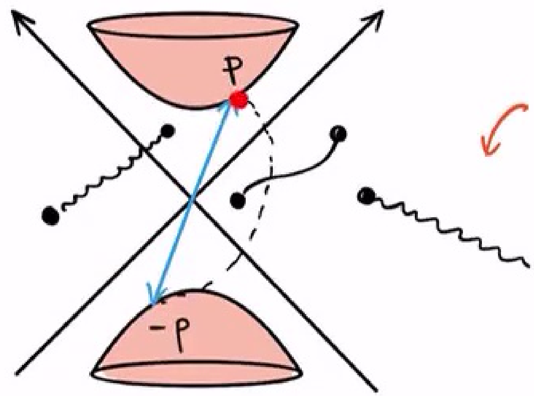
The two processes are defined in disjoint regions of the momentum space. We need to complexify it.



singularities
(worldline saddle points)

Scattering amplitudes become multi-valued functions with a host of new singularities. Crossing symmetry

momentum space. We need to complexify it.



singularities

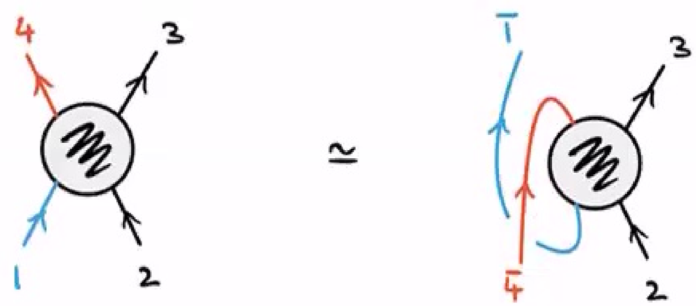
(worldline saddle points)

Scattering amplitudes become multi-valued functions with a host of new singularities. Crossing symmetry has to explain how to navigate around such singularities.

There has been a lot of work in attempting to prove crossing symmetry non-perturbatively.

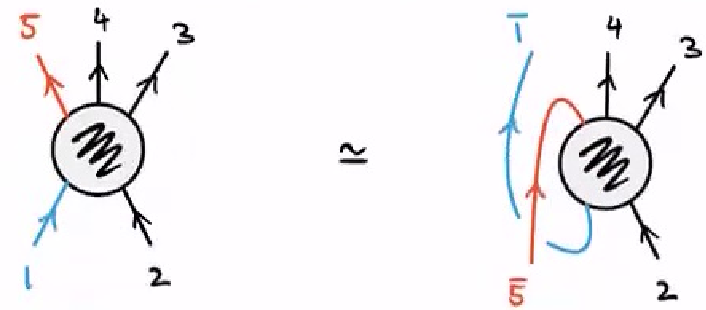
Under the assumptions of microcausality, locality, unitarity, and the mass gap:

$2 \to 2$:



[Bros, Epstein, Glaser '65]

$2 \to 3$:



[Bros '86]

So why don't we learn about it in the PSI Quantum Field Theory courses?

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→ The proofs are not physically illuminating:

$$\underbrace{\text{position-space correlators} \xrightarrow{\text{LSZ}} \text{off-shell Green's functions}}_{\text{all the physics input}} \xrightarrow{\text{complex analysis}} \text{on-shell scattering amplitudes}$$

- LSZ: all the physics input
- complex analysis: all the meat of the proof

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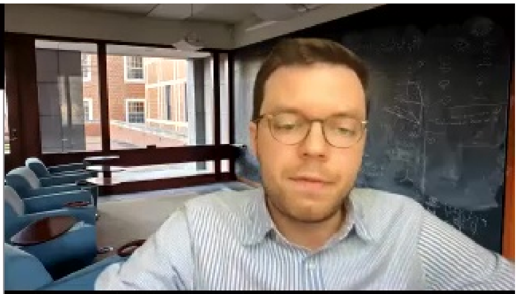
position-space correlators $\xrightarrow{\text{LSZ}}$ off-shell Green's functions $\xrightarrow{\text{analysis}}$ on-shell scattering amplitudes

all the physics input

all the meat of the proof

→ The proofs are so technically complicated, they were not carried out in many physically-important situations



The simplest cases where we don't really know if crossing symmetry holds:

→ Any process involving massless particles

→ Different number of in/out states, e.g.,

$$12 \to 34 \quad \text{and} \quad 1 \to \bar{2}34$$

$$12 \to 345 \quad \text{and} \quad 12\bar{3} \to 45$$

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$$12 \to 345 \quad \text{and} \quad 12\bar{3} \to 45$$

→ Any amplitude with $n>5$ external particles

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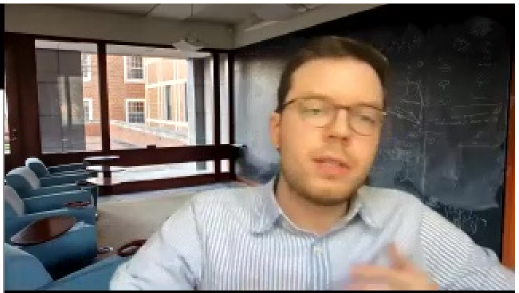
The bottom line is: we need a better approach...



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The idea is to prove crossing symmetry within the framework of perturbation theory, where one might reasonably hope to circumvent the aforementioned issues.

→ Singularities have a clear physical meaning as worldline saddle points, where the intermediate states go on-shell

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the aforementioned issues.

→ Singularities have a clear physical meaning as worldline saddle points, where the intermediate states go on-shell

→ Standard ways of dealing with UV/IR divergences

→ The same Feynman rules for any multiplicity

(we can focus on 4-pt scattering)

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→ The same Feynman rules for any multiplicity

(we can focus on 4-pt scattering)

This will allow us to

→ Identify the classes of singularities that can pose an obstruction to crossing symmetry

→ Explain why they don't exist in planar scattering processes, thus proving crossing in those cases (still work to all loops, multiplicities, masses, ...)

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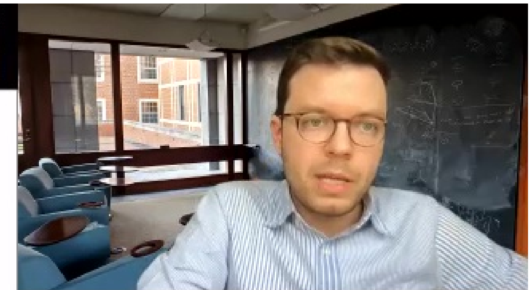

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masses, ...)

We will show that if an amplitude in the channel $AB \to CD$ exists, the crossed amplitude is given by its analytic continuation,

$$S_{AB\to CD} = S_{B\bar{C}\to D\bar{A}}$$

explaining what this means is the content of this talk

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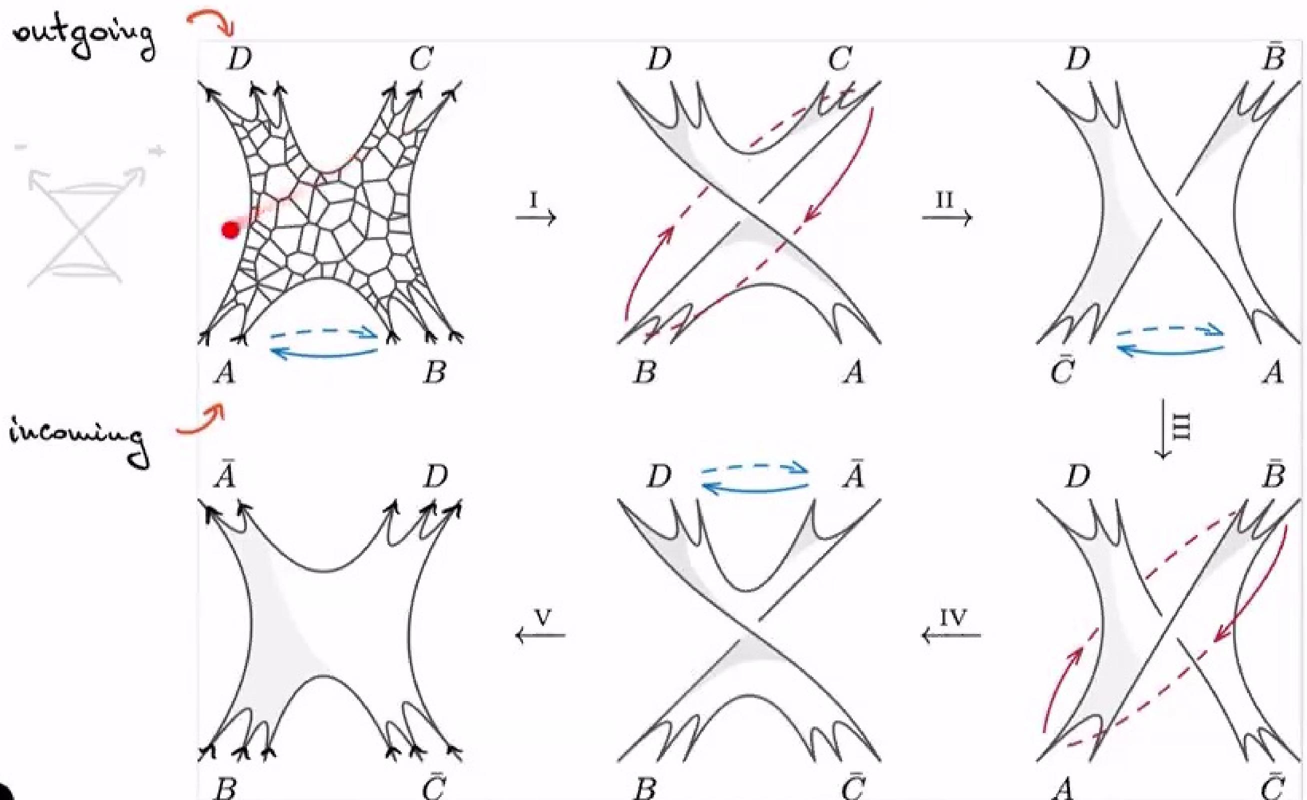
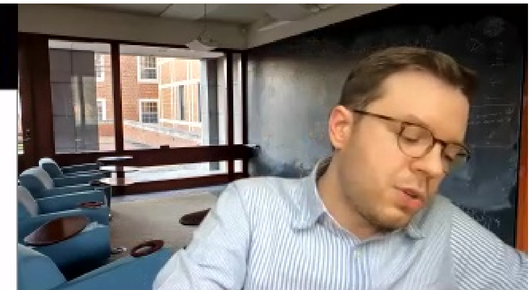
Continuation on a cartoon level:





