Title: Jordan algebras: from QM to 5D supergravity to â€¦ Standard Model?

Speakers: Paul Townsend

Collection: Octonions and the Standard Model

Date: May 03, 2021 - 12:00 PM

URL: http://pirsa.org/21050003

Abstract: This talk will be about two applications of Jordan algebras. The first, to quantum mechanics, follows on from the talk of John Baez. I will explain how time dependence makes use of the associator, and how this is related to the commutator in the standard density matrix formulation.

The associator of a Jordan algebra also determines the curvature of a Riemannian metric on its positive cone, invariant under the symmetry group of the norm (mentioned in the talk of John Baez); the cone is foliated by hypersurfaces of constant norm. This geometry is relevant to a class of N=2 5D supergravity theories (from the early 1980s) which arise (in some cases, at least) from Calabi-Yau compactification of 11D supergravity. The 5D interactions are determined by the structure constants of a euclidean Jordan algebra with cubic norm. The exceptional JA of 3x3 octonionic matrices yields an ``exceptionalâ€™â€™ 5D supergravity which yields, on reduction to 4D, an ``exceptionalâ€™â€™ N=2 supergravity with many similarities to N=8 supergravity, such as a non-compact global E7 symmetry. However, it has a compact `compositeâ€™ E6 gauge invariance (in contrast to the SU(8) of N=8 supergravity). An old speculation is that non-perturbative effects break the N=2 supersymmetry and cause the E6 gauge potentials to become the dynamical fields of an E6 GUT. Potentially (albeit improbably) this provides a connection between M-theory, the exceptional Jordan algebra, and the Standard Model.

Jordan algebras: from QM to 5D supergravity to ... Standard Model?

Paul K. Townsend

University of Cambridge

- Part 1. JA and QM. Time dependence
  [Follow-up to John Baez talk]
- Interlude. Geometry of JA cones. $H_3(\mathbb{O})$ case.
- Part 2. 5D Maxwell-Einstein supergravities. $H_3(\mathbb{O})$ case.
  Reduction to 4D. Composite E6 GUT?
  Günaydin, Sierra, PKT, '83-'85]

"Octonions & SM" workshop, 3 May '21

Jordan formulation of QM: observables I

A QM observable is an element of a ‘formally real” (aka “Euclidean”) Jordan algebra J with identity element. Some properties:

- Jordan identity is equivalent to power-associativity. In particular, the exponential map is defined; this takes $J \to \mathscr{C}(J) \subset J$, the positive convex cone of J.
- Every J is a direct sums of simple Js
- For simple J with identity e_0, we can choose basis $\{e_0, e_i; i = 1, \ldots, n\}$ such that

$$e_0 \circ e_0 = e_0 \,, \qquad e_0 \circ e_i = e_i \,, \qquad e_i \circ e_j = \delta_{ij}\, e_0 + T_{ijk}\, e_k$$

  where T_{ijk} is totally symmetric.
- Automorphisms: $\mathrm{Aut}(J) \subset SO(n)$ for which T_{ijk} is invariant tensor.

The Norm

We may view J as $(n+1)$-dimensional Euclidean space, and $\mathscr{C}(J)$ as a subspace. The characteristic function ω of $\mathscr{C}(J)$ is defined by

$$\omega^{-1}(x) = \int_{\mathscr{C}(J)} e^{-\mathrm{tr}(x\circ y)} d^{n+1}y \qquad (x = x^I e_I \equiv x^0 e_0 + x^i e_i)$$

Notice that $\omega(Ax) = (\det A)\omega(x)$ for automorphism $x \mapsto Ax$ of cone: $\Rightarrow$ $\omega(x)$ is homogeneous of degree $n+1$. For $\mathscr{C}(J)$,

$$\omega^2(x) = [\mathscr{N}(x)]^{2(n+1)/\nu}$$

for homogeneous polynomial $\mathscr{N}(x)$ of degree $\nu \in \mathbb{N}$, called the norm

➻ $\mathscr{N}$ is invariant under automorphisms of $\mathscr{C}(J)$ with $\det A = 1$. These form the reduced structure group: $\boxed{\mathrm{Str}_0(J) \supset \mathrm{Aut}(J)}$

Classification of simple Euclidean Jordan algebras

- $\nu = 1$: $J = \mathbb{R}$.
- $\nu = 2$: $J = J(Q)$. In this case $T_{ijk} = 0$, so $\mathrm{Aut}(J) = SO(n)$.
  Norm is $Q = x_0^2 - |\mathrm{x}|^2$, so $\mathrm{Str}_0 = SO(1,n)$.

