Title: A tale of two geometries

Speakers: Tzu Chen Huang

Series: Particle Physics

Date: April 20, 2021 - 1:00 PM

URL: http://pirsa.org/21040030

Abstract: It is known that constraints imposed by causality and unitarity of four-particle scattering amplitudes lead to non-trivial requirements on the low energy effective field theory coefficients. We introduce families of linear and nonlinear inequalities resulting from a systematic study of positive geometry structure hidden in those constraints.

A tale of two geometries

Tzu-Chen Huang

California Institute of Technology

April 20, 2021

Outline

Bounding EFTs

- Integrating out massive UV states produces IR effective couplings
- Naive EFTs can be in conflict with fundamental physical requirements[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi][Penedones, Silva, Zhiboedov][de Rham, Melville, Tolley, Zhou][Caron-Huot, Duong][Hebbar,Karateev, Penedones]...
- Dispersion relation

$$\frac{i}{2\pi}\oint_{\mathcal{C}_0}\frac{ds}{s^{n+1}}M(s,t^*)=\sum_j\frac{1}{s'^{n+1}}\mathrm{Res}_{s=s'}M(s,t^*)+\int\frac{ds'}{s'^{n+1}}\mathrm{Disc}M.$$

together with an IR amplitude applicable when $\mathcal{C}_0$ is near origin

$$M^{\mathrm{IR}}(s,t)=(\text{massless poles})+\sum c_{a,b}s^at^b$$

connects our knowledge o spectral density and residues/discontinuities to Wilson coefficients $c_{a,b}$.

Bounding EFTs

- Integrating out massive UV states produces IR effective couplings
- Naive EFTs can be in conflict with fundamental physical requirements[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi][Penedones, Silva, Zhiboedov][de Rham, Melville, Tolley, Zhou][Caron-Huot, Duong][Hebbar,Karateev, Penedones]...
- Dispersion relation

$+ \oint_{\infty} \frac{ds}{s^{n+1}} M.$

$$\frac{i}{2\pi}\oint_{\mathcal{C}_0}\frac{ds}{s^{n+1}}M(s,t^*) = \sum_j \frac{1}{s'^{n+1}}\text{Res}_{s=s'}M(s,t^*) + \int \frac{ds'}{s'^{n+1}}\text{Disc}M.$$

together with an IR amplitude applicable when $\mathcal{C}_0$ is near origin

$$M^{\text{IR}}(s,t) = (\text{massless poles}) + \sum c_{a,b}s^a t^b$$

connects our knowledge o spectral density and residues/discontinuities to Wilson coefficients $c_{a,b}$.

Gegenbauer positivity

- Tree-level Four point massless superstring amplitude[Kawai, Lewellen, Tye]:

$$M = -\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1+u)} K_{\alpha_1\alpha_2\alpha_3\alpha_4}\eta_1^{\alpha_1}\eta_2^{\alpha_2}\eta_3^{\alpha_3}\eta_4^{\alpha_4}.$$

- Expanding on Gegenbauer polynomial basis

$$\mathrm{Res}_{s=m_i^2} M = \sum_j c_j(D) G_j^{(\frac{D-2}{2})}(1+\frac{2t}{m_i^2}), \quad c_j(D) \geq 0.$$

- Forward limit positivity bound: $G_n^{(\alpha)}(1) > 0$.
- Beyond forward limit: $G_n^{(\alpha)}(1+x) = \sum_i v_{n,i}^{(\alpha)} x^i$. Consider $(1, t, t^2)$. Collect contribution from each spin into the following table

$$\sum_i c_i \begin{pmatrix} 1 & & \\ & \frac{1}{m_i^2} & \\ & & \frac{1}{m_i^4} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 2 & 6 & 12 & \dots \\ 0 & 0 & 6 & 30 & \dots \end{pmatrix}$$

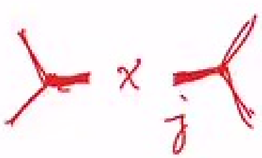
- m_i^2 dependence. Needs better way of organizing coefficients.

