Title: An Information-Theoretic Approach to Contextuality

Speakers: Iman Marvian

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Abstract: Classical probabilistic models of quantum systems are not only relevant for understanding the non-classical features of quantum mechanics, but they are also useful for determining the possible advantage of using quantum resources for information processing tasks.  A common feature of these models is the presence of inaccessible information, as captured by the concept of preparation contextuality:  There are ensembles of quantum states described by the same density operator, and hence operationally indistinguishable, and yet in any probabilistic (ontological) model, they should be described by distinct probability distributions.  In this talk, I discuss a method for quantifying this inaccessible information and present a family of lower bounds on this quantity in terms of experimentally measurable quantities. These bounds, which can also be interpreted as a new class of robust non-contextuality inequalities, are obtained based on a family of guessing games.  As an application of this result, I derive a noise threshold for the presence of contextually in a noisy system, in terms of the average gate fidelity of the noise channel.

# An Information-Theoretic Approach to Contextuality

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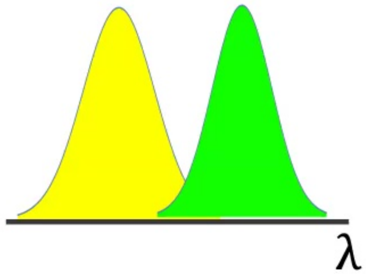
May 2021

IM, arXiv:2003.05984 [quant-ph]

**Overview:**




From an information-theoretic point of view, what is the most *efficient* ontological model (hidden-variable model), that describes a (possibly noisy) quantum-mechanical system?

What is the minimum amount of **inaccessible information** about the preparation process?

How this minimum inaccessible information depends on the noise in the system?

Our guiding principle for quantifying inaccessible information is the principle of Preparation Non-Contextuality. In fact, inaccessible information can also be interpreted as a **measure of preparation contextuality**.

IM, arXiv:2003.05984 [quant-ph]

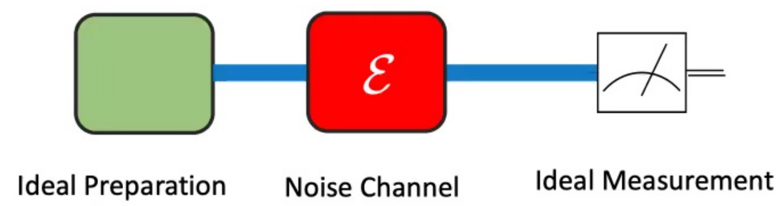
**Motivations:**

- Probing non-classical features of (noisy) Quantum-Mechanical Systems (Theory and Experiment)
- Understanding the limits of Quantum advantage for information-processing tasks
- Investigating the effects of noise on quantum systems

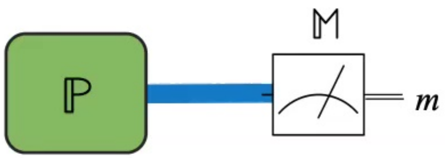
**Related Previous works:** Hardy (2004), Spekkens (2005), Montina (2007), Leifer (2105), Harrigan-Rudolph (2007)… ..

# Outline

- Preliminaries
    - Operational Theories, Ontological models, Preparation Non-Contextuality
- Inaccessible Information (Lower and upper bounds)
    - **Step 1:** A new proof of Preparation Contextuality of Quantum Mechanics
    - **Step 2:** A new family of robust non-contextuality inequalities from guessing games
        - Operational Max-Relative entropy
    - **Step 3:** Lower bounds on inaccessible information
- Noise Thresholds for Contextuality



# Operational Theories




The list of probabilities that can be directly measured in experiments.

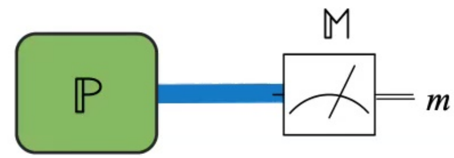
Probability of outcome $m$ in measurement $\mathbb{M}$ and preparation $\mathbb{P}$: $P(m|\mathbb{M},\mathbb{P})$

In Quantum Mechanics $P(m|\mathbb{M},\mathbb{P}) = \mathbf{Tr}(\rho F_m)$

Set of all preparations: $\mathcal{P}$

Set of all measurements: $\mathcal{M}$ Tomographically complete.

# Operational Theories

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In Quantum Mechanics $P(m|\mathbb{M},\mathbb{P}) = \mathrm{Tr}(\rho F_m)$

Set of all preparations: $\mathcal{P}$

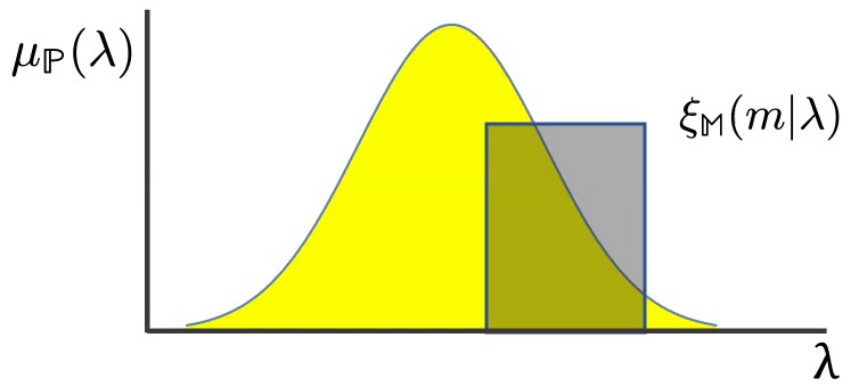
Set of all measurements: $\mathcal{M}$ Tomographically complete.

**Ensemble of preparations:** $\mathbb{P} = \{(p_i, \mathbb{P}_i)\}$

**Equivalent preparations:**

$$\{(p_i, \mathbb{P}_i)\} \sim \{(p'_j, \mathbb{P}'_j)\} \quad \Longleftrightarrow \quad \forall \mathbb{M} \in \mathcal{M}, \forall m : \sum_i p_i P(m|\mathbb{M},\mathbb{P}_i) = \sum_j p'_j P(m|\mathbb{M},\mathbb{P}'_j)$$

# Ontological Models




Ontic space $\Lambda = \{\lambda\}$

Ontic state $\lambda$

Preparation $\mathbb{P}$ $\mu_{\mathbb{P}} : \Lambda \mapsto \mathbb{R}$

$$\forall \lambda \in \Lambda : \mu_{\mathbb{P}}(\lambda) \geq 0 \text{ , and } \quad \sum_{\lambda \in \Lambda} \mu_{\mathbb{P}}(\lambda) = 1$$

Measurement $\mathbb{M}$ $\{\xi_{\mathbb{M}}(m|\lambda)\}$

$$\forall \lambda \in \Lambda : \xi_{\mathbb{M}}(m|\lambda) \geq 0 \, , \quad \sum_{m} \xi_{\mathbb{M}}(m|\lambda) = 1$$

Ensemble of Preparations $\mathbb{P} = \{(p_i, \mathbb{P}_i)\}$ $\mu_{\mathbb{P}} = \sum_i p_i \mu_i$

Convex Linearity

$$P(m|\mathbb{M}, \mathbb{P}) = \sum_{\lambda \in \Lambda} \xi_{\mathbb{M}}(m|\lambda) \times \mu_{\mathbb{P}}(\lambda)$$

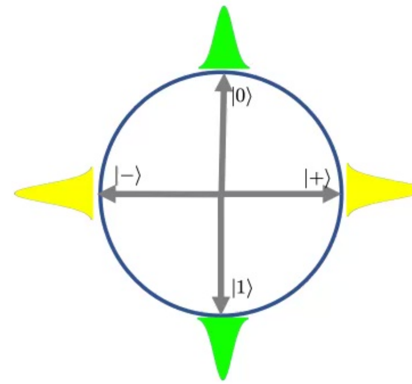
(??Spekkens, PRA 2005)

Experimental Data

| Operational Theory | Quantum Mechanics | Ontological Model |
|---|---|---|
| Preparation $\mathbb{P}_i$ | Density Operator $\rho_i$ | Probability Distribution $\mu_i$ |
| Measurement $\mathbb{M}$ | POVM $\{F_m\}$ | Response functions $\{\xi_{\mathbb{M}}(m\|\lambda)\}$ |

$$P(m|\mathbb{M}, \mathbb{P}_i) = \mathbf{Tr}(\rho_i F_m) = \sum_{\lambda \in \Lambda} \xi_{\mathbb{M}}(m|\lambda) \times \mu_i(\lambda)$$

## Preparation Contextuality




There are ensembles of quantum states described by the same density operator, and hence operationally indistinguishable, and yet in the ontological model, they are described by distinct probability distributions.