  ➸ Viewing Q as Minkowski metric, $\mathscr{C}(J)$ is positive light-cone

- $\nu = 3$: $J = \mathbb{H}_3(\mathbb{K})$, i.e 3×3 hermitian matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.
  For $H \in \mathbb{H}_3(\mathbb{K})$, cubic norm is

$$\mathscr{N} = \det H \qquad \Rightarrow \mathrm{Str}_0 = \begin{cases} Sl(3;\mathbb{K}) & \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \\ E_{6(-26)} & \mathbb{K} = \mathbb{O} \end{cases}$$

- $\nu > 3$: $J = H_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, [det not defined for $\mathbb{K} = \mathbb{O}$]

The Trace

We can introduce a "trace-form" with the property that

$$\mathrm{tr}(e_i) = 0\,, \qquad \mathrm{tr}(e_0) = \nu \in \mathbb{N}$$

Relation to matrix trace:

- $\nu = 2$. Trace is matrix trace for realization by generalized Pauli matrices [of dimension $N(n)$; e.g. $N(3) = 2$]

$$(e_0, e_i) \to \frac{2}{N}(\mathbb{I}_N\,, \sigma_i)\,, \qquad x \circ y \to \frac{N}{4}\{x, y\}$$

- $\nu = 3$. Trace form is matrix trace for realization as 3×3 hermitian matrices with $x \circ y \to \frac{1}{2}\{x, y\}$

Jordan formulation of QM: states I

To understand states we need the primitive idempotents:

$$P^2 = P \ (\text{idempotent}), \qquad \text{tr}P = 1 \ (\text{primitive})$$

(P_1, P_2) are orthogonal if $P_1 \circ P_2 = 0$. There are at most ν mutually orthogonal primitive idempotents, and any such set provides a decomposition of the identity: $e_0 = P_1 + \cdots + P_\nu$. For example

- $\nu = 2$. $P_\pm = \frac{1}{2}(e_0 \pm e_1)$ is one of many possible choices
- $\nu = 3$. for 3×3 matrix realization, one obvious choice is

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Jordan formulation of QM: states II

Any observable a can be written as

$$a = \sum_{\alpha=1}^{\nu} a_\alpha P_\alpha(a)$$

for some set $\{P_\alpha(a)\}$ of mutually orthogonal primitive idempotents and real coefficients a_α. This theorem has the following interpretation:

- the elements $P_\alpha(a)$ correspond to eigenstates of a
- the coefficients a_α are the corresponding eigenvalues.

As expected, 'eigenstates' are unique only if 'eigenvalues' are distinct.

➸ Every primitive idempotent is a pure state.

Jordan formulation of QM: mixed states

A generic normalized mixed state can be represented by

$$\rho \in \mathscr{C}(J)\,, \qquad \mathrm{tr}\rho = 1$$

The state is pure iff $\rho^2 = \rho$; these are extremal (and hence boundary) points of $\mathscr{C}(J)$. The expectation value of observable a in the state ρ is $\langle a\rangle_\rho = \mathrm{tr}(a \circ \rho)$.

This is essentially the density matrix formulation of QM, except that there is no obvious analog of the Heisenberg equation of motion

$$\dot\rho = -i\,[H,\rho] \qquad (\rho = \text{density matrix}\,,\, H = \text{Hamiltonian})$$

because the RHS is not defined for $\rho \in J$.