Gegenbauer positivity

- Tree-level Four point massless superstring amplitude[Kawai, Lewellen, Tye]:

$$M = -\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1+u)} K_{\alpha_1\alpha_2\alpha_3\alpha_4}\eta_1^{\alpha_1}\eta_2^{\alpha_2}\eta_3^{\alpha_3}\eta_4^{\alpha_4}.$$

- Expanding on Gegenbauer polynomial basis



$$\mathrm{Res}_{s=m_i^2} M = \sum_j c_j(D) G_j^{(\frac{D-2}{2})}\left(1+\frac{2t}{m_i^2}\right), \quad c_j(D) \geq 0.$$

- Forward limit positivity bound: $G_n^{(\alpha)}(1) > 0$.
- Beyond forward limit: $G_n^{(\alpha)}(1+x) = \sum_i v_{n,i}^{(\alpha)} x^i$. Consider $(1, t, t^2)$. Collect contribution from each spin into the following table

$$\sum_i c_i \begin{pmatrix} 1 & & \\ & \frac{1}{m_i^2} & \\ & & \frac{1}{m_i^4} \end{pmatrix} \begin{array}{ccccc} 0 & 1 & 2 & & \\ \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 2 & 6 & 12 & \dots \\ 0 & 0 & 6 & 30 & \dots \end{pmatrix}$$

$d = 3 + 1.$

- m_i^2 dependence. Needs better way of organizing coefficients.

Spectral density

- Recall the Källén-Lehmann representation:

$$\Delta(p) = \int d\mu^2 \rho(\mu^2) \frac{1}{\mu^2 - s} \to \sum_n \underbrace{\left(\int d\mu^2 \rho(\mu^2) \frac{1}{\mu^{2n+2}} \right)}_{m_n} s^n .$$

- We already know that $\rho(\mu^2) \geq 0$, guaranteeing non-negative coefficients in small s expansion.
- Moment problem: given a sequence of numbers $(m_0, m_1, \ldots)$, does there exist a positive Borel measure σ such that

$$m_n = \int_{\mathcal{K}} x^n d\sigma(x) \quad ?$$

$$\mathcal{K} = \begin{pmatrix} \mathbb{R} \\ \mathbb{R}_{\geq 0} \\ [0,1] \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \text{Hamburger} \\ \text{Stieltjes} \\ \text{Hausdorff} \end{pmatrix} \text{moment criteria.}$$

- m_n being non-negative is not enough.

Bounding EFTs

- Integrating out massive UV states produces IR effective couplings
- Naive EFTs can be in conflict with fundamental physical requirements[Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi][Penedones, Silva, Zhiboedov][de Rham, Melville, Tolley, Zhou][Caron-Huot, Duong][Hebbar,Karateev, Penedones]...
- Dispersion relation

$$+ \oint_{\infty} \frac{ds}{s^{n+1}} M.$$

$$\frac{i}{2\pi}\oint_{\mathcal{C}_0}\frac{ds}{s^{n+1}}M(s,t^*) = \sum_j \frac{1}{s'^{n+1}}\mathrm{Res}_{s=s'}M(s,t^*) + \int \frac{ds'}{s'^{n+1}}\mathrm{Disc}M.$$

together with an IR amplitude applicable when $\mathcal{C}_0$ is near origin

$$M^{\mathrm{IR}}(s,t) = (\text{massless poles}) + \sum c^{(n)}_{a,b} s^a t^b$$

connects our knowledge o spectral density and residues/discontinuities to Wilson coefficients $c_{a,b}$.

Spectral density

- Recall the Källén-Lehmann representation:

$$\Delta(p) = \int d\mu^2 \rho(\mu^2) \frac{1}{\mu^2 - s} \to \sum_n \underbrace{\left(\int d\mu^2 \rho(\mu^2) \frac{1}{\mu^{2n+2}} \right)}_{m_n} s^n .$$

- We already know that $\rho(\mu^2) \geq 0$, guaranteeing non-negative coefficients in small s expansion.
- Moment problem: given a sequence of numbers $(m_0, m_1, \ldots)$, does there exist a positive Borel measure σ such that

$$m_n = \int_{\mathcal{K}} x^n d\sigma(x) \quad ?$$

$$\mathcal{K} = \begin{pmatrix} \mathbb{R} \\ \mathbb{R}_{\geq 0} \\ [0,1] \end{pmatrix} \Rightarrow \begin{pmatrix} \text{Hamburger} \\ \text{Stieltjes} \\ \text{Hausdorff} \end{pmatrix} \text{ moment criteria.}$$

- m_n being non-negative is not enough.