**Example:**

$$\frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}\left(|+\rangle\langle +| + |-\rangle\langle -|\right)$$

**Preparation Non-Contextuality (PNC):** Any two operationally indistinguishable scenarios should have the same descriptions in the model.

A model is preparation contextual, if there are distinct ensembles of states which yield the same average density operator, and yet in the model they are represented by different probability distributions.

$$\mathbb{P}_a \sim \mathbb{P}_b \iff \mu_a = \mu_b$$

**Theorem:** An ideal quantum-mechanical system cannot be described by a model satisfying PNC. (Spekkens 2005)

In this talk:

"Preparation Contextual" ⟷ "Inaccessible Information"

**Total Variational Distance** $$d_{\mathrm{TV}}(\mu_a, \mu_b) \equiv \frac{1}{2} \sum_{\lambda} |\mu_a(\lambda) - \mu_b(\lambda)|$$

**Preparation Non-Contextuality** $$\mathbb{P}_a \sim \mathbb{P}_b \iff d_{\mathrm{TV}}(\mu_a, \mu_b) = 0$$

Assuming measurements have finite number of outcomes

**Definition: Inaccessible information of an ontological model**

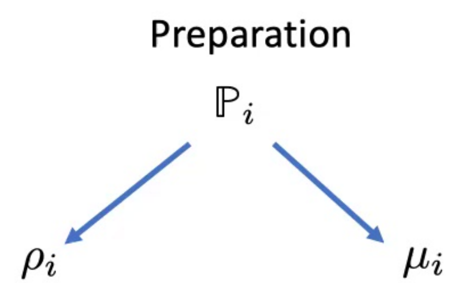
The largest distance between distributions associated to pairs of equivalent preparations

$$C_{\mathrm{prep}} \equiv \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{\mathrm{TV}}(\mu_a, \mu_b)$$

$$C_{\mathrm{prep}} \equiv \sup d_{\mathrm{TV}}\left( \sum_i p_i \mu_i \ , \ \sum_j p'_j \mu'_j \right)$$

where the supremum is taken over all pairs of $\{(p_i, \mathbb{P}_i)\}$ and $\{(p'_j, \mathbb{P}'_j)\}$

satisfying the constraint $$\sum_i p_i \rho_i = \sum_j p'_j \rho'_j$$



**Inaccessible information of an ontological model**

$$C_{\text{prep}} \equiv \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{\text{TV}}(\mu_a, \mu_b)$$

**Inaccessible information of an operational theory**

$$C_{\text{prep}}^{\text{min}} \equiv \inf_{\text{Models}} C_{\text{prep}} = \inf_{\text{Models}} \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{\text{TV}}(\mu_a, \mu_b)$$

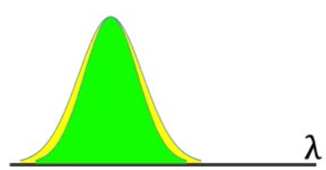
where the infimum is taken over all ontological models of the operational theory

$$\left(\Lambda \,,\ \{\mu_{\mathbb{P}} : \mathbb{P} \in \mathcal{P}\} \,, \{\xi_{\mathbb{M}} : \mathbb{M} \in \mathcal{M}\}\right)$$

**Interpretation:** In any ontological model that describes the operational theory, one can find two preparations that are indistinguishable under all possible measurements, and yet, a hypothetical observer who can observe the ontic state λ, can distinguish them with probability of success (at least) equal to

$$\frac{1 + C_{\text{prep}}^{\text{min}}}{2}$$

(assuming the two preparations are given with equal probability). Furthermore, there exists a model for the operational theory, such that the hypothetical observer cannot distinguish two equivalent preparations with probability larger than this number.



**Inaccessible information of an ontological model**

$$C_{\text{prep}} \equiv \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{\text{TV}}(\mu_a, \mu_b)$$

**Inaccessible information of an operational theory**

$$C_{\text{prep}}^{\min} \equiv \inf_{\text{Models}} C_{\text{prep}} = \inf_{\text{Models}} \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{\text{TV}}(\mu_a, \mu_b)$$

where the infimum is taken over all ontological models of the operational theory

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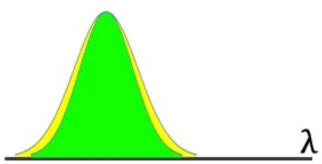
$$\frac{1 + C_{\text{prep}}^{\min}}{2}$$

(assuming the two preparations are given with equal probability). Furthermore, there exists a model for the operational theory, such that the hypothetical observer cannot distinguish two equivalent preparations with probability larger than this number.

Model satisfies Preparation Non-Contextuality → $C_{\text{prep}} = 0$



## How about inequivalent preparations?

$$d_{\text{trace}}(\rho_a, \rho_b) \equiv \frac{1}{2}\|\rho_a - \rho_b\|_1$$

$$d_{\text{TV}}(\mu_a, \mu_b) = 0 \quad \overset{C_{\text{prep}} = 0}{\Longleftrightarrow} \quad d_{\text{trace}}(\rho_a, \rho_b) = 0$$

**Trace distance in terms of equivalency relations:** There exists density operators $\rho'_a$ and $\rho'_b$ such that

$$\frac{1}{1 + d_{\text{trace}}(\rho_a, \rho_b)}\rho_a + \frac{d_{\text{trace}}(\rho_a, \rho_b)}{1 + d_{\text{trace}}(\rho_a, \rho_b)}\rho'_a = \frac{1}{1 + d_{\text{trace}}(\rho_a, \rho_b)}\rho_b + \frac{d_{\text{trace}}(\rho_a, \rho_b)}{1 + d_{\text{trace}}(\rho_a, \rho_b)}\rho'_b$$

$d_{\text{trace}}(\rho_a, \rho_b)$ is the minimum value of $r \geq 0$ such that there exists density operators $\rho'_a$ and $\rho'_b$ satisfying

$$\frac{1}{1+r}\rho_a + \frac{r}{1+r}\rho'_a = \frac{1}{1+r}\rho_b + \frac{r}{1+r}\rho'_b$$

or, equivalently,

$$\rho_a - \rho_b = r(\rho'_a - \rho'_b)$$

A generalization of trace distance to all operational theories and a geometric interpretation of trace distance.
IM, arXiv:2003.05984 [quant-ph]

**The gap between the operational and ontological distinguishabilities:**

Assuming $\rho'_a$ and $\rho'_b$ can be prepared in the operational theory

$$\boxed{d_{\text{TV}}(\mu_a, \mu_b) - d_{\text{trace}}(\rho_a, \rho_b) \leq C_{\text{prep}}[1 + d_{\text{trace}}(\rho_a, \rho_b)] \leq 2C_{\text{prep}}}$$

For an ideal Quantum-Mechanical System, Inaccessible information is determined by the dimension of the Hilbert space.

**Theorem:** For the operational theory corresponding to an ideal quantum system with finite-dimensional Hilbert space, the inaccessible information $C_{\text{prep}}^{\min} \equiv \inf_{\text{Models}} C_{\text{prep}} = \inf_{\text{Models}} \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{\text{TV}}(\mu_a, \mu_b)$ satisfies

$$0.07 \approx \frac{2-\sqrt{2}}{8} \leq C_{prep}^{min} < 1$$

Furthermore, in the case of a single qubit, $C_{prep}^{min} \leq 0.5$ .

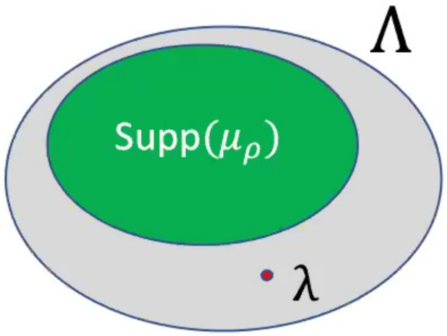
**Proofs:**

1) The upper bound $C_{\text{prep}}^{\min} < 1$ can be proven using an ontological model introduced by of Aaronson et al. and Lewis et. al in which non-orthogonal pure states are described by overlapping probability distributions.
2) The upper bound $C_{\text{prep}}^{\min} \leq 1/2$ in the qubit case can be shown using an ontological model introduced by Kochen and Specker (1975).
3) The lower bound $C_{prep}^{min} \geq \frac{2-\sqrt{2}}{8} \approx 0.07$ is established using a family of guessing games.

IM, arXiv:2003.05984 [quant-ph]

Let $\Lambda$ be the set of all ontic states that have non-zero probability for some quantum states. Assume $\Lambda$ is a finite set.

**Claim 1:** PNC implies that for any mixed state ρ the corresponding probability distribution should have full support in $\Lambda$.