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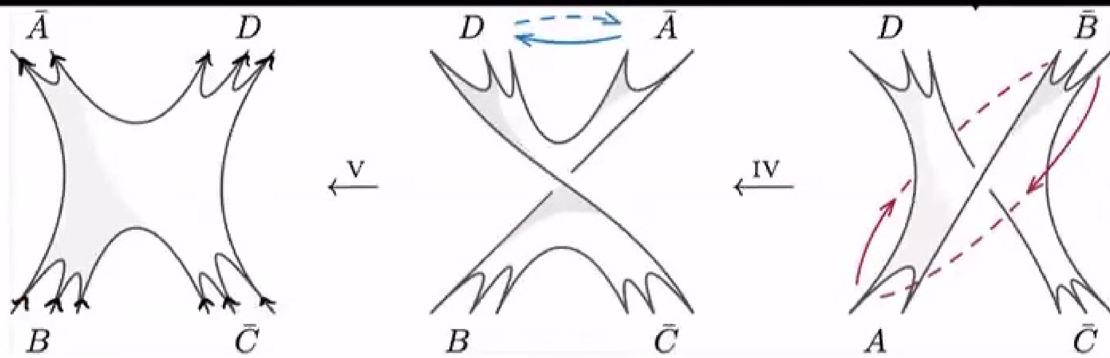

→ Each diagram represents a configuration that cannot have singularities

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→ Each diagram represents a configuration that cannot have singularities

We can compose these deformations to obtain, e.g.,

$$S_{\underbrace{1}_{A}\underbrace{2}_{B}\to\underbrace{34}_{C}\underbrace{5}_{D}} = S_{\underbrace{2\bar{3}}_{A}\underbrace{\bar{4}}_{B}\to\underbrace{5}_{C}\underbrace{\bar{1}}_{D}} = S_{\underbrace{\bar{4}}_{A}\underbrace{\bar{5}}_{B}\to\underbrace{\bar{1}\bar{2}}_{C}\underbrace{3}_{D}} = S_{\bar{\bar{5}}12\to34}$$

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We can compose these deformations to obtain, e.g.,

$$\underbrace{S_{\underbrace{1}_{A}\underbrace{2}_{B}\to\underbrace{34}_{C}\underbrace{5}_{D}}}_{\text{particle } 5} = S_{\underbrace{2\bar{3}}_{A}\underbrace{\bar{4}}_{B}\to\underbrace{5}_{C}\underbrace{\bar{1}}_{D}} = S_{\underbrace{\bar{4}}_{A}\underbrace{\bar{5}}_{B}\to\underbrace{\bar{1}\bar{2}}_{C}\underbrace{3}_{D}} = \underbrace{S_{\bar{5}12\to34}}_{\text{anti-particle } \bar{5}}$$

particle 5

anti-particle $\bar{5}$

Crossing symmetry!

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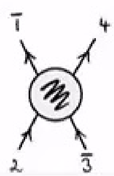
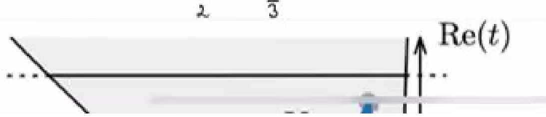
For concreteness we will focus on the simplest 4-pt example with $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$.

Two Mandelstam invariants $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$





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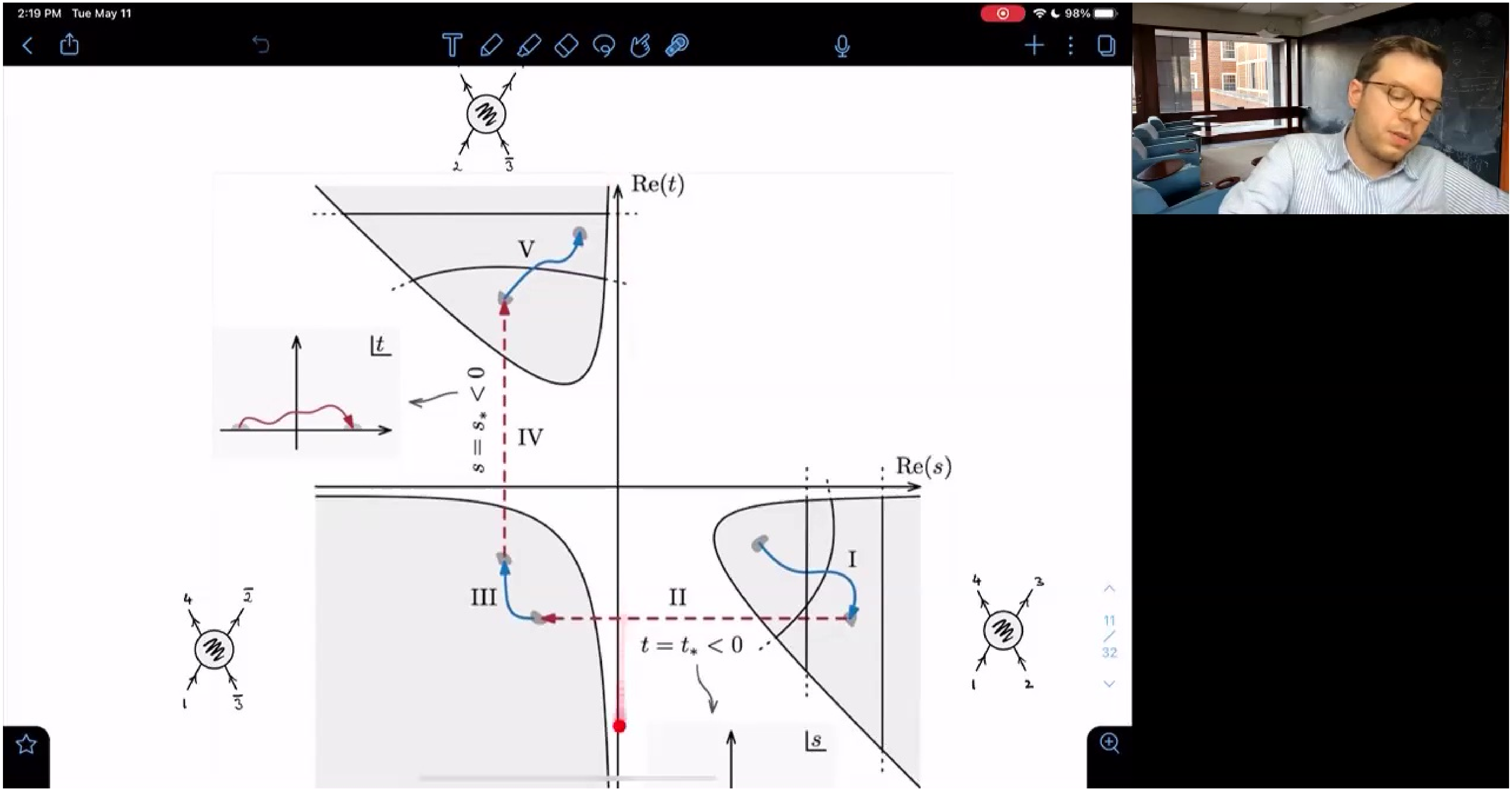
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Re(t)
V
IV
$s = s_* < 0$
t
Re(s)
I
II
III
$t = t_* < 0$
s
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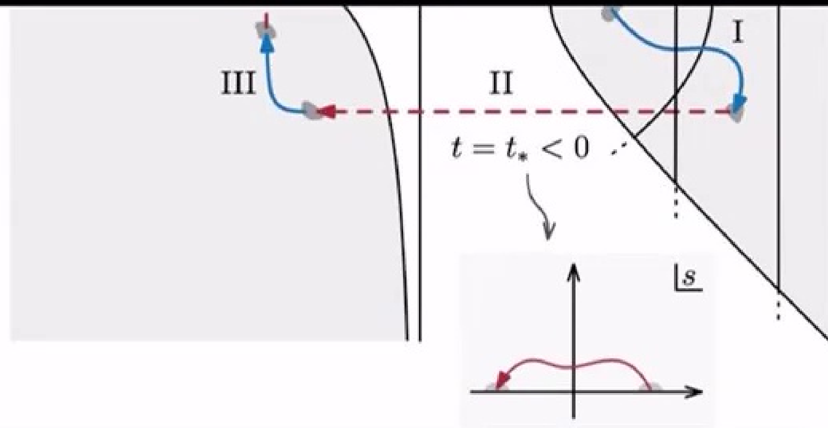