Jordan formulation of QM: time dependence I

For the time evolution to take a pure state into another pure state we need $\dot{\rho} = D(\rho)$ for some derivation D of J. All derivations of a Jordan algebra take the form

$$D_{x,y} = [L_x, L_y] \qquad (L_x \;=\; \text{left multiplication by } x \in J)$$

but

$$[L_x, L_y]\rho = x \circ (y \circ \rho) - y \circ (x \circ \rho) = x \circ (\rho \circ y) - (x \circ \rho) \circ y$$

which is the associator of ρ with x and y, so $\boxed{\dot{\rho} = \{x, \rho, y\}}$ [Nambu]

- Notice invariance under the combination of $t \to -t$ and $x \leftrightarrow y$.

Jordan formulation of QM: time dependence II

For a matrix realization of J with $x \circ y = \frac{1}{2}\{x, y\}$, we have

$$\dot{\rho} = \frac{1}{4}\left[[x, y], \rho\right] .$$

For complex Hermitian matrices we recover the Heisenberg equation via the identification

$$H = H = \frac{i}{4}[x, y] + \lambda \mathbb{I}\,, \qquad (\lambda \in \mathbb{R})$$

A simple example

Take J to be the algebra of 2×2 hermitian matrices with basis $\{\mathbb{I}_2, \sigma_x, \sigma_y, \sigma_z\}$, and choose $(x, y) = (\sigma_x, \sigma_y)$ to get

$$\dot{\rho} = \{\sigma_y, \rho, \sigma_x\} = -i[H, \rho]\,, \qquad H = \sigma_z/2$$

i.e. Hamiltonian for a spin-$\frac{1}{2}$ particle in a uniform magnetic field

Geometry of $\mathscr{C}(J)$: I

By homogeneity of $\mathscr{N}$, the dual $(n+1)$-vector $x^*_I = \nu^{-1}\partial_I \ln \mathscr{N}(x)$ satisfies

$$x^I x^*_I = 1$$

Differentiating we deduce that

$$x^*_I = h_{IJ}(x)\, x^J\,, \qquad \boxed{h(x) = -\nu^{-1} d^2 \ln \mathscr{N}(x)}$$

The metric h is positive, so $\mathscr{C}(J)$ is a self-dual cone. Every vector in $\mathscr{C}(J)$ has its dual vector in $\mathscr{C}$, except the basepoint $x = e_0$, which is its own dual since

$$h_{IJ}(x) = \delta_{IJ} \;\Leftrightarrow\; x = e_0.$$

The curvature of h at the basepoint is given by the associator of J:

$$\boxed{\{e_I, e_K, e_J\} = R_{IJK}{}^L(e_0)e_L} \qquad (*)$$

Geometry of $\mathscr{C}(J)$: II

The cone $\mathscr{C}(J)$ is foliated by scaled copies of the symmetric space

$$\mathscr{M}(J) \equiv \mathscr{C}(J)\big|_{\mathscr{N}=1} = \mathrm{Str}_0(J)/\mathrm{Aut}(J)$$

- $\nu = 2$. $\mathscr{M} = SO(1,n)/SO(n)$ (a maximally symmetric space)
- $\nu = 3$, e.g. $\mathrm{H}_3(\mathbb{O})$. In this case $\boxed{\mathscr{M} = E_{6(-26)}/F_4}$.

The only non-zero components of $R_{IKJL}(e_0)$ are [from $(*)$]

$$R_{ijkl} = \delta_{k[i}\delta_{j]l} + T^m{}_{k[i} T_{j]lm}$$

This is curvature tensor (at basepoint) of metric on $\mathscr{M}(J)$ induced by h.

Jordan formulation of QM: time dependence II

For a matrix realization of J with $x \circ y = \frac{1}{2}\{x, y\}$, we have

$$\dot{\rho} = \frac{1}{4}\left[[x, y], \rho\right] .$$

For complex Hermitian matrices we recover the Heisenberg equation via the identification

$$H = H = \tfrac{i}{4}[x, y] + \lambda \mathbb{I} \,, \qquad (\lambda \in \mathbb{R})$$

A simple example

Take J to be the algebra of 2×2 hermitian matrices with basis $\{\mathbb{I}_2, \sigma_x, \sigma_y, \sigma_z\}$, and choose $(x, y) = (\sigma_x, \sigma_y)$ to get

$$\dot{\rho} = \{\sigma_y, \rho, \sigma_x\} = -i[H, \rho] \,, \qquad H = \sigma_z/2$$

i.e. Hamiltonian for a spin-$\frac{1}{2}$ particle in a uniform magnetic field

5D Maxwell-Einstein Supergravity I

Omitting fermions, the susy multiplets are

- Graviton multiplet: (metric g + one-form potential A^0)
- Maxwell multiplets: $(A^i + \text{ scalars } \varphi^i) \qquad (i = 1, \ldots, n)$