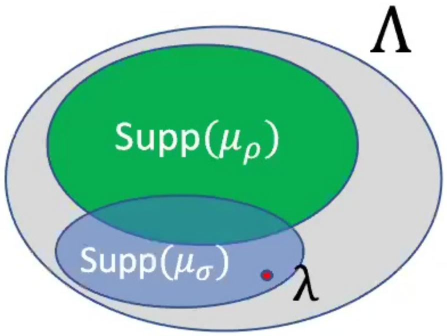


# A new proof of
# Preparation Contextuality of QM

Let $\Lambda$ be the set of all ontic states that have non-zero probability for some quantum states. Assume $\Lambda$ is a finite set.

**Claim 1:** PNC implies that for any mixed state ρ the corresponding probability distribution should have full support in $\Lambda$.

For any $\lambda \in \Lambda$ there is a density operator σ, such that $\lambda \in$ Supp($\mu_\sigma$) .
Since $\rho$ has full support, there exists $p > 0$ and density operator $\rho'$, such that




$$\rho = p\sigma + (1-p)\rho'$$

PNC: $$\mu_\rho(\lambda) = p\mu_\sigma(\lambda) + (1-p)\mu_{\rho'}(\lambda)$$

$$\mu_\sigma(\lambda) > 0 \implies \mu_\rho(\lambda) > 0$$

Let $\Lambda$ be the set of all ontic states that have non-zero probability for some quantum states. Assume $\Lambda$ is a finite set.

**Claim 1:** PNC implies that for any mixed state ρ the corresponding probability distribution should have full support in $\Lambda$.

For any $\lambda \in \Lambda$ there is a density operator σ, such that $\lambda \in \text{Supp}(\mu_\sigma)$ .
Since $\rho$ has full support, there exists $p > 0$ and density operator $\rho'$, such that

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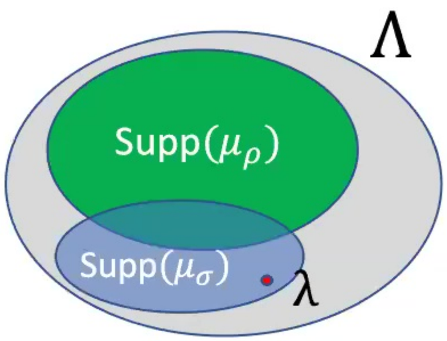



**Claim 2:** PNC implies that for any ontic state λ ∈ Λ, there is, at most, one pure state ψ, such that $\mu_\psi(\lambda) = 0$.

**Claim 3:** PNC implies that a qubit cannot have more than one pairs of perfectly distinguishable states, in contradiction with QM.

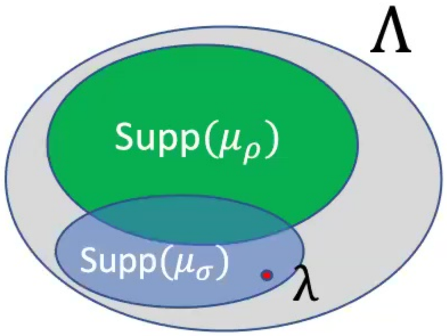
Consider two sets of orthonormal states $\{|\psi\rangle, |\psi^\perp\rangle\}$ and $\{|\phi\rangle, |\phi^\perp\rangle\}$
For any λ ∈ Λ

$\mu_\psi(\lambda)\mu_{\psi^\perp}(\lambda) > 0$ or $\mu_\phi(\lambda)\mu_{\phi^\perp}(\lambda) > 0$ → At most, one pair of perfectly distinguishable states.

Let $\Lambda$ be the set of all ontic states that have non-zero probability for some quantum states. Assume $\Lambda$ is a finite set.

**Claim 1:** PNC implies that for any mixed state ρ the corresponding probability distribution should have full support in Λ.

For any $\lambda \in \Lambda$ there is a density operator σ, such that $\lambda \in$ Supp$(\mu_\sigma)$ .
Since $\rho$ has full support, there exists $p > 0$ and density operator $\rho'$, such that

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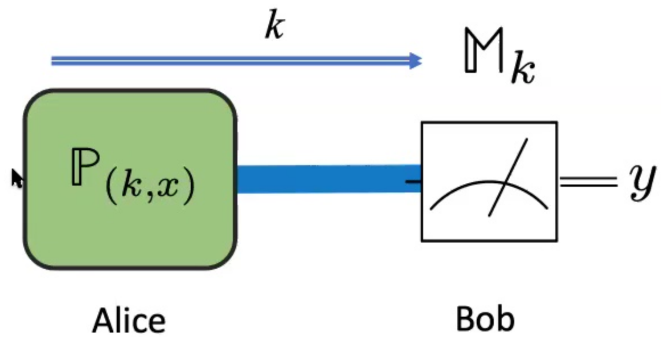
**Claim 2:** PNC implies that for any ontic state λ ∈ Λ, there is, at most, one pure state ψ, such that $\mu_\psi(\lambda) = 0$.

## A new family of non-contextuality inequalities from guessing games

# A guessing game

| | | Alphabet | Message |
|---|---|---|---|
| Preparation | $\mathbb{P}_{(k,x)} \in \mathcal{P}$ , | $k = 1, \cdots, n;$ | $x = 1, \cdots, d$ |
| Measurement | $\mathbb{M}_k \in \mathcal{M}$ , | $k = 1, \cdots, n$ , | |



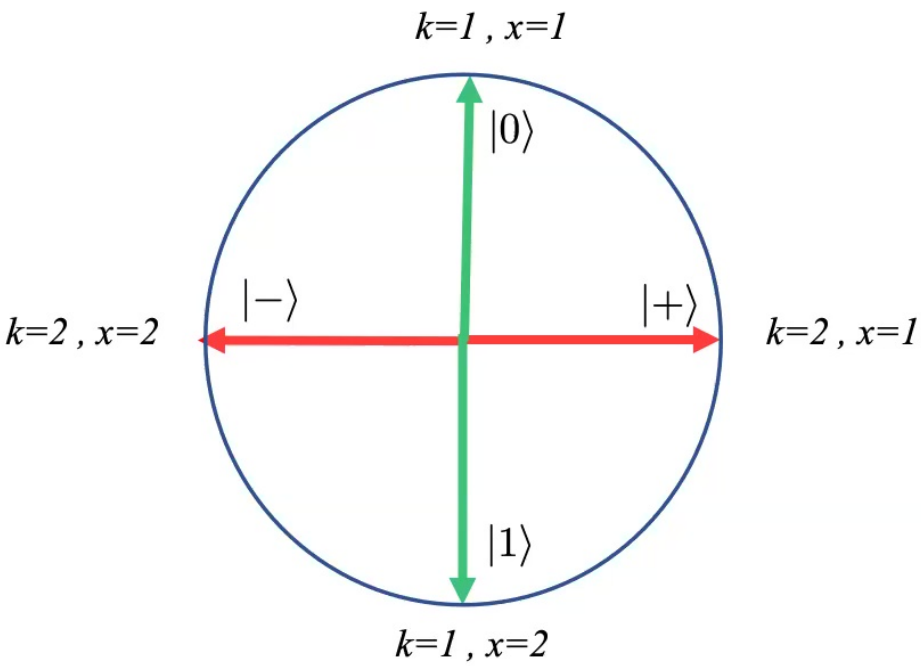

Alice chooses alphabet $k=1,\ldots, n$ and message $x=1,\ldots, d$ uniformly at random and independent of each other. She reveals $k$ to Bob and asks him to guess $x$. Bob performs a measurement and obtains outcome $y$. He wins if $y=x$, i.e., his guess coincides with Alice's choice, which happens with probability

$$P_{\text{guess}} \equiv \frac{1}{nd} \times \sum_{k=1}^{n} \sum_{x=1}^{d} P\left(y = x \middle| \mathbb{M}_k, \mathbb{P}_{(k,x)}\right)$$

## Example:

If Bob measures *Z* for *k*=1 and measures *X* for *k=2,* then he wins with probability 1.

$$P_{\text{guess}} \equiv \frac{1}{nd} \times \sum_{k=1}^{n} \sum_{x=1}^{d} P\big(y = x \big| \mathbb{M}_k, \mathbb{P}_{(k,x)}\big)$$