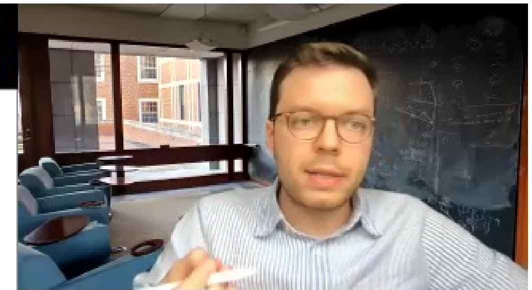

It should be stressed that $n=4$ continuation can be formulated more simply, but this one

→ has physical interpretation

→ generalizes to arbitrary n

→ has a chance of working for non-planar amplitudes

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Outline

1. Review of Landau equations
2. Energy flow in planar diagrams
3. Analytic continuation near physical regions (steps I, III, IV)
4. Analytic continuation in crossing domains (steps II, IV)
5. Putting everything together

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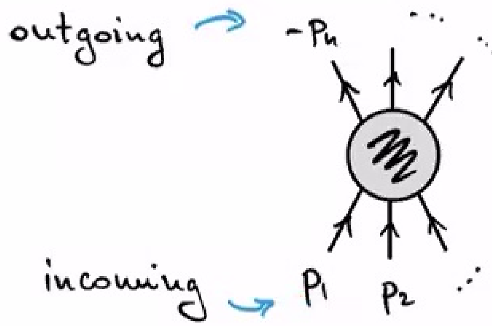
Conventions:



→ momentum conservation reads

$$\sum_{i=1}^{n} p_i^\mu = 0$$ and on-shell conditions $p_i^2 = M_i^2$.

→ $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.

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Review of Landau equations

In order to make the question of crossing symmetry well-posed we need to assume the amplitude exists in the first place.

In other words, all the overall divergences have been

→ renormalized (e.g. BPHZ renorm.) ; and/or

→ regularized (e.g. analytic / dimensional reg.)

In a CPT-invariant quantum field theory admitting

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well-posed we need to assume the amplitude exists in the first place.

In other words, all the overall divergences have been

→ renormalized (e.g. BPHZ renorm.); and/or

→ regularized (e.g. analytic / dimensional reg.)

In a CPT-invariant quantum field theory admitting local Feynman rules, scattering amplitudes are linear combinations of Feynman integrals

$$I = \int d^{DL}\ell_a \, N \prod_{e=1}^{E} \frac{i\hbar}{q_e^2 - m_e^2 + i\epsilon}$$

linear combinations of Feynman integrals

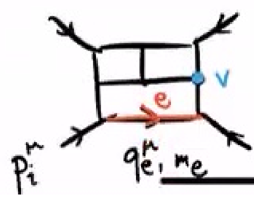
$$I = \int d^{DL}\ell_a \, N \prod_{e=1}^{E} \frac{i\hbar}{q_e^2 - m_e^2 + i\epsilon}$$

set $N=1$ for now (scalar)

D = space-time dimension

L = # loops

E = # internal edges (propagators)



Mom. cons. at every vertex v:

$$p_v^\mu + \sum_{e \ni v} \pm q_e^\mu = 0$$

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$$I = \int d^{DL}\ell_a \; N \prod_{e=1}^{E} \frac{i\hbar}{q_e^2 - m_e^2 + i\epsilon}$$

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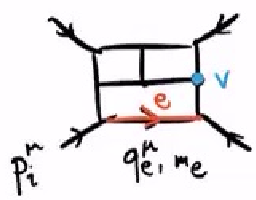

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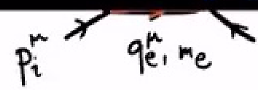
$$p_v^\mu + \sum_{e \ni v} \pm q_e^\mu = 0$$

ext' mom.

orientation



$$p_v^\mu + \sum_{e \ni v} \pm q_e^\mu = 0$$

ext' mom.   orientation

Introduce Schwinger parameters α_e :

$$\frac{i\hbar}{q_e^2 - m_e^2 + i\epsilon} = \int_0^\infty d\alpha_e \; e^{\frac{i}{\hbar}(q_e^2 - m_e^2 + i\epsilon)\,\alpha_e}$$

Convergence at ∞

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$$\frac{\cdots}{q_e^2 - m_e^2 + i\epsilon} = \int_0 d\alpha_e \; e^{\cdots}$$

convergence at ∞

Applying it to every propagator we obtain

$$I = \int d^{DL}\ell_a \; d^{E}\alpha_e \; e^{\frac{i}{\hbar}\left(V + i\epsilon \sum_e \alpha_e \right)}$$

where

$$V(\alpha_e, q_e^\mu) = \sum_{e=1}^{E} \left(q_e^2 - m_e^2 \right) \alpha_e$$

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$$I = \int d^{DL} \ell_a \, d^{E} \alpha_e \; e^{\frac{i}{\hbar}\left(V + i\epsilon \sum_e \alpha_e \right)}$$

where

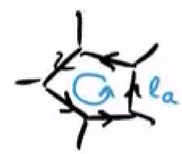
$$V(\alpha_e, q_e^\mu) = \sum_{e=1}^{E} \left(q_e^2 - m_e^2 \right) \alpha_e$$

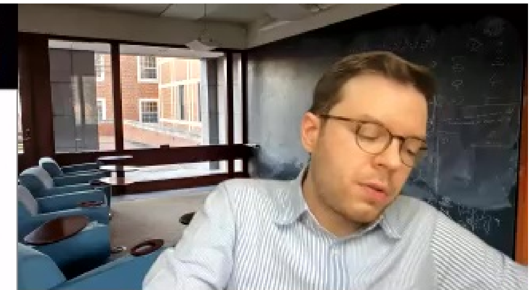
In the classical limit, $\hbar \to 0$, they are dominated by saddle points; ignoring boundaries:

→ Vary ℓ_a^μ :

$$\sum_{e \ni \ell} \pm q_e^\mu \alpha_e = 0 .$$

orientation





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$$V(\alpha_e, q_e^\mu) = \sum_{e=1}^{E} \left(q_e^2 - m_e^2 \right) \alpha_e$$

In the classical limit, $\hbar \to 0$, they are dominated by saddle points; ignoring boundaries:

→ Vary ℓ_a^μ:

$$\sum_{e \in \ell} \pm q_e^\mu \alpha_e = 0 .$$

orientation

→ Vary α_e:

$$q_e^2 - m_e^2 = 0 .$$

particle going on-shell

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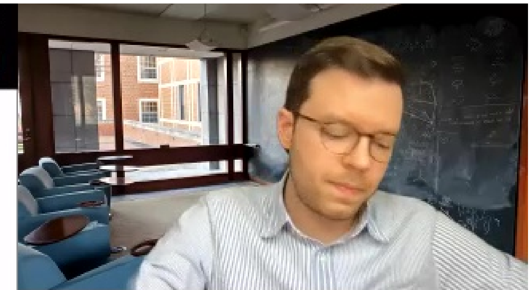
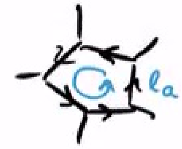