Spectral density

- Recall the Källén-Lehmann representation:

$$\Delta(p) = \int d\mu^2 \rho(\mu^2) \frac{1}{\mu^2 - s} \to \sum_n \underbrace{\left(\int d\mu^2 \rho(\mu^2) \frac{1}{\mu^{2n+2}} \right)}_{m_n} s^n .$$

- We already know that $\rho(\mu^2) \geq 0$, guaranteeing non-negative coefficients in small s expansion.
- Moment problem: given a sequence of numbers $(m_0, m_1, \ldots)$, does there exist a positive Borel measure σ such that

$$m_n = \int_{\mathcal{K}} x^n d\sigma(x) \quad ?$$

$$\mathcal{K} = \begin{pmatrix} \mathbb{R} \\ \mathbb{R}_{\geq 0} \\ [0,1] \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \text{Hamburger} \\ \text{Stieltjes} \\ \text{Hausdorff} \end{pmatrix} \text{ moment criteria.}$$

- m_n being non-negative is not enough.

No assumption on spectrum: Stieltjes

- Since μ^2 must be non-negative for physical states, we take $\mathcal{K}$ to be $\mathbb{R}_{\geq 0}$.
- Infinite sequence case:

$$K[\vec{m}] = \begin{pmatrix} m_0 & m_1 & \cdots \\ m_1 & m_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0, \quad \tilde{K}[\vec{m}] = \begin{pmatrix} m_1 & m_2 & \cdots \\ m_2 & m_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0.$$

- For truncated case, the above is sufficient when both matrices are non-singular. Otherwise one needs an extra condition.[Curto, Fialkow]
- Example(massless superstring):

$$\vec{m} = \begin{pmatrix} \frac{2}{5}\zeta_2^2 \\ \zeta_5 \\ \frac{8}{35}\zeta_2^3 \\ \zeta_7 \\ \frac{24}{175}\zeta_2^4 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{2}{5}\zeta_2^2 & \zeta_5 & \frac{8}{35}\zeta_2^3 \\ \zeta_5 & \frac{8}{35}\zeta_2^3 & \zeta_7 \\ \frac{8}{35}\zeta_2^3 & \zeta_7 & \frac{24}{175}\zeta_2^4 \end{pmatrix} \geq 0, \quad \begin{pmatrix} \zeta_5 & \frac{8}{35}\zeta_2^3 \\ \frac{8}{35}\zeta_2^3 & \zeta_7 \end{pmatrix} \geq 0.$$

No assumption on spectrum: Stieltjes

- Since μ^2 must be non-negative for physical states, we take $\mathcal{K}$ to be $\mathbb{R}_{\geq 0}$.
- Infinite sequence case:

$x = \frac{1}{\mu^2}$

$$K[\vec{m}] = \begin{pmatrix} m_0 & m_1 & \cdots \\ m_1 & m_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0, \quad \tilde{K}[\vec{m}] = \begin{pmatrix} m_1 & m_2 & \cdots \\ m_2 & m_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0.$$

- For truncated case, the above is sufficient when both matrices are non-singular. Otherwise one needs an extra condition [Curto, Fialkow]
- Example(massless superstring):

$$\vec{m} = \begin{pmatrix} \frac{2}{5}\zeta_2^2 \\ \zeta_5 \\ \frac{8}{35}\zeta_2^3 \\ \zeta_7 \\ \frac{24}{175}\zeta_2^4 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{2}{5}\zeta_2^2 & \zeta_5 & \frac{8}{35}\zeta_2^3 \\ \zeta_5 & \frac{8}{35}\zeta_2^3 & \zeta_7 \\ \frac{8}{35}\zeta_2^3 & \zeta_7 & \frac{24}{175}\zeta_2^4 \end{pmatrix} \geq 0, \quad \begin{pmatrix} \zeta_5 & \frac{8}{35}\zeta_2^3 \\ \frac{8}{35}\zeta_2^3 & \zeta_7 \end{pmatrix} \geq 0.$$