Consider an arbitrary ontological model for this operational theory. $\mathbb{P}_{(k,x)} \longrightarrow \mu_{(k,x)}$

$$P_{\text{guess}} \equiv \frac{1}{nd} \times \sum_{k=1}^{n} \sum_{x=1}^{d} P\big(y = x \big| \mathbb{M}_k, \mathbb{P}_{(k,x)}\big)$$

$$= \frac{1}{n} \sum_{k=1}^{n} \frac{1}{d} \sum_{x=1}^{d} \sum_{\lambda} \xi_{\mathbb{M}_k}(x|\lambda)\mu_{(k,x)}(\lambda) \leq \frac{1}{d} \sum_{\lambda} \frac{1}{n} \sum_{k=1}^{n} \max_x \mu_{(k,x)}(\lambda) = \frac{1}{d} \sum_{\lambda} \max_{\mathbf{x}} \mu_{\mathbf{x}}(\lambda)$$

$$\sum_{x=1}^{d} \xi_{\mathbb{M}}(x|\lambda) = 1 \qquad \mu_{(k,x)}(\lambda) \leq \max_x \mu_{(k,x)}(\lambda)$$

$$\mu_{\mathbf{x}} \equiv \frac{1}{n} \sum_{k} \mu_{(k,x_k)}$$

$$\mathbf{x} = x_1, \cdots, x_n \in \{1, \cdots, d\}^n$$

$$P_{\text{guess}} \leq \min \left\{ \frac{1}{d} \sum_{\lambda} \max_{\mathbf{x}} \mu_{\mathbf{x}}(\lambda), 1 - \frac{d-1}{d} \sum_{\lambda} \min_{\mathbf{x}} \mu_{\mathbf{x}}(\lambda) \right\}$$

Ensemble $\mathbb{P}_{\mathbf{x}} = \{(\frac{1}{n}, \mathbb{P}_{(k,x_k)}) : k = 1, \cdots, n\}$

$\mathbf{x} = x_1, \cdots, x_n \in \{1, \cdots, d\}^n$

$$\rho_{\mathbf{x}} = \frac{1}{n} \sum_{k=1}^{n} \rho_{(k,x_k)}$$

$$\mu_{\mathbf{x}} \equiv \frac{1}{n} \sum_{k} \mu_{(k,x_k)}$$

$$P_{\text{guess}} \leq \min\left\{\frac{1}{d}\sum_{\lambda}\max_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda), 1-\frac{d-1}{d}\sum_{\lambda}\min_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda)\right\}$$

Ensemble $\left\{(\frac{1}{n}, \mathbb{P}_{(k,x_k)}) : k = 1, \cdots, n\right\}$

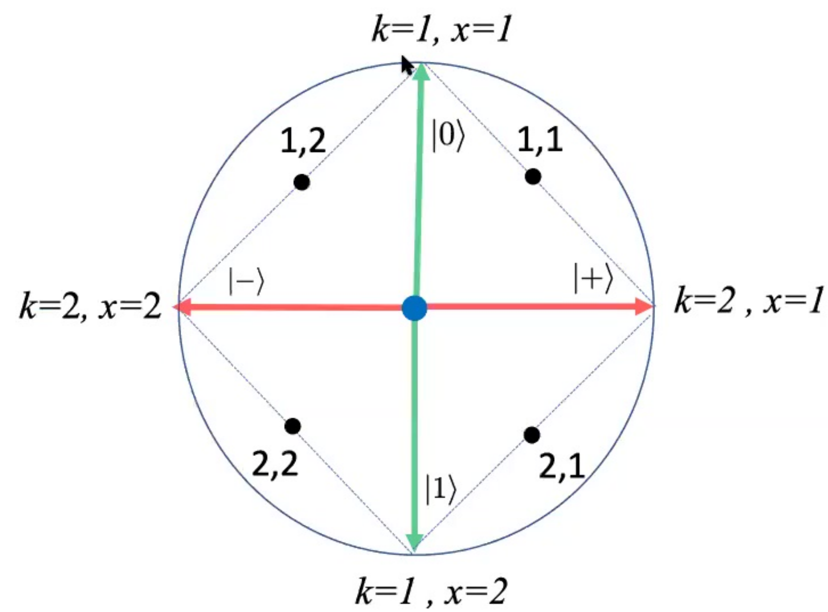
$$\rho_{\mathbf{x}} = \frac{1}{n}\sum_{k=1}^{n}\rho_{(k,x_k)}$$

$$\mu_{\mathbf{x}} \equiv \frac{1}{n}\sum_{k}\mu_{(k,x_k)}\,.$$

$$\mathbf{x} = x_1, \cdots, x_n \in \{1, \cdots, d\}^n$$

**Example:**

$$\rho_{x_1 x_2} = \frac{1}{2}\left(X^{x_1}|1\rangle\langle 1|X^{x_1} + Z^{x_2}|-\rangle\langle -|Z^{x_2}\right) \qquad x_1, x_2 \in \{1, 2\}$$



$$P_{\text{guess}} \leq \min \left\{ \frac{1}{d} \sum_{\lambda} \max_{\mathbf{x}} \mu_{\mathbf{x}}(\lambda), 1 - \frac{d-1}{d} \sum_{\lambda} \min_{\mathbf{x}} \mu_{\mathbf{x}}(\lambda) \right\}$$

Ensemble $\left\{ \left( \frac{1}{n}, \mathbb{P}_{(k,x_k)} \right) : k = 1, \cdots, n \right\}$

$$\rho_{\mathbf{x}} = \frac{1}{n} \sum_{k=1}^{n} \rho_{(k,x_k)}$$

$$\mu_{\mathbf{x}} \equiv \frac{1}{n} \sum_{k} \mu_{(k,x_k)} .$$

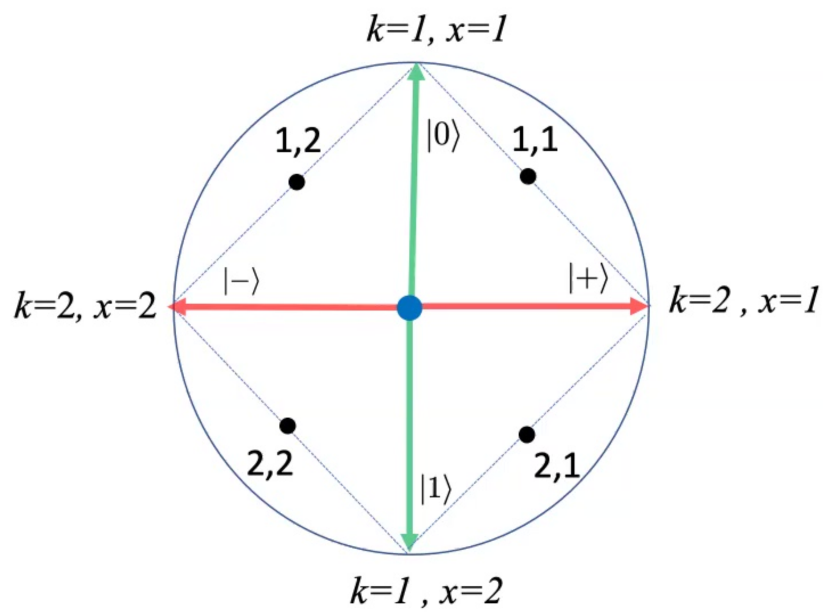
$$\mathbf{x} = x_1, \cdots, x_n \in \{1, \cdots, d\}^n$$

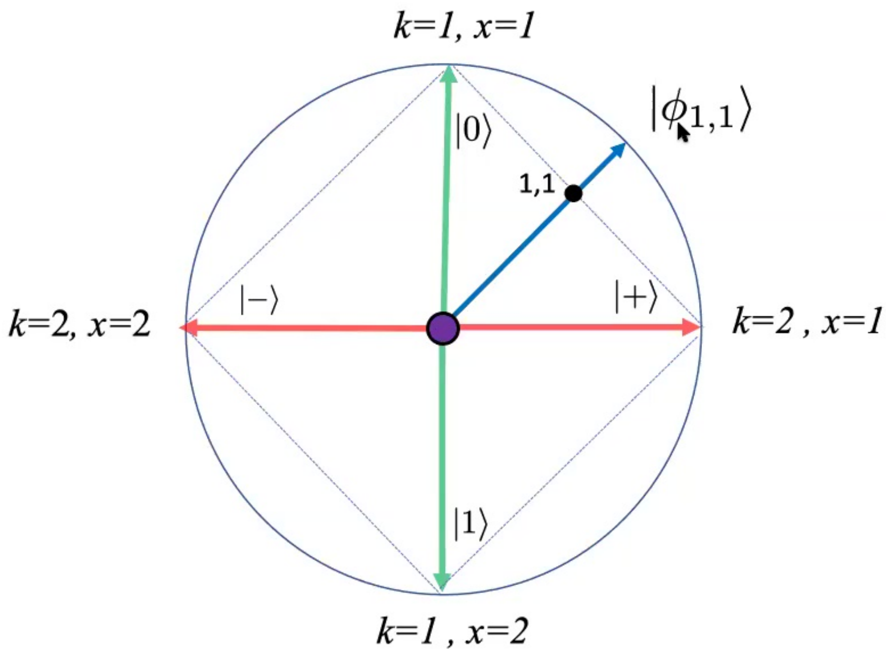
**Example:**

$$\rho_{x_1 x_2} = \frac{1}{2} \left( X^{x_1} |1\rangle\langle 1| X^{x_1} + Z^{x_2} |-\rangle\langle -| Z^{x_2} \right) \qquad x_1, x_2 \in \{1, 2\}$$