→ Vary ℓ_a^μ : $\displaystyle\sum_{e \ni \ell} \pm q_e^\mu \alpha_e = 0$.

orientation



→ Vary α_e : $q_e^2 - m_e^2 = 0$.

particle going on-shell

These are known as the leading Landau equations:

necessary cond^s for singularities

linear
$$\begin{cases} p_v^\mu + \displaystyle\sum_e \pm q_e^\mu = 0 & \forall \text{ vertices } v \\ \displaystyle\sum_e \pm q_e^\mu \alpha_e = 0 & \forall \text{ loops } a \end{cases}$$

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particle going on-shell

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necessary cond^s for singularities

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$$\begin{cases} p_v^\mu + \sum_e \pm q_e^\mu = 0 & \forall \text{ vertices } v \\ \sum_e \pm q_e^\mu \alpha_e = 0 & \forall \text{ loops } a \end{cases}$$

quadratic
$$\left\{ q_e^2 - m_e^2 = 0 \quad \forall \text{ edges } e \right.$$

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There are also boundary saddle points

→ **Subleading LE** ($\alpha_e \to 0, \infty$):

same as leading LE for a simpler diagram, but we already consider all (planar) diagrams

→ **Second-type LE** ($\ell_a^\mu \to \pm\infty$):

only matter after performing large contour deformations (e.g. discontinuities) which we won't do

(& only at special kinematic configurations)

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At this stage the integral is Gaussian in the loop momenta, so we can just integrate them out:

$$I = \# \int_0^\infty \frac{d^E \alpha_e}{\mathcal{U}^{D/2}} \, N \, e^{\frac{i}{\hbar}\left(V + i\epsilon \sum_e \alpha_e\right)}.$$

inclusion of spin interactions cannot introduce new singularities

where

$$V(\alpha_e) = \sum_{e=1}^{E} \left(q_e^2 - m_e^2\right) \alpha_e \Big|_{\text{linear LE}}$$

$q_e^2 = q_e^2(s_{ij}, \alpha_e)$

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which is only a ~~function of Schw~~inger parameters

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At this stage the integral is Gaussian in the loop momenta, so we can just integrate them out:

$$I = \# \int_0^\infty \frac{d^E \alpha_e}{\mathcal{U}^{D/2}} \, N \, e^{\frac{i}{\hbar}\left(V + i\epsilon \sum_e \alpha_e\right)}.$$

inclusion of spin interactions cannot introduce new singularities

where

$$V(\alpha_e) = \sum_{e=1}^{E} \left(q_e^2 - m_e^2\right) \alpha_e \Big|_{\text{linear LE}}$$

$q_e^2 = q_e^2(s_{ij}, \alpha_e)$

which is only a function of Schwinger parameters and Mandelstam invariants.

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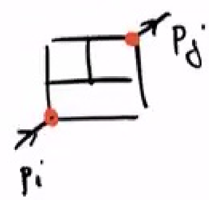
In fact, V is nothing more than the worldline action.

$$V = - \sum_{i<j} p_i \cdot p_j \, \mathcal{G}_{ij} - \sum_e m_e^2 \alpha_e$$

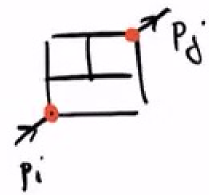
Green's function on the graph



worldline action.

$$V = -\sum_{i<j} p_i \cdot p_j \, G_{ij} - \sum_e m_e^2 \alpha_e$$

Green's function on the graph



The action is homogeneous

$$V(\lambda \alpha_e) = \lambda \, V(\alpha_e),$$

which has three important consequences:

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$$V(\lambda \alpha_e) = \lambda V(\alpha_e),$$

which has three important consequences:

→ The action vanishes on the saddle points:

$$\frac{\partial V}{\partial \alpha_e} = q_e^2 - m_e^2 = 0$$

$$\Rightarrow \quad V = \sum_e \alpha_e \frac{\partial V}{\partial \alpha_e} = 0.$$

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which has three important consequences:

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$$\frac{\partial V}{\partial \alpha_e} = q_e^2 - m_e^2 = 0$$

$$\Rightarrow \quad V = \sum_e \alpha_e \frac{\partial V}{\partial \alpha_e} = 0.$$

→ Projectivity $\alpha_e \in \mathbb{CP}^{E-1}$ means we get E constraints on $E-1$ variables; giving generically one constraint on the external kinematics.

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→ Projectivity $\alpha_e \in \mathbb{CP}^{E-1}$ means we get E constraints on E−1 variables; giving generically one constraint on the external kinematics.

→ Factoring out the overall scale $\alpha_e \to \lambda \alpha_e$:

$$V \to \lambda V \,, \quad \mathcal{U} \to \lambda^L \mathcal{U} \,, \quad d^E\alpha_e \longrightarrow \frac{d^E\alpha_e}{GL(1)} \frac{d\lambda}{\lambda^{1-E}}$$

makes the integral proportional to

$$\int_0^\infty \frac{d\lambda}{\lambda^{1-\gamma}} \; e^{\frac{i\lambda}{\hbar}(V + i\epsilon)} \quad \propto \quad \frac{\Gamma(\gamma)}{(V+i\epsilon)^\gamma}$$

degree of divergence $\gamma = E - LD/2$

" 0 on saddles

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$V \to \lambda V$, $\quad \mathcal{U} \to \lambda^{L}\,\mathcal{U}$, $\quad d^{E}\alpha_e \longrightarrow \dfrac{d^{E}\alpha_e}{GL(1)}\,\dfrac{d\lambda}{\lambda^{1-E}}$

makes the integral proportional to

$$\int_0^\infty \frac{d\lambda}{\lambda^{1-\gamma}}\; e^{\frac{i\lambda}{\hbar}(V+i\epsilon)} \;\propto\; \frac{\Gamma(\gamma)}{(V+i\epsilon)^{\gamma}}$$

degree of divergence $\gamma = E - LD/2$

$(V+i\epsilon) = 0$ on saddles

Hence worldline saddle points give singularities

(historically referred to as anomalous thresholds).

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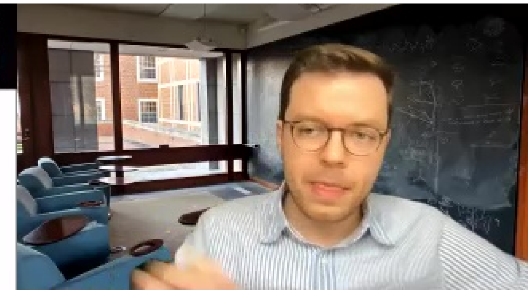

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Hence worldline saddle points give singularities

(historically referred to as anomalous thresholds).

Causality is imposed by inserting $i\epsilon$ "by hand".

We will instead:

→ deform the contour

→ deform the external kinematics

to the same effect, i.e., $\mathrm{Im}\, V > 0$.

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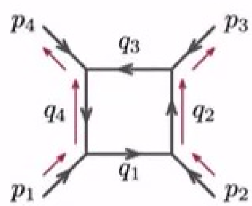
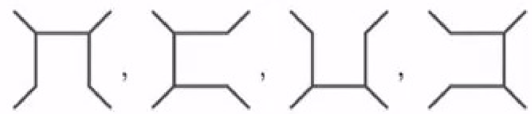
Simple example and energy flow in planar diagrams



Spanning trees

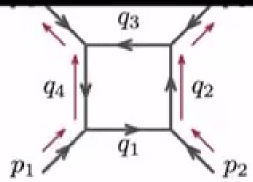
$T \in \{$

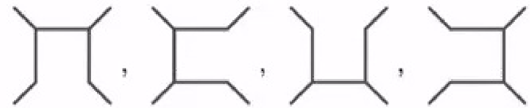


$\}$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > 0$$

Linear Landau equations



$T \in \{$  $\}$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > 0$$

Linear Landau equations

$$p_i^\mu - q_i^\mu + q_{i-1}^\mu = 0 \qquad i = 1,2,3,4$$

$$\sum_{e=1}^{4} \alpha_e q_e^\mu = 0$$

have the solution

$$q_1^\mu = \frac{-p_2^\mu \alpha_2 - p_{23}^\mu \alpha_3 + p_1^\mu \alpha_4}{\quad}, \qquad q_2^\mu = \frac{p_2^\mu \alpha_1 - p_3^\mu \alpha_3 + p_{12}^\mu \alpha_4}{\quad},$$

$$p_i^\mu - q_i^\mu + q_{i-1}^\mu = 0 \qquad i = 1,2,3,4$$

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$$q_3^\mu = \frac{p_{23}^\mu \alpha_1 + p_3^\mu \alpha_2 + p_{123}^\mu \alpha_4}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, \qquad q_4^\mu = \frac{-p_1^\mu \alpha_1 - p_{12}^\mu \alpha_2 - p_{123}^\mu \alpha_3}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}.$$