Spectrum with gap: Hausdorff

- Without loss of generality let's suppose $\mu_{\text{gap}}^2 = 1$.
- Again consider the sequence $\vec{m}$. Criteria for feasibility is that of Stieltjes plus

$$K[\vec{m}] \geq \tilde{K}[\vec{m}].$$

- (Alternative characterization) Introduce the difference operator

$$\Delta \vec{m} = (m_1 - m_0, m_2 - m_1, \ldots).$$

Then,

$$(-1)^n \Delta^n \vec{m}$$

must have non-negative entries for every $n \in \mathbb{N}$. [Englert, Giudice, Greljo, McCullough][Bellazzini,Miró,Rattazzi,Riembau,Riva]

- Example:

$$-\Delta \vec{m} = \left(\frac{\pi^4}{90} - \zeta_5, -\frac{\pi^6}{945} + \zeta_5, \frac{\pi^6}{945} - \zeta_7, -\frac{\pi^8}{9450} + \zeta_7\right) \geq 0,$$

$$\Delta^2 \vec{m} = \left(\frac{\pi^4}{90} + \frac{\pi^6}{945} - 2\zeta_5, -2\frac{\pi^6}{945} + \zeta_5 + \zeta_7, \frac{\pi^6}{945} + \frac{\pi^8}{9450} - 2\zeta_7\right) \geq 0.$$

Spectrum with gap: Hausdorff

- Without loss of generality let's suppose $\mu_{\text{gap}}^2 = 1$.
- Again consider the sequence $\vec{m}$. Criteria for feasibility is that of Stieltjes plus

$$K[\vec{m}] \geq \tilde{K}[\vec{m}].$$

- (Alternative characterization) Introduce the difference operator

$$\Delta\vec{m} = (m_1 - m_0, m_2 - m_1, \ldots).$$

Then,

$$(-1)^n \Delta^n \vec{m}$$

must have non-negative entries for every $n \in \mathbb{N}$. [Englert, Giudice, Greljo, McCullough][Bellazzini,Miró,Rattazzi,Riembau,Riva]

- Example:

$$-\Delta\vec{m} = (\frac{\pi^4}{90} - \zeta_5, -\frac{\pi^6}{945} + \zeta_5, \frac{\pi^6}{945} - \zeta_7, -\frac{\pi^8}{9450} + \zeta_7) \geq 0,$$

$$\Delta^2\vec{m} = (\frac{\pi^4}{90} + \frac{\pi^6}{945} - 2\zeta_5, -2\frac{\pi^6}{945} + \zeta_5 + \zeta_7, \frac{\pi^6}{945} + \frac{\pi^8}{9450} - 2\zeta_7) \geq 0.$$

Revisiting the Gegenbauer positivity

- Now that we know the geometries that arise when expanding in s and t directions separately, let us try putting them together

$$a_{k,q} = \frac{1}{q!}\frac{d^q}{dt^q}\left(\int_{\mathcal{K}\times\mathbb{N}} d\sigma(\mu^2,\ell)\frac{G_\ell(1+\frac{2t}{\mu^2})}{\mu^{2k-2q+2}}\right)\Bigg|_{t=0}.$$

- Consider terms with the same total mass dimension($k=4$):

$$a_{4,0}s^4 + a_{4,1}s^3t + a_{4,2}s^2t^2 + a_{4,3}st^3 + a_{4,4}t^4$$

- mass dependence can be factored out and gives

$$\Rightarrow \begin{pmatrix} a_{4,0} \\ a_{4,1} \\ a_{4,2} \end{pmatrix} = \int_{\mathcal{K}\times\mathbb{N}} d\sigma(\mu^2,\ell)\frac{1}{\mu^{10}}\begin{pmatrix} v_{\ell,0} \\ v_{\ell,1} \\ v_{\ell,2} \end{pmatrix}.$$