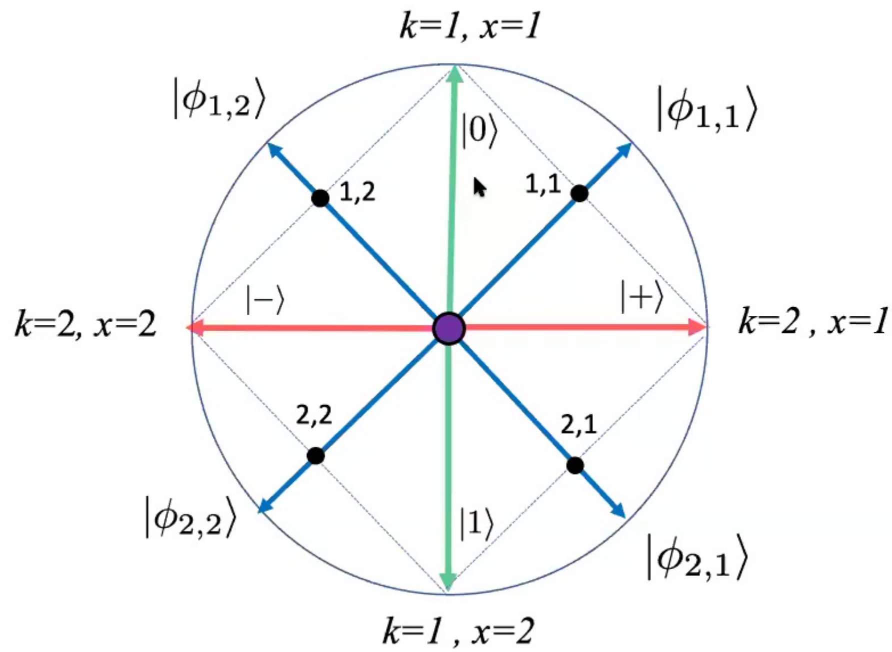
$$P_{\text{guess}} \leq \frac{1}{2} \sum_{\lambda} \max \left\{ \mu_{1,1}(\lambda), \mu_{1,2}(\lambda), \mu_{2,1}(\lambda), \mu_{2,2}(\lambda) \right\}$$

$$P_{\text{guess}} \leq 1 - \frac{1}{2} \sum_{\lambda} \min \left\{ \mu_{1,1}(\lambda), \mu_{1,2}(\lambda), \mu_{2,1}(\lambda), \mu_{2,2}(\lambda) \right\}$$






$$\rho_{1,1} = \frac{1}{2}\big(|0\rangle\langle 0| + |+\rangle\langle +|\big) = (1 - \frac{\sqrt{2}}{2})\frac{I}{2} + \frac{\sqrt{2}}{2}|\phi_{1,1}\rangle\langle\phi_{1,1}|$$

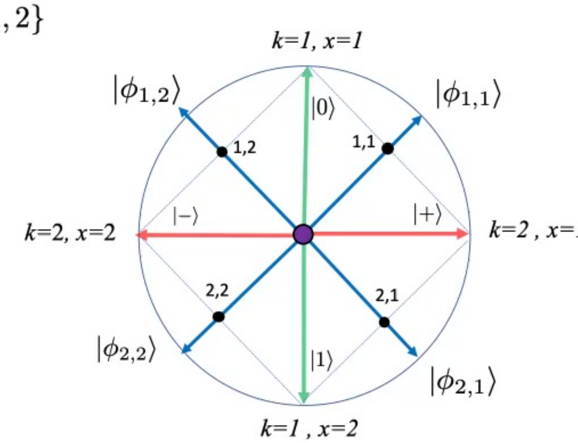



$$\rho_{1,1} = \frac{1}{2}\big(|0\rangle\langle 0| + |+\rangle\langle +|\big) = (1 - \frac{\sqrt{2}}{2})\frac{I}{2} + \frac{\sqrt{2}}{2}|\phi_{1,1}\rangle\langle\phi_{1,1}| \quad \xrightarrow{\text{PNC}} \quad \mu_{1,1} = (1 - \frac{\sqrt{2}}{2})\nu + \frac{\sqrt{2}}{2}\mu'_{1,1}$$

$$\rho_{x_1 x_2} = \frac{1}{2}\big(X^{x_1}|1\rangle\langle 1|X^{x_1} + Z^{x_2}|-\rangle\langle -|Z^{x_2}\big) = (1 - \frac{\sqrt{2}}{2})\frac{I}{2} + \frac{\sqrt{2}}{2}|\phi_{x_1,x_2}\rangle\langle\phi_{x_1,x_2}| \quad \xrightarrow{\text{PNC}} \quad \mu_{x_1,x_2} = (1 - \frac{\sqrt{2}}{2})\nu + \frac{\sqrt{2}}{2}\mu'_{x_1,x_2}$$

$$x_1, x_2 \in \{1, 2\}$$

$$\rho_{x_1x_2} = \frac{1}{2}(X^{x_1}|1\rangle\langle 1|X^{x_1} + Z^{x_2}|-\rangle\langle -|Z^{x_2}) = (1-\frac{\sqrt{2}}{2})\frac{I}{2} + \frac{\sqrt{2}}{2}|\phi_{x_1,x_2}\rangle\langle\phi_{x_1,x_2}| \qquad x_1, x_2 \in \{1,2\}$$




PNC →

$$\mu_{x_1,x_2} = (1-\frac{\sqrt{2}}{2})\nu + \frac{\sqrt{2}}{2}\mu'_{x_1,x_2}$$

$$\mu_{x_1,x_2}(\lambda) \geq (1-\frac{\sqrt{2}}{2})\nu(\lambda)$$

$$P_{\text{guess}} \leq \min\left\{\frac{1}{d}\sum_{\lambda}\max_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda), 1 - \frac{d-1}{d}\sum_{\lambda}\min_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda)\right\}$$

$$P_{\text{guess}} \leq 1 - \frac{1}{2}\sum_{\lambda}\min_{x_1,x_2}\mu_{x_1,x_2} \leq 1 - \frac{1}{2}\sum_{\lambda}(1-\frac{\sqrt{2}}{2})\nu(\lambda) = \frac{2+\sqrt{2}}{4}$$

PNC

$$P_{\text{guess}} = 1$$

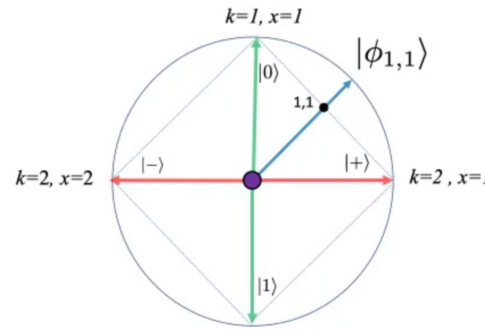
QM

The inequality imposed by PNC is violated in QM.

$$P_{\text{guess}} \leq \min\left\{\frac{1}{d}\sum_{\lambda}\max_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda), 1 - \frac{d-1}{d}\sum_{\lambda}\min_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda)\right\}$$

$$P_{\text{guess}} \leq 1 - \frac{1}{2}\sum_{\lambda}\min_{x_1,x_2}\mu_{x_1,x_2} \leq 1 - \frac{1}{2}\sum_{\lambda}(1 - \frac{\sqrt{2}}{2})\nu(\lambda) = \frac{2+\sqrt{2}}{4}$$

PNC

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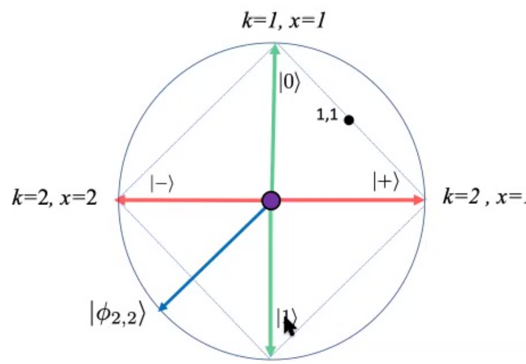
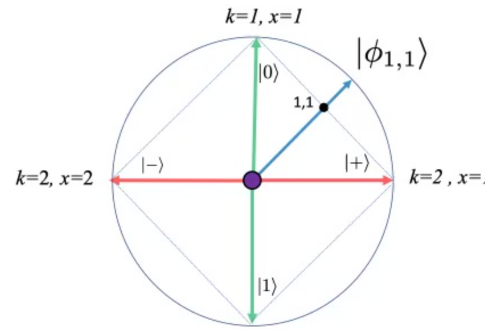
$$P_{\text{guess}} \leq \min\left\{\frac{1}{d}\sum_{\lambda}\max_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda), 1 - \frac{d-1}{d}\sum_{\lambda}\min_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda)\right\}$$

$$\frac{I}{2} = \frac{2}{2+\sqrt{2}}\rho_{1,1} + \frac{\sqrt{2}}{2+\sqrt{2}}|\phi_{2,2}\rangle\langle\phi_{2,2}|$$



$$P_{\text{guess}} \leq \min\left\{\frac{1}{d}\sum_{\lambda}\max_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda), 1 - \frac{d-1}{d}\sum_{\lambda}\min_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda)\right\}$$

$$P_{\text{guess}} \leq 1 - \frac{1}{2}\sum_{\lambda}\min_{x_1,x_2}\mu_{x_1,x_2} \leq 1 - \frac{1}{2}\sum_{\lambda}(1 - \frac{\sqrt{2}}{2})\nu(\lambda) = \frac{2+\sqrt{2}}{4}$$ PNC