Let's look at the energy component $\mu = 0$ for $12 \to 34$ scattering, i.e., $p_1^0, p_2^0 > 0$ and $p_3^0, p_4^0 < 0$ with $\alpha_e > 0$:

$$\sum_{e=1} \alpha_e q_e^\mu = 0$$

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$$q_2^0 > 0 \quad \text{and} \quad q_4^0 < 0$$

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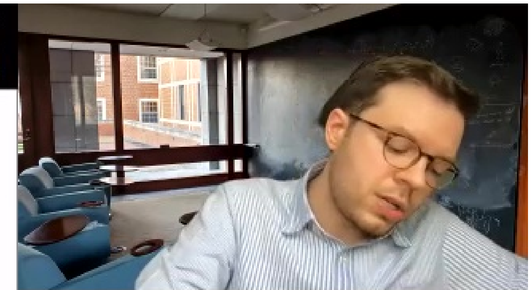

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Let's look at the energy component $\mu=0$ for $12 \to 34$ scattering, i.e., $p_1^0, p_2^0 > 0$ and $p_3^0, p_4^0 < 0$ with $\alpha_e > 0$:

$$q_2^0 > 0 \quad \text{and} \quad q_4^0 < 0$$

If there's a singularity, the energy can only flow in the causal direction along its sides!

This is a general fact. One explanation is that on the saddle point the total Lorentzian length of the diagram wants to be minimized:

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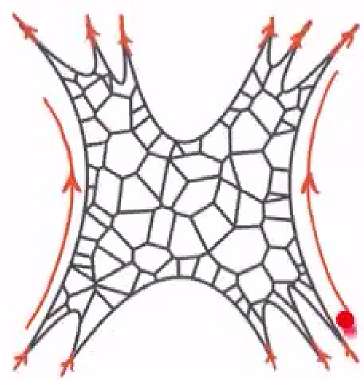

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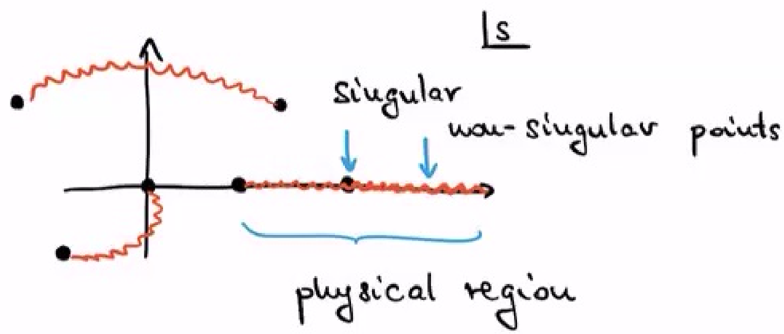
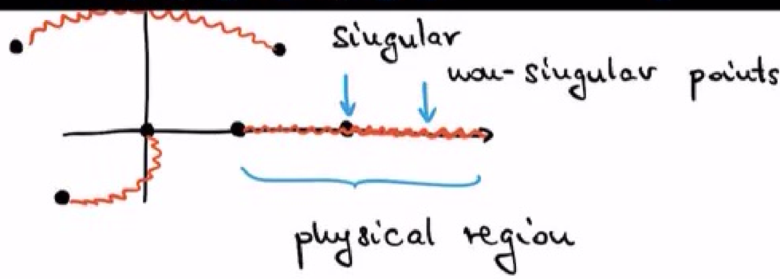
Analyticity near the physical regions

doesn't assume planarity



Near non-singular points we can simply deform the integration contour $\alpha_e \mapsto \check{\alpha}_e$. There exists a canonical deformation:



Near non-singular points we can simply deform the integration contour $\alpha_e \mapsto \check{\alpha}_e$. There exists a canonical deformation:

$$\check{\alpha}_e = \alpha_e \, e^{i\epsilon(q_e^2 - m_e^2)}$$

$$= \alpha_e + i\epsilon(q_e^2 - m_e^2)\,\alpha_e + \mathcal{O}(\epsilon^2)$$

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The deformed action reads

$$\check{V}(\check{\alpha}_e) = V(\alpha_e) + i\epsilon \sum_{e=1}^{E} (q_e^2 - m_e^2)\,\alpha_e \overbrace{\frac{\partial V}{\partial \alpha_e}}^{q_e^2 - m_e^2} + \mathcal{O}(\epsilon^2)$$

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$$\check{V}(\check{\alpha}_e) = V(\alpha_e) + i\epsilon \sum_{e=1}^{E} (q_e^2 - m_e^2)\, \alpha_e \underbrace{\frac{\partial V}{\partial \alpha_e}}_{q_e^2 - m_e^2} + \mathcal{O}(\epsilon^2)$$

$$= V + i\epsilon \sum_{e=1}^{E} (q_e^2 - m_e^2)^2 \alpha_e + \mathcal{O}(\epsilon^2).$$

Its imaginary part is

$$\mathrm{Im}\, \check{V} = \epsilon \underbrace{\sum_{e=1}^{E} (q_e^2 - m_e^2)^2 \alpha_e}_{>0} + \dots$$

So for sufficiently small ϵ it implements the correct causality conditions.

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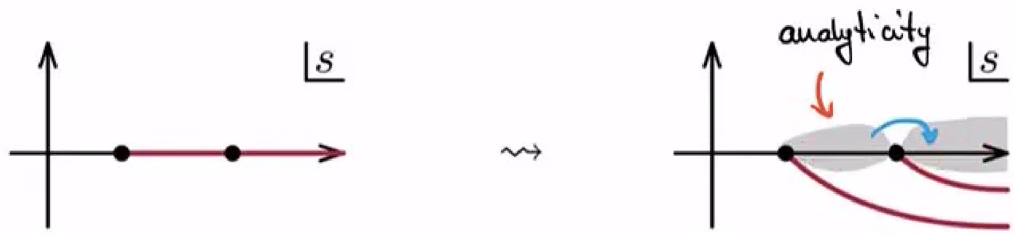
Moreover, we can make small $\mathcal{O}(\epsilon^2)$ deformations of external kinematics. ~~which alter~~ subleading

> 0

So for sufficiently small ϵ it implements the correct causality conditions.

Moreover, we can make small $O(\epsilon^2)$ deformations of external kinematics, which gives subleading corrections to V $\Rightarrow$ Contour deformation still valid.