Revisiting the Gegenbauer positivity

- Now that we know the geometries that arise when expanding in s and t directions separately, let us try putting them together

$$a_{k,q} = \frac{1}{q!}\frac{d^q}{dt^q}\left(\int_{K\times\mathbb{N}} d\sigma(\mu^2,\ell)\frac{G_\ell(1+\frac{2t}{\mu^2})}{\mu^{2k-2q+2}}\right)\Bigg|_{t=0}.$$

- Consider terms with the same total mass dimension($k=4$):

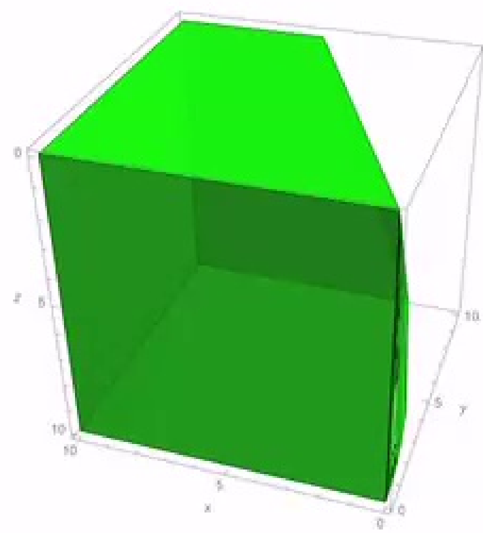
$$\boxed{a_{4,0}s^4 + a_{4,1}s^3t + a_{4,2}s^2t^2} + a_{4,3}st^3 + a_{4,4}t^4$$

- mass dependence can be factored out and gives

$$\Rightarrow \begin{pmatrix} a_{4,0} \\ a_{4,1} \\ a_{4,2} \end{pmatrix} = \boxed{\int_{K\times\mathbb{N}} d\sigma(\mu^2,\ell)\frac{1}{\mu^{10}}\begin{pmatrix} v_{\ell,0} \\ v_{\ell,1} \\ v_{\ell,2} \end{pmatrix}}$$

Revisiting the Gegenbauer positivity

- The allowed region for coefficients corresponding to a fixed mass dimension is a positive cone of $\vec{v}_{\ell,*}$, where ℓ runs over spins.
- $(a_{4,0}, a_{4,1}, a_{4,2})$ must lie inside



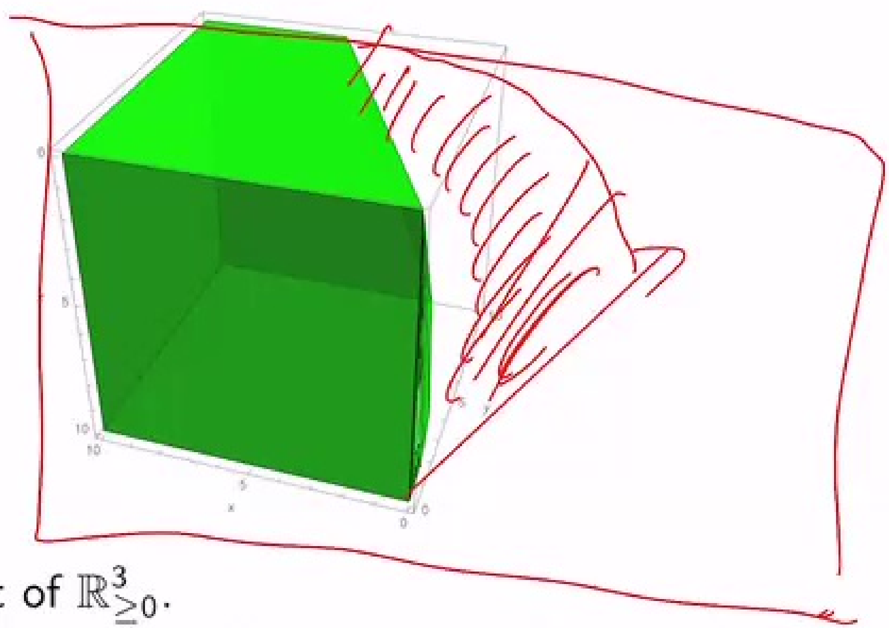
which is a proper subset of $\mathbb{R}^3_{\geq 0}$.