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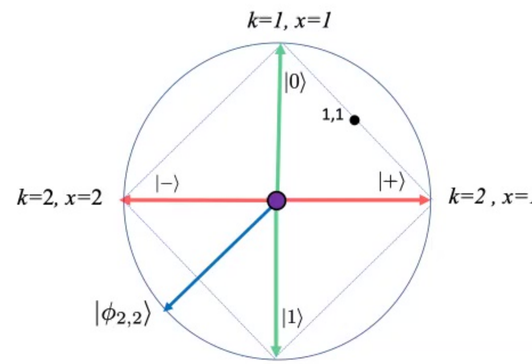
$$P_{\text{guess}} \leq \min\left\{\frac{1}{d}\sum_{\lambda}\max_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda), 1 - \frac{d-1}{d}\sum_{\lambda}\min_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda)\right\}$$

$$\frac{I}{2} = \frac{2}{2+\sqrt{2}}\rho_{1,1} + \frac{\sqrt{2}}{2+\sqrt{2}}|\phi_{2,2}\rangle\langle\phi_{2,2}|$$

$$\nu = \frac{2}{2+\sqrt{2}}\mu_{1,1} + \frac{\sqrt{2}}{2+\sqrt{2}}\mu'_{2,2}$$ PNC




$$\mu_{1,1}(\lambda) \leq \frac{2+\sqrt{2}}{2}\nu(\lambda)$$

$$P_{\text{guess}} \leq \frac{1}{2}\sum_{\lambda}\max_{x_1,x_2}\mu_{x_1,x_2}(\lambda) \leq \frac{2+\sqrt{2}}{4}$$ PNC

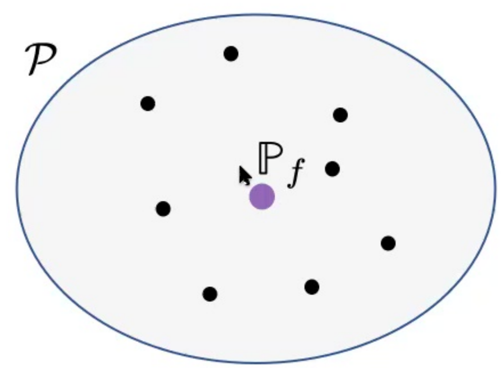
# Generalization: Minimizing the maximum "distance"

$$P_{\text{guess}} \leq \min\left\{\frac{1}{d}\sum_{\lambda}\max_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda), 1-\frac{d-1}{d}\sum_{\lambda}\min_{\mathbf{x}}\mu_{\mathbf{x}}(\lambda)\right\}$$

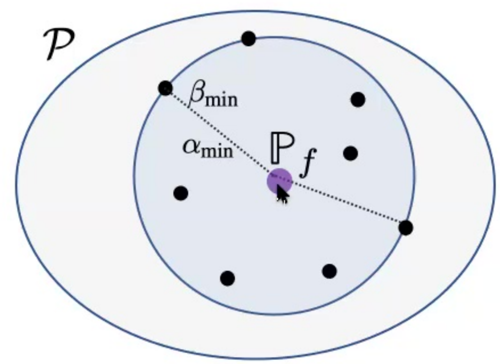
We find the preparation with the maximum "overlap" with preparations

$$\mathbb{P}_{\mathbf{x}} = \{(\frac{1}{n}, \mathbb{P}_{(k,x_k)}) : k = 1, \cdots, n\} \qquad \mathbf{x} = x_1, \cdots, x_n \in \{1, \cdots, d\}^n$$

In other words, we find the preparation that has the minimum of the maximum "distance" with these preparations.



**Theorem:** If PNC holds, i.e., if $C_{\text{prep}}^{\min} = 0$, then




$$P_{\text{guess}} \equiv \frac{1}{nd} \times \sum_{k=1}^{n} \sum_{x=1}^{d} P\big(y = x \big| \mathbb{M}_k, \mathbb{P}_{(k,x)}\big) \leq \min\{\frac{\alpha_{\min}}{d}, 1 - \frac{d-1}{d}\beta_{\min}^{-1}\}$$

where

$$\alpha_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_{\mathbf{x}} \| \mathbb{P}_f)} \,,$$

$$\beta_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_f \| \mathbb{P}_{\mathbf{x}})} \,.$$

Preparation radius

and

$$\mathbb{P}_{\mathbf{x}} = \{(\frac{1}{n}, \mathbb{P}_{(k,x_k)}) : k = 1, \cdots, n\} \qquad \mathbf{x} = x_1, \cdots, x_n \in \{1, \cdots, d\}^n$$

---

**Qubit Example:** $d$=2 (Binary guessing games)

$$\rho_{(k,x)} = \frac{I + \vec{n}_{(k,x)} \cdot \vec{\sigma}}{2} \qquad \frac{1}{2n} \sum_k \sum_x \rho_{(k,x)} = \frac{I}{2}$$

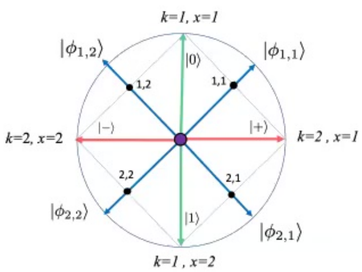
$$\vec{n}_{\mathbf{x}} = \frac{1}{n} \sum_{k=1}^{n} \vec{n}_{(k,x_k)} \quad \mathbf{x} \in \{1,2\}^n$$

$$\alpha_{\min} = 1 + \max_{\mathbf{x}} \|n_{\mathbf{x}}\| \qquad \beta_{\min} = \frac{1}{1 - \max_{\mathbf{x}} \|\vec{n}_{\mathbf{x}}\|}$$

$$P_{\text{guess}} \leq \frac{1 + \max_{\mathbf{x}} \|n_{\mathbf{x}}\|}{2} = \frac{\alpha_{\min}}{2} = 1 - \frac{1}{2}\beta_{\min}^{-1}$$

### PNC inequality with 8 preparations

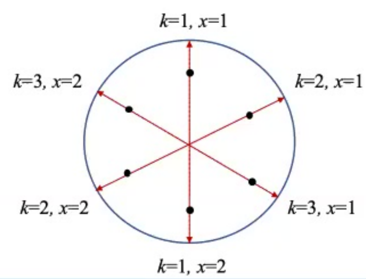
$$P_{\text{guess}} \leq \frac{1+\sqrt{2}/2}{2} = \frac{2+\sqrt{2}}{4} \approx 0.85$$




### PNC inequality with 6 preparations

$$P_{\text{guess}} \leq \frac{1+2/3}{2} = \frac{5}{6} \approx 0.83$$

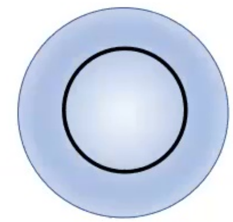
This bound was previously found in Mazurek et al. NCOMM 2016 (Assuming both PNC and MNC).