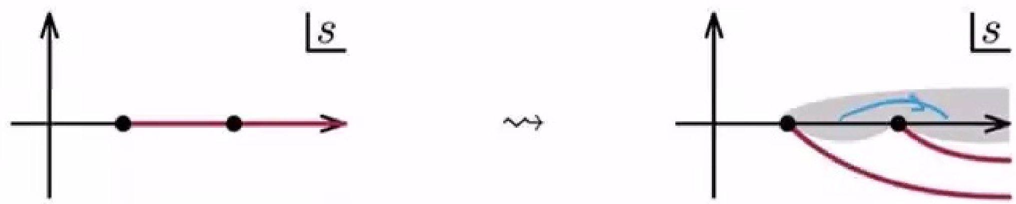
only <=> definition of Landau curve

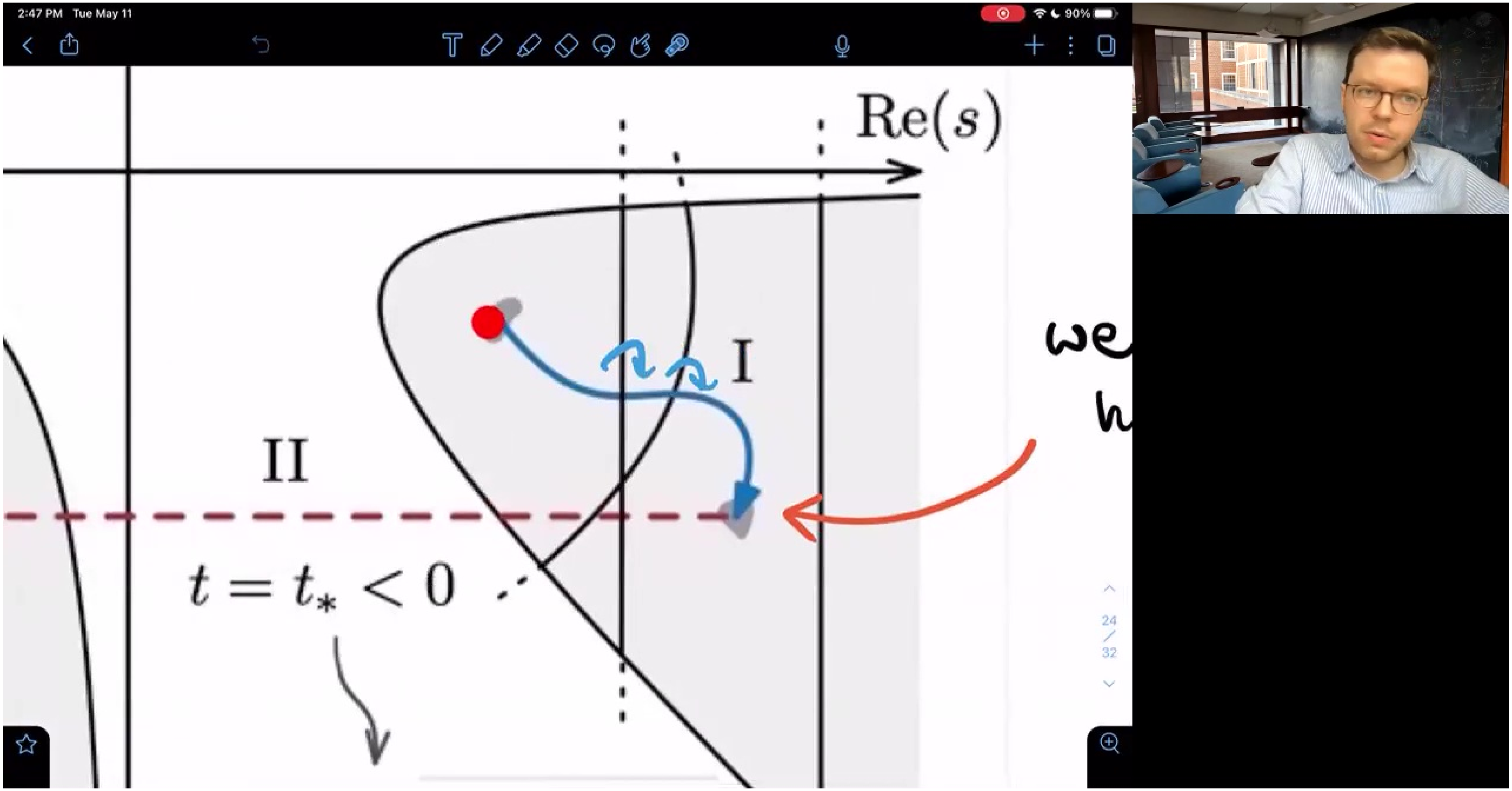
Now it's clear how to continue around this Landau singularity; go in

$$\operatorname{Im}\left(\sum_{e=1}^{E} \Delta q_e^* \cdot q_e^* \, \alpha_e^* \right) > 0$$

for sufficiently small ϵ.

for sufficiently small ϵ.





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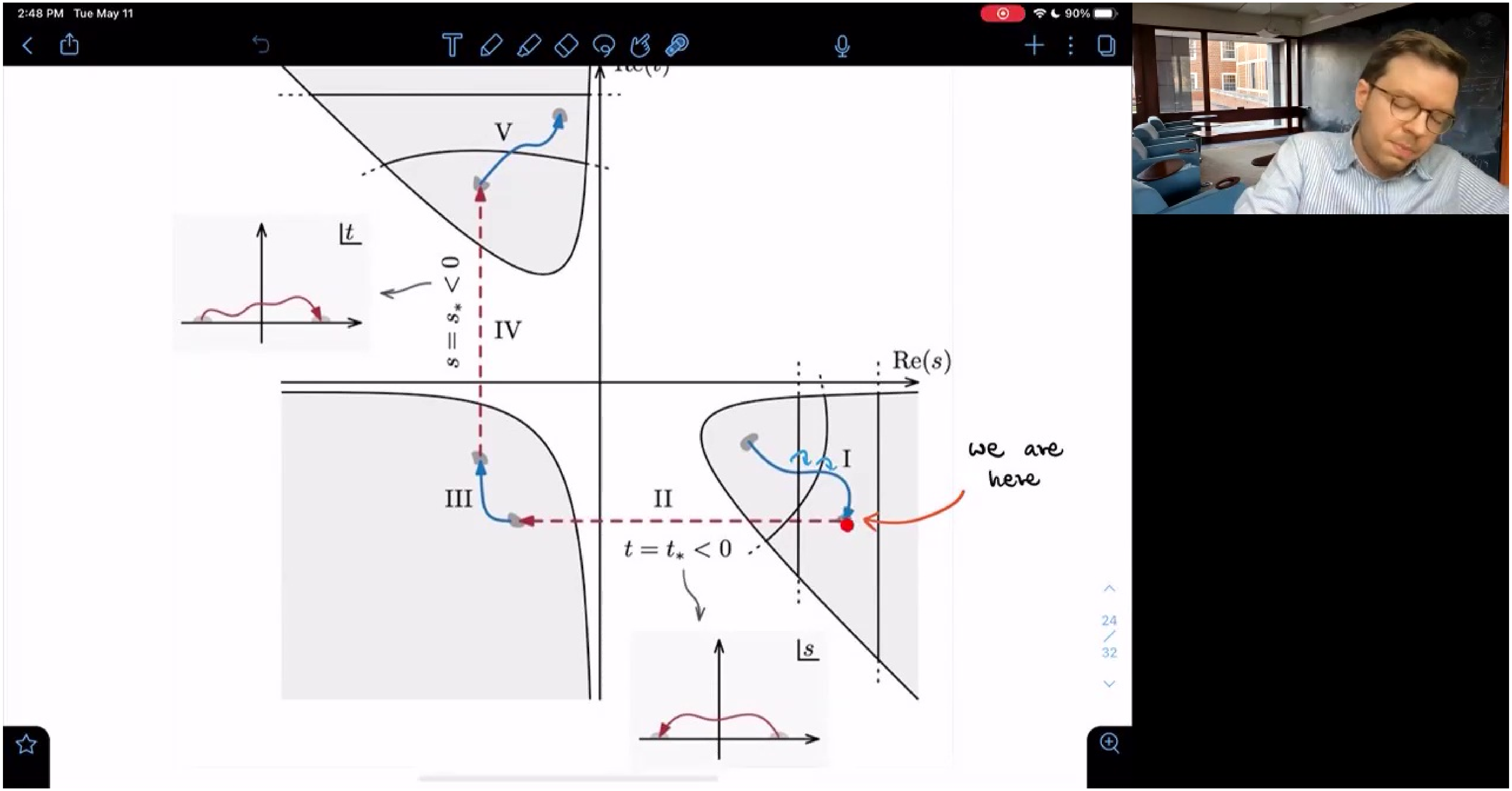
Re(s)
I
II
$t = t_* < 0$
we
24
32



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V
IV
III
II
I
$s = s_* < 0$
$t = t_* < 0$
Re(s)
$\lfloor t$
$\lfloor s$
we are here
24 / 32

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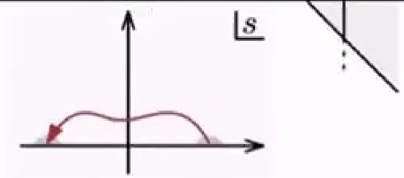

Analyticity in the crossing domains

Go to the Lorentz frame (in lightcone coord^s)

$$ds^2 = dx^+ dx^- - d\vec{x}^{\,2}$$

$$p_1^\mu = (p_1^+, \; p_1^-, \; \vec{p}_1)$$

$$p_2^\mu = (p_2^+, \; p_2^-, \; \vec{p}_2)$$

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$$p_4^\mu = (-p_1^+ , -p_1^- , \vec{p}_4)$$

$$\vec{p}_2 \neq -\vec{p}_3$$

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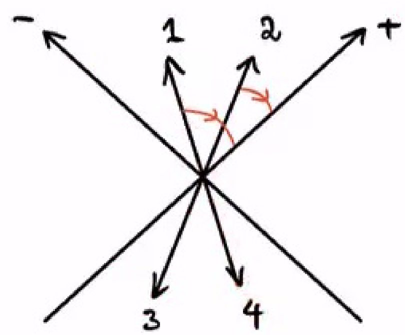
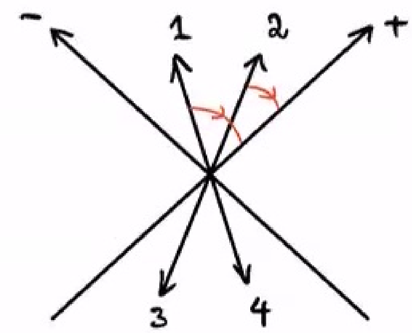
$$\vec{p}_2 \neq -\vec{p}_3$$

such that all particles are on-shell, $M_i^2 = p_i^+ p_i^- - \vec{p}_i^{\,2}$

and mom. cons. $\sum_i \vec{p}_i = 0$



Suppose that 2 & 3 are closer to the +ve side of the lightcone.