- How to determine the facets of a polytope with infinite number of vertices?

Revisiting the Gegenbauer positivity

- The allowed region for coefficients corresponding to a fixed mass dimension is a positive cone of $\vec{v}_{\ell,*}$, where ℓ runs over spins.
- $(a_{4,0}, a_{4,1}, a_{4,2})$ must lie inside

$a_{4,0} \geq 0$.

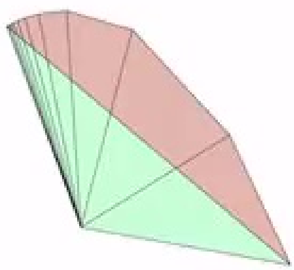


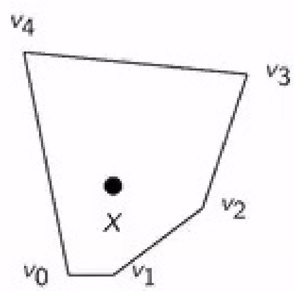
which is a proper subset of $\mathbb{R}^3_{\geq 0}$.

- How to determine the facets of a polytope with infinite number of vertices?

Cyclicity of Gegenbauer polytope

- Cyclicity$\Rightarrow$ co-dimension 1 facets given by adjacent spin vertices:





$$\Rightarrow \quad \langle X, \vec{v}_i, \vec{v}_{i+1} \rangle \geq 0.$$

- In the 3-component example, we have

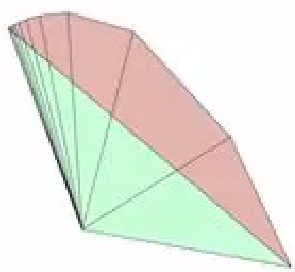
$$\mathcal{W}^T \cdot \begin{pmatrix} a_{4,0} \\ a_{4,1} \\ a_{4,2} \end{pmatrix} \geq 0, \quad \mathcal{W} = \begin{pmatrix} 0 & 6 & 18 & 120 & \ldots \\ 0 & -3 & -4 & -15 & \ldots \\ 1 & 2 & 1 & 2 & \ldots \end{pmatrix}$$

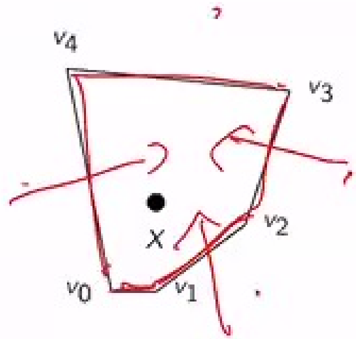
where $\mathcal{W}$ is given by collecting normal vectors of co-dimension 1 facets.

- The fact that one can derive the exact form of the walls are crucial to deriving bounds.

Cyclicity of Gegenbauer polytope

- Cyclicity$\Rightarrow$ co-dimension 1 facets given by adjacent spin vertices:





$$\Rightarrow \quad \langle X, \vec{v}_i, \vec{v}_{i+1} \rangle \geq 0.$$

- In the 3-component example, we have

$$\mathcal{W}^T \cdot \begin{pmatrix} a_{4,0} \\ a_{4,1} \\ a_{4,2} \end{pmatrix} \geq 0, \quad \mathcal{W} = \begin{pmatrix} 0 & 6 & 18 & 120 & \dots \\ 0 & -3 & -4 & -15 & \dots \\ 1 & 2 & 1 & 2 & \dots \end{pmatrix}$$

where $\mathcal{W}$ is given by collecting normal vectors of co-dimension 1 facets.

- The fact that one can derive the exact form of the walls are crucial to deriving bounds.