### PNC inequality with infinite states

Qubit: $P_{\text{guess}} \leq \frac{3}{4} = 0.75$

Qudit $\mathbb{C}^D$: $P_{\text{guess}} \leq \frac{1}{D}(1 + \frac{1}{2} + \cdots + \frac{1}{D})$



**Qubit Example:** $d$=2 (Binary guessing games)

$$\rho_{(k,x)} = \frac{I + \vec{n}_{(k,x)} \cdot \vec{\sigma}}{2} \qquad \frac{1}{2n}\sum_k \sum_x \rho_{(k,x)} = \frac{I}{2}$$

$$\vec{n}_{\mathbf{x}} = \frac{1}{n}\sum_{k=1}^{n} \vec{n}_{(k,x_k)} \quad \mathbf{x} \in \{1,2\}^n \qquad \alpha_{\text{min}} = 1 + \max_{\mathbf{x}} \|n_{\mathbf{x}}\| \qquad \beta_{\text{min}} = \frac{1}{1 - \max_{\mathbf{x}} \|\vec{n}_{\mathbf{x}}\|}$$

$$P_{\text{guess}} \leq \frac{1 + \max_{\mathbf{x}} \|n_{\mathbf{x}}\|}{2} = \frac{\alpha_{\text{min}}}{2} = 1 - \frac{1}{2}\beta_{\text{min}}^{-1}$$

**Theorem:** $P_{\text{guess}} \equiv \frac{1}{nd} \times \sum_{k=1}^{n} \sum_{x=1}^{d} P\big(y = x \big| \mathbb{M}_k, \mathbb{P}_{(k,x)}\big)$ satisfies

$$P_{\text{guess}} \leq \frac{\alpha_{\text{min}}}{d}(1 + C_{\text{prep}}^{\text{min}} \times d^n)\ ,$$

$$P_{\text{guess}} \leq 1 - \frac{d-1}{d}\beta_{\text{min}}^{-1} + (d-1)d^{n-1} \times C_{\text{prep}}^{\text{min}}$$

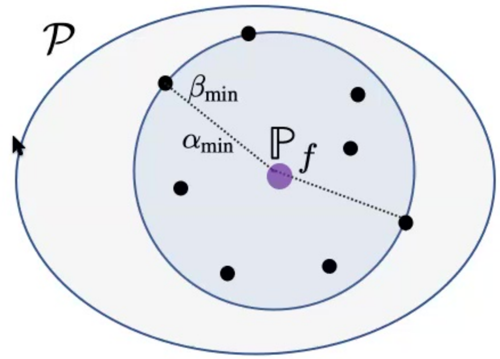
where $C_{prep}^{min} \equiv \inf_{Models} \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{TV}(\mu_a, \mu_b)$

and $\alpha_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_{\mathbf{x}} || \mathbb{P}_f)}\ ,$

$\beta_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_f || \mathbb{P}_{\mathbf{x}})}\ .$

Preparation radius

$\mathbb{P}_{\mathbf{x}} = \{(\frac{1}{n}, \mathbb{P}_{(k,x_k)}) : k = 1, \cdots, n\}$ $\quad$ $\mathbf{x} = x_1, \cdots, x_n \in \{1, \cdots, d\}^n$



**Theorem:** $P_{\text{guess}} \equiv \frac{1}{nd} \times \sum_{k=1}^{n} \sum_{x=1}^{d} P\big(y = x \big| \mathbb{M}_k, \mathbb{P}_{(k,x)}\big)$ satisfies

$$P_{\text{guess}} \leq \frac{\alpha_{\min}}{d}(1 + C_{\text{prep}}^{\min} \times d^n)\ ,$$
$$P_{\text{guess}} \leq 1 - \frac{d-1}{d}\beta_{\min}^{-1} + (d-1)d^{n-1} \times C_{\text{prep}}^{\min}$$

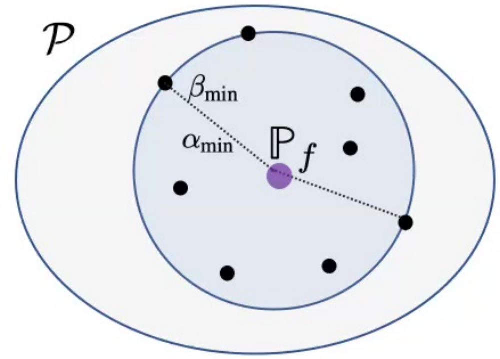
where $C_{prep}^{min} \equiv \inf_{Models} \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{TV}(\mu_a, \mu_b)$

and $\alpha_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_{\mathbf{x}} || \mathbb{P}_f)}\ ,$

$\beta_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_f || \mathbb{P}_{\mathbf{x}})}\ .$

Preparation radius

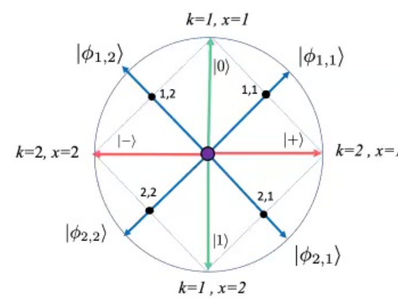
$\mathbb{P}_{\mathbf{x}} = \{(\frac{1}{n}, \mathbb{P}_{(k,x_k)}) : k = 1, \cdots, n\}$ $\qquad \mathbf{x} = x_1, \cdots, x_n \in \{1, \cdots, d\}^n$



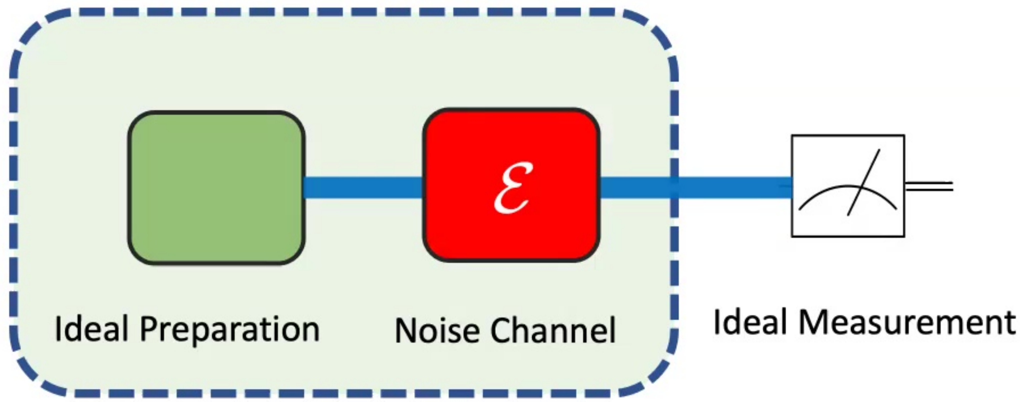

$$C_{\text{prep}}^{\min} \geq \frac{1}{d^n}\left[\frac{d \times P_{\text{guess}}}{\alpha_{\min}} - 1\right] = \frac{1}{d^n}\left[\frac{P_{\text{guess}}}{P_{\text{guess}}^{\text{PNC}}} - 1\right]$$

$$C_{\text{prep}}^{\min} \geq \frac{P_{\text{guess}} - (1 - \frac{d-1}{d}\beta_{\min}^{-1})}{(d-1)d^{n-1}} = \frac{P_{\text{guess}} - P_{\text{guess}}^{\text{PNC}}}{(d-1)d^{n-1}}$$

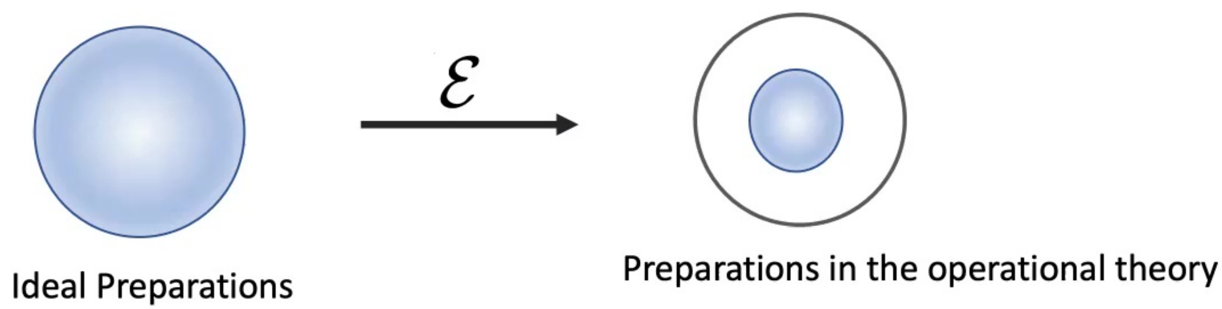
$\longrightarrow$ $C_{prep}^{min} \geq \frac{2-\sqrt{2}}{8} \approx 0.07$



# Noise Thresholds for Contextuality




Preparation in the operational theory



$$F(\mathcal{E}) \equiv \int d\eta \, \langle \eta | \mathcal{E}(|\eta\rangle\langle\eta|) |\eta\rangle$$ Average Gate Fidelity

Average gate fidelity can be interpreted as the probability of success in a guessing game, where the alphabet (basis) is chosen according to Haar measure.

$$F(\mathcal{E}) = P_{\text{guess}} \equiv \frac{1}{nd} \times \sum_{k=1}^{n} \sum_{x=1}^{d} P\big(y = x \big| \mathbb{M}_k, \mathbb{P}_{(k,x)}\big) \overset{\text{PNC}}{\leq} \min\{\frac{\alpha_{\min}}{d}, 1 - \frac{d-1}{d}\beta_{\min}^{-1}\}$$