These combine to

$$\boxed{(g;\ A^I;\ \varphi^i) \qquad (I = 0, i)}$$

The scalars map spacetime to some n-dimensional target space $\mathscr{M}$. The Lagrangian 5-form includes

$$L = \cdots - \frac{1}{2}\left[h_{ij}(\varphi)d\varphi^i \wedge *d\varphi^j + h_{IJ}(\varphi)F^I \wedge *F^J\right] + C_{IJK}A^I \wedge F^J \wedge F^K$$

where $F^I = dA^I$, and C_{IJK} are constants

Geometry of $\mathscr{C}(J)$: II

The cone $\mathscr{C}(J)$ is foliated by scaled copies of the symmetric space

$$\mathscr{M}(J) \equiv \mathscr{C}(J)\big|_{\mathscr{N}=1} = \mathrm{Str}_0(J)/\mathrm{Aut}(J)$$

- $\nu = 2$. $\mathscr{M} = SO(1,n)/SO(n)$ (a maximally symmetric space)
- $\nu = 3$, e.g. $\mathrm{H}_3(\mathbb{O})$. In this case $\boxed{\mathscr{M} = E_{6(-26)}/F_4}$.

The only non-zero components of $R_{IKJL}(e_0)$ are [from $(*)$]

$$R_{ijkl} = \delta_{k[i}\delta_{j]l} + T^m{}_{k[i}T_{j]lm}$$

This is curvature tensor (at basepoint) of metric on $\mathscr{M}(J)$ induced by h.

5D Maxwell-Einstein Supergravity I

Omitting fermions, the susy multiplets are

- Graviton multiplet: (metric g + one-form potential A^0)
- Maxwell multiplets: $(A^i + \text{scalars } \varphi^i) \qquad (i = 1, \ldots, n)$

These combine to

$$\boxed{(g;\ A^I;\ \varphi^i) \qquad (I = 0, i)}$$

The scalars map spacetime to some n-dimensional target space $\mathscr{M}$. The Lagrangian 5-form includes

$$L = \cdots - \frac{1}{2}\left[h_{ij}(\varphi)d\varphi^i \wedge *d\varphi^j + h_{IJ}(\varphi)F^I \wedge *F^J\right] + C_{IJK}A^I \wedge F^J \wedge F^K$$

where $F^I = dA^I$, and C_{IJK} are constants

5D Maxwell-Einstein Supergravity II

Let ξ^I be cartesian coordinates for $\mathbb{R}^{n+1}$ and define $\mathcal{N} = C_{IJK}\xi^I\xi^J\xi^K$. Let φ^i be coordinates for $\mathcal{N} = 1$ hypersurface. Then $(\mathcal{N}, \varphi^i)$ are coordinates for a convex cone with metric $\propto -d^2 \ln \mathcal{N}$

- $h_{ij}(\varphi)$ is pullback to $\mathcal{N} = 1$ hypersurface of cone metric
- $h_{IJ}(\varphi)$ is restriction to $\mathcal{N} = 1$ hypersurface of cone metric

Pullback of C_{IJK} yields target space symmetric tensor $T_{ijk}(\varphi)$, but compatibility of 5D susy and constancy of C_{IJK} requires

$$T_{ijk;l} = g_{(ij}g_{k)l} - 2T_{(ij}{}^m T_{k)lm} \Rightarrow \boxed{R_{ijkl} = g_{k[i}g_{j]l} + T_{k[i}{}^m T_{j]lm}}$$

➽ When T_{ijk} is covariantly constant it equals (at the basepoint) the reduced structure constants of a Jordan algebra with cubic norm $\mathcal{N}$

The exceptional $H_3(\mathbb{O})$ case

The 'exceptional' 5D Maxwell-Einstein supergravity associated to $H_3(\mathbb{O})$ has scalar-field target space

$$\mathrm{Str}_0(J)/\mathrm{Aut}(J) = E_6(-26)/F_4$$

The non-compact E_6 symmetry group acts linearly, and irreducibly, on the abelian 1-form potential $A^I, I = 1, \ldots, 27$.