Cyclicity of Gegenbauer polytope

- Cyclicity is determined by examining the non-negativity of ordered minors of the matrix formed by collecting corresponding vectors.
- Consider a matrix where entries are Taylor coefficients of Gegenbauer polynomials

$$v^D_{\ell,q} = \frac{(\Delta)_\ell}{q!(\ell)!}\frac{(\ell)_{-q}(\ell+\Delta)_q}{\prod_{a=1}^{q}(\Delta+2a-1)}, \quad \Delta := D-3.$$

Factors that depend on either ℓ or q alone can be factored out.

$$\Rightarrow \mathrm{Det}[v^D_{\ell_i,j}]_{i,j=0,\cdots,n} = \prod_{i=1}^{n+1}\frac{(\Delta)_{\ell_i}}{\ell_i!}\frac{1}{\prod_{a=1}^{q}(\Delta+2a-1)a!} \cdot \mathrm{Det}\left[(\ell_i)_{-j}(\ell_i+\Delta)_j\right]_{i,j=0,\cdots,n}$$

- The last determinant should have a factor $\prod_{i>j}(\ell_i-\ell_j)$ due to alternating symmetry. Together with the invariance under $\ell_i \to -\ell_i-\Delta$ from

$$(-a)_b = (-1)^b(a)_{-b}$$

we deduce that the last determinant is just $\prod_{i>j}(\ell_i-\ell_j)(\Delta+\ell_i+\ell_j)$, a positive factor.

Cyclicity of Gegenbauer polytope

- Cyclicity is determined by examining the non-negativity of ordered minors of the matrix formed by collecting corresponding vectors.
- Consider a matrix where entries are Taylor coefficients of Gegenbauer polynomials

$$v^D_{\ell,q} = \frac{(\Delta)_\ell}{q!(\ell)!}\frac{(\ell)_{-q}(\ell+\Delta)_q}{\prod_{a=1}^{q}(\Delta+2a-1)}, \quad \Delta := D-3.$$

Factors that depend on either ℓ or q alone can be factored out.

$$\Rightarrow \mathrm{Det}[v^D_{\ell_i,j}]_{i,j=0,\cdots,n} = \prod_{i=1}^{n+1}\frac{(\Delta)_{\ell_i}}{\ell_i!}\frac{1}{\prod_{a=1}^{q}(\Delta+2a-1)a!}$$
$$\mathrm{Det}\left[(\ell_i)_{-j}(\ell_i+\Delta)_j\right]_{i,j=0,\cdots,n}$$

- The last determinant should have a factor $\prod_{i>j}(\ell_i-\ell_j)$ due to alternating symmetry. Together with the invariance under $\ell_i \to -\ell_i - \Delta$ from

$$(-a)_b = (-1)^b(a)_{-b}$$

$a\cdot(a-1)(a \cdots ?$

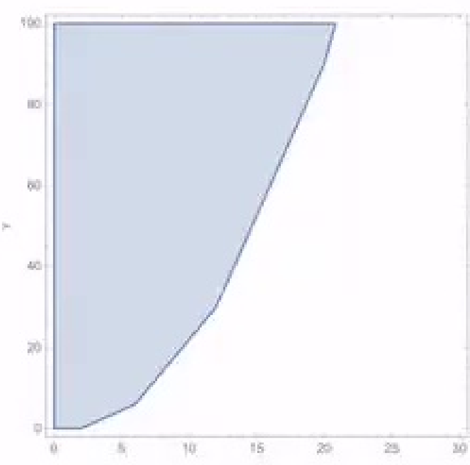
$(-a)(1-a)(2-a) \cdots$

we deduce that the last determinant is just $\prod_{i>j}(\ell_i-\ell_j)(\Delta+\ell_i+\ell_j)$, a positive factor.

General Gegenbauer polytope

- Expand to higher order in Mandelstam variables, a coefficient vector $\in \mathbb{P}^p$ should lie within the higher dimensional Gegenbauer polytope. Boundary structure:
$\langle X, \vec{v}_i, \vec{v}_{i+1}, \vec{v}_j, , \vec{v}_{j+1}, \cdots \rangle \geq 0$ for even p
$\langle 0, X, \vec{v}_i, \vec{v}_{i+1}, \vec{v}_j, , \vec{v}_{j+1}, \cdots \rangle \geq 0$ and $\langle X, \vec{v}_i, \vec{v}_{i+1}, \cdots, \infty \rangle \geq 0$ for odd p.
- Example: $(1, x, y, y) \in \mathbb{P}^3$ that satisfies permutation invariance.