Suppose that 2 & 3 are closer to the +ve side of the lightcone,

$$\frac{p_1^+}{p_1^-} < \frac{p_2^+}{p_2^-}$$

This defines the starting point of step II

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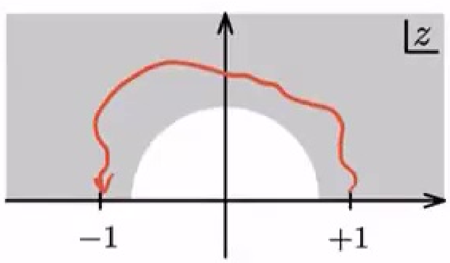
Now let's rotate the energies:

$$\hat{p}_2^\mu = \left(z\, p_2^+ ,\ \frac{1}{z} p_2^- ,\ \vec{p}_2 \right)$$

$$\hat{p}_3^\mu = \left(-z\, p_2^+ ,\ -\frac{1}{z} p_2^- ,\ \vec{p}_3 \right)$$

$$\hat{p}_i^{\,2} = \mu_i^2$$



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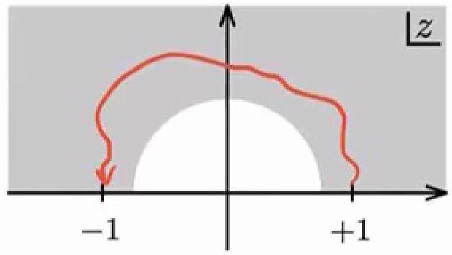

In terms of Mandelstam invariants

$$\mathrm{Im}\,\hat{s} = \mathrm{Im}\,(p_1 + \hat{p}_2)^2$$

$$= \underbrace{\mathrm{Im}\,z}_{>0} \underbrace{\left(p_2^+ p_1^- - \overbrace{\frac{1}{|z|^2}}^{<1} p_1^+ p_2^- \right)}_{>0} > 0 .$$

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$$= \underbrace{\operatorname{Im} z}_{>0} \underbrace{\left(p_2^+ p_1^- - \frac{1}{|z|^2} p_1^+ p_2^- \right)}_{>0} > 0.$$

$$\hat{t} = (\hat{p}_2 + \hat{p}_3)^2 = t \quad \text{fixed}$$

$$\Rightarrow \operatorname{Im} \hat{t} = 0$$

Hence in the imaginary directions it looks like highly energetic process, even though all the Mandelstam invariants remain finite

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We want to show Landau equations cannot have solutions along the path of deformation.

→ Linear LE:

$$\hat{q}_e^{\pm} = \underbrace{z^{\pm 1}\, p_2^{\pm}}\, f_e + \underbrace{p_1^{\pm}\, g_e}$$

28 / 32

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$$\hat{q}_e^{\pm} = \underbrace{z^{\pm 1}\, p_2^{\pm}}\; f_e + \underbrace{p_1^{\pm}\; g_e}$$

→ Quadratic LE:

$$\mathrm{Im}\left(\hat{q}_e^2 - m_e^2\right) = \mathrm{Im}\Big(\left(z\, p_2^{+} f_e + p_1^{+} g_e\right)\left(\tfrac{1}{z}\, p_2^{-} f_e + p_1^{-} g_e\right)\Big)$$

$$= \underbrace{\mathrm{Im}\, z}_{>0}\; \underbrace{f_e\, g_e}_{?}\; \underbrace{\left(p_2^{+} p_1^{-} - \frac{1}{|z|^2}\, p_1^{+} p_2^{-} \right)}_{>0}$$

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$p_2^\pm$ $p_1^\pm$

→ **Quadratic LE**:

$$\mathrm{Im}\left(\hat{q}_e^2 - m_e^2\right) = \mathrm{Im}\left(\left(z\, p_2^+ f_e + p_1^+ g_e\right)\left(\frac{1}{z}\, p_2^- f_e + p_1^- g_e\right)\right)$$

$$= \underbrace{\mathrm{Im}\, z}_{>0}\ \underbrace{f_e\, g_e}_{?}\ \underbrace{\left(p_2^+ p_1^- - \frac{1}{|z|^2}\, p_1^+ p_2^- \right)}_{>0}$$

If there was a solution, how would it look like?

$$\underline{f_e = 0} \quad \text{or} \quad \underline{g_e = 0} \quad \text{or} \quad \underline{f_e = g_e} = 0$$

$$\hat{q}_e^\pm \propto p_1^\pm \qquad \hat{q}_e^\pm \propto p_2^\pm \qquad q_e^\pm = 0$$

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>0 $\qquad$? $\qquad$ >0

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$\underline{f_e = 0}$ or $\underline{g_e = 0}$ or $\underline{f_e = g_e} = 0$

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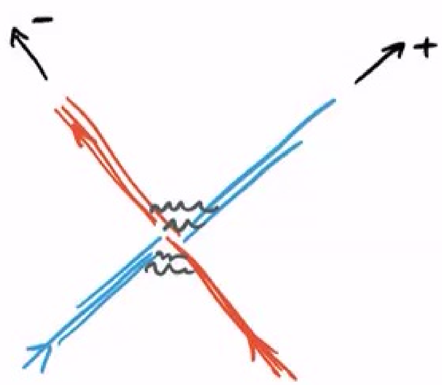

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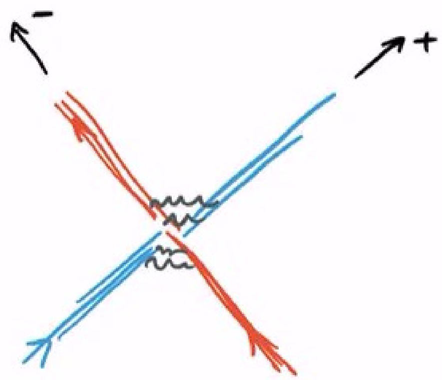

But we already know this cannot happen for planar diagrams because momenta in the $\pm$ directions can only flow in one direction along the perimeter:

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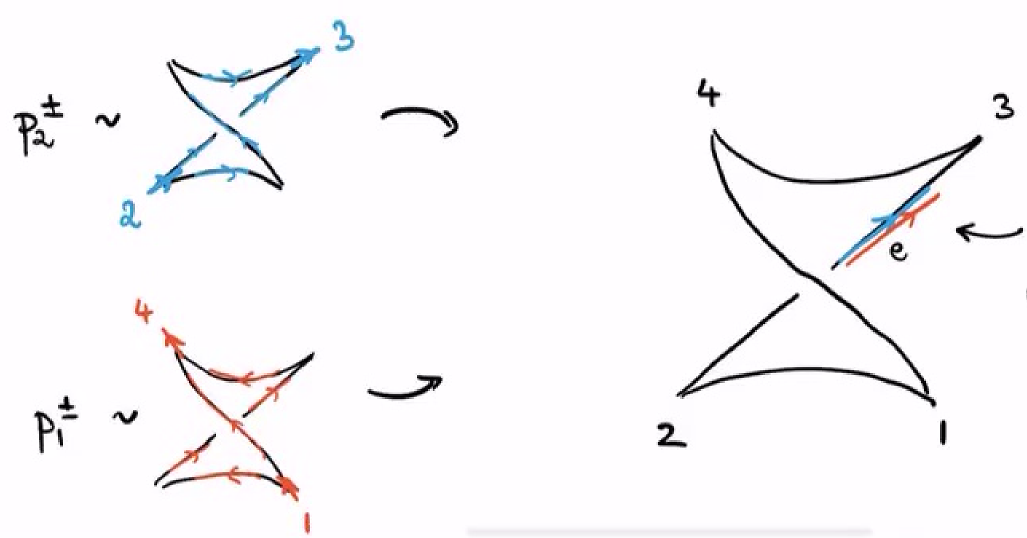
But we already know ==this cannot happen for planar diagrams== because momenta in the ± directions can only flow in one direction along the perimeter:



edges with non-zero $\hat{q}_e^\pm$ compon...

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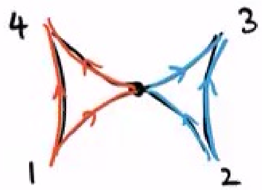
Therefore for the perimeter edges:

$$\mathrm{Im}\left(\hat{q}_e^2 - m_e^2\right) \propto \underbrace{f_e}_{\gtrless 0}\ \underbrace{g_e}_{\gtrless 0} \neq 0 \quad \text{definite sign}$$

and the amplitude is analytic.