M-theory origin via Calabi-Yau compactification?

CY compactification of 11D to 5D supergravity gives

$$C_{IJK} = \int_{CY} \Omega_I \wedge \Omega_J \wedge \Omega_K$$

where $\{\Omega_I; I = 0, \ldots n\}$ are harmonic (1,1) forms (Ω_0 is Kahler 2-form).
Is there a CY manifold that gives the exceptional 5D sugra?

5D Maxwell-Einstein Supergravity I

Omitting fermions, the susy multiplets are

- Graviton multiplet: (metric g + one-form potential A^0)
- Maxwell multiplets: (A^i + scalars φ^i) $\qquad (i = 1, \ldots, n)$

These combine to

$$\boxed{(g;\ A^I;\ \varphi^i) \qquad (I = 0, i)}$$

The scalars map spacetime to some n-dimensional target space $\mathscr{M}$. The Lagrangian 5-form includes

$$L = \cdots - \frac{1}{2}\left[h_{ij}(\varphi)d\varphi^i \wedge *d\varphi^j + h_{IJ}(\varphi)F^I \wedge *F^J\right] + C_{IJK}A^I \wedge F^J \wedge F^K$$

where $F^I = dA^I$, and C_{IJK} are constants

Reduction from 5D to 4D

Bosonic fields are now

- Spacetime metric g,
- $(n+2)$ Abelian 1-forms (A^I, A)
- $n+1$ complex scalars Z^i

Scalar field target space is now "Koecher half-space" $\mathscr{D} = J + i\mathscr{C}(J)$, with Kähler metric derived from Kähler potential $\ln \mathscr{N}(Z - \bar{Z})$.
It is the coset space

$$\mathrm{M\ddot{o}(J)}/\widetilde{Str}(J) \qquad \text{(tilde indicates compact form)}$$

where Mö(J) denotes the Möbius group of J. For example

- For $J = H_2(\mathbb{O})$: Mö(J) is conformal group in $D = 10$
- For $J = H_3(\mathbb{O})$: $\mathrm{M\ddot{o}(J)} = E_{7(-25)}$

Composite E_6 GUT conjecture

For $J = H_3(\mathbb{O})$ we get 'exceptional' $N = 2$ 4D sugra with target space

$$\boxed{E_{7(-25)}/[E_6 \times U(1)]} \qquad (cf\ E_{7(7)}/SU(8) \text{ for N} = 8 \text{ sugra})$$

Linear realization of $E_{7(-25)}$ requires $E_6 \times U(1)$ gauge invariance but gauge fields are composite.

Popular in early 80's was suggestion that composite $SU(8)$ gauge fields of N=8 sugra become dynamical but not easy to get phenomenologically plausible GUT.

A further suggestion from 1984 (but not pursued!) was for dynamical $E_6 \times U(1)$ in 'exceptional' sugra. It was also observed that this sugra 'contains' another "octonionic" one (related to Jordan triple system)

$$\boxed{E_{6(-14)}/[\mathrm{Spin}(10) \times U(1)]}$$

Reduction from 5D to 4D

Bosonic fields are now

- Spacetime metric g,
- $(n+2)$ Abelian 1-forms (A^I, A)
- $n+1$ complex scalars Z^i

Scalar field target space is now "Koecher half-space" $\mathscr{D} = J + i\mathscr{C}(J)$, with Kähler metric derived from Kähler potential $\ln \mathscr{N}(Z-\bar{Z})$.
It is the coset space

$$\mathrm{M\ddot{o}(J)}/\widetilde{Str}(J) \qquad \text{(tilde indicates compact form)}$$

where Mö(J) denotes the Möbius group of J. For example

- For $J = H_2(\mathbb{O})$: Mö(J) is conformal group in $D=10$
- For $J = H_3(\mathbb{O})$: $\mathrm{M\ddot{o}(J)} = E_{7(-25)}$