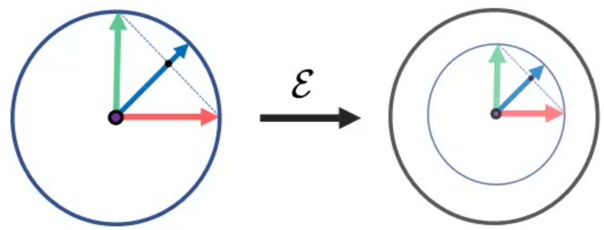
$$\alpha_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_{\mathbf{x}} \| \mathbb{P}_f)} ,$$

$$\beta_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_f \| \mathbb{P}_{\mathbf{x}})} .$$

If the noise channel is an invertible function, then it does not change the equivalency relations of the operational theory.

$$\rho_{1,1} = \frac{1}{2}\big(|0\rangle\langle 0| + |+\rangle\langle +|\big) = (1 - \frac{\sqrt{2}}{2})\frac{I}{2} + \frac{\sqrt{2}}{2}|\phi_{1,1}\rangle\langle\phi_{1,1}|$$

$$\mathcal{E}(\rho_{1,1}) = \frac{1}{2}(\mathcal{E}(|0\rangle\langle 0|) + \mathcal{E}(|1\rangle\langle 1|)) = (1 - \frac{\sqrt{2}}{2})\mathcal{E}(\frac{I}{2}) + \frac{\sqrt{2}}{2}\mathcal{E}(|\phi_{1,1}\rangle\langle\phi_{1,1}|)$$




Ideal Preparations Noisy Preparations

$$\mathbb{D}_{\max}(\mathbb{P}_a, \mathbb{P}_b) = \mathbb{D}_{\max}(\mathbb{P}'_a, \mathbb{P}'_b)$$

$$D_{\max}(\rho_a, \rho_b) > D_{\max}(\rho'_a, \rho'_b)$$

**Theorem:** $P_{\text{guess}} \equiv \frac{1}{nd} \times \sum_{k=1}^{n} \sum_{x=1}^{d} P\big(y = x \big| \mathbb{M}_k, \mathbb{P}_{(k,x)}\big)$ satisfies

$$P_{\text{guess}} \leq \frac{\alpha_{\text{min}}}{d}(1 + C_{\text{prep}}^{\text{min}} \times d^n) \,,$$

$$P_{\text{guess}} \leq 1 - \frac{d-1}{d}\beta_{\text{min}}^{-1} + (d-1)d^{n-1} \times C_{\text{prep}}^{\text{min}}$$

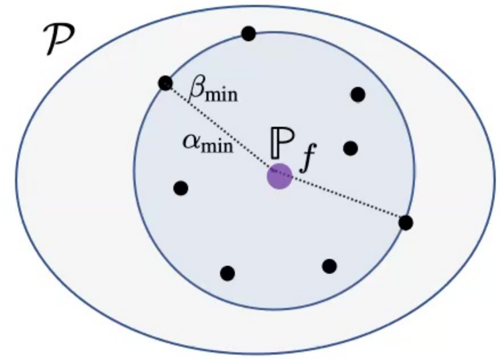
where $C_{prep}^{min} \equiv \inf_{Models} \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{TV}(\mu_a, \mu_b)$

and $\alpha_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_{\mathbf{x}} \| \mathbb{P}_f)} \,,$

$\beta_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_f \| \mathbb{P}_{\mathbf{x}})} \,.$

Preparation radius

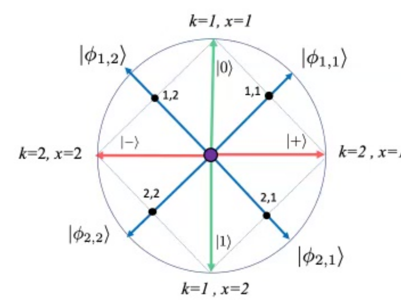
$\mathbb{P}_{\mathbf{x}} = \{(\frac{1}{n}, \mathbb{P}_{(k,x_k)}) : k = 1, \cdots, n\}$ $\quad$ $\mathbf{x} = x_1, \cdots, x_n \in \{1, \cdots, d\}^n$

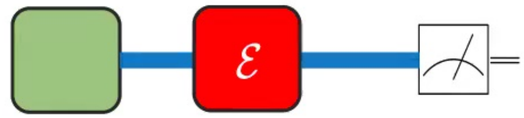



$$C_{\text{prep}}^{\text{min}} \geq \frac{1}{d^n}\left[\frac{d \times P_{\text{guess}}}{\alpha_{\text{min}}} - 1\right] = \frac{1}{d^n}\left[\frac{P_{\text{guess}}}{P_{\text{guess}}^{\text{PNC}}} - 1\right]$$

$$C_{\text{prep}}^{\text{min}} \geq \frac{P_{\text{guess}} - (1 - \frac{d-1}{d}\beta_{\text{min}}^{-1})}{(d-1)d^{n-1}} = \frac{P_{\text{guess}} - P_{\text{guess}}^{\text{PNC}}}{(d-1)d^{n-1}}$$

$$\longrightarrow \quad C_{prep}^{min} \geq \frac{2-\sqrt{2}}{8} \approx 0.07$$





**Theorem:** For a system with Hilbert space of dimension $D$

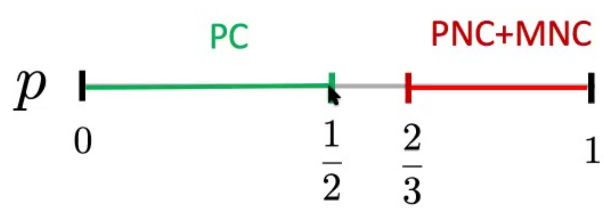
$$F(\mathcal{E}) \equiv \int d\eta \, \langle \eta | \mathcal{E}(|\eta\rangle\langle\eta|) |\eta\rangle > \frac{1}{D}(1 + \frac{1}{2} + \cdots + \frac{1}{D}) \implies$$ The operational theory is preparation contextual.

**Example:** Depolarizing channel $\mathcal{E}(\rho) = (1-p)\rho + p\frac{I}{D}$

$$p < \frac{D - [1 + \cdots + D^{-1}]}{D-1} \implies \text{Not PNC}$$

$$p \geq \frac{D}{D+1} \iff \text{PNC + MNC} \iff \text{Entanglement-Breaking}$$

The bound is **tight** in the qubit case.

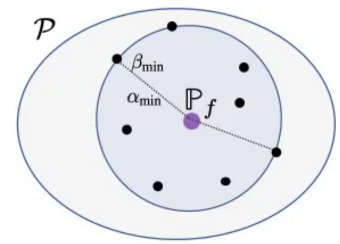


# Summary

## Non-contextuality inequalities based on guessing games

$$P_{\text{guess}} \equiv \frac{1}{nd} \times \sum_{k=1}^{n} \sum_{x=1}^{d} P\big(y=x \big| \mathbb{M}_k, \mathbb{P}_{(k,x)}\big) \leq \min\{\frac{\alpha_{\min}}{d}, 1 - \frac{d-1}{d}\beta_{\min}^{-1}\}$$

$$\alpha_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_{\mathbf{x}} \| \mathbb{P}_f)},$$
$$\beta_{min} \equiv \inf_{\mathbb{P}_f \in \mathcal{P}} \max_{\mathbf{x}} 2^{\mathbb{D}_{max}(\mathbb{P}_f \| \mathbb{P}_{\mathbf{x}})}.$$




## Inaccessible information

$$C_{prep}^{min} \equiv \inf_{Models} \sup_{\mathbb{P}_a \sim \mathbb{P}_b} d_{TV}(\mu_a, \mu_b)$$

$$0.07 \approx \frac{2-\sqrt{2}}{8} \leq C_{prep}^{min} < 1$$

$$d_{\text{TV}}(\mu_a, \mu_b) - d_{\text{trace}}(\rho_a, \rho_b) \leq C_{\text{prep}}[1 + d_{\text{trace}}(\rho_a, \rho_b)] \leq 2C_{\text{prep}}$$

$$C_{\text{prep}}^{\text{min}} \geq \frac{1}{d^n}\left[\frac{d \times P_{\text{guess}}}{\alpha_{\text{min}}} - 1\right] = \frac{1}{d^n}\left[\frac{P_{\text{guess}}}{P_{\text{guess}}^{\text{PNC}}} - 1\right]$$

$$C_{\text{prep}}^{\text{min}} \geq \frac{P_{\text{guess}} - (1 - \frac{d-1}{d}\beta_{\text{min}}^{-1})}{(d-1)d^{n-1}} = \frac{P_{\text{guess}} - P_{\text{guess}}^{\text{PNC}}}{(d-1)d^{n-1}}$$

## Noise thresholds for contextuality


Preparation Contextual if the Average Gate Fidelity is larger than

$$F(\mathcal{E}) > \frac{1}{D}\big(1 + \frac{1}{2} + \cdots + \frac{1}{D}\big)$$