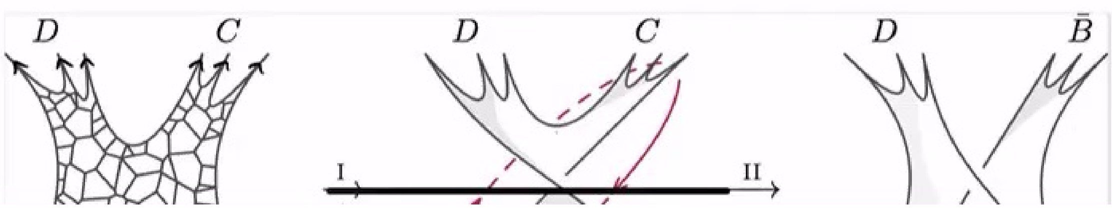
(The only exception are 1-vertex reducible diagrams

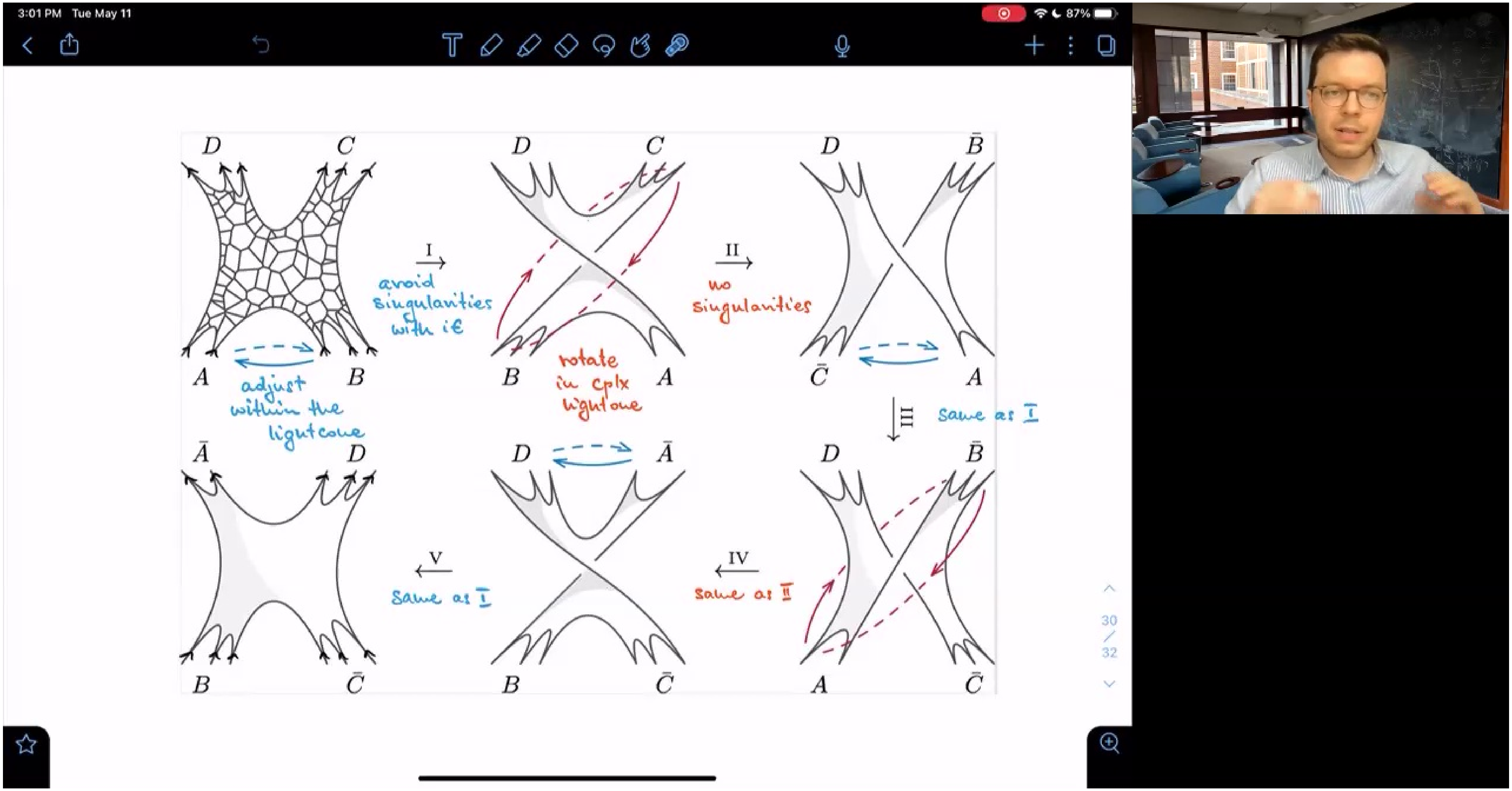


which are s-independent anyway)

Putting everything together

The path of analytic continuation for planar diagrams is as follows:





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87%
D C
A B
I
avoid singularities with iε
adjust within the lightcone
D C
B A
rotate in cplx lightcone
II
no singularities
D B̄
C̄ A
III
same as I
Ā D
B C̄
V
same as I
D Ā
B C̄
IV
same as II
D B̄
A C̄
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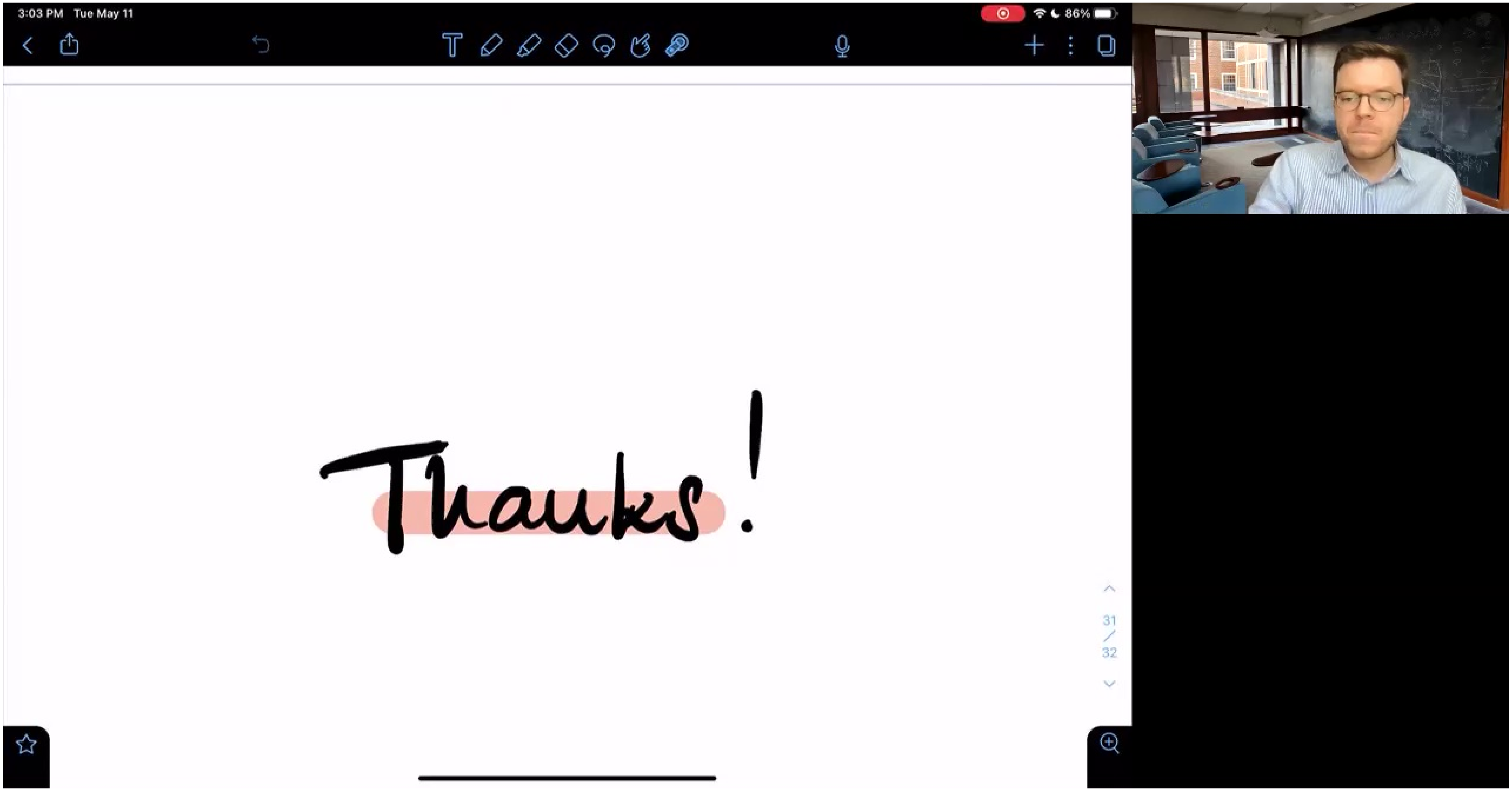
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Thanks!
31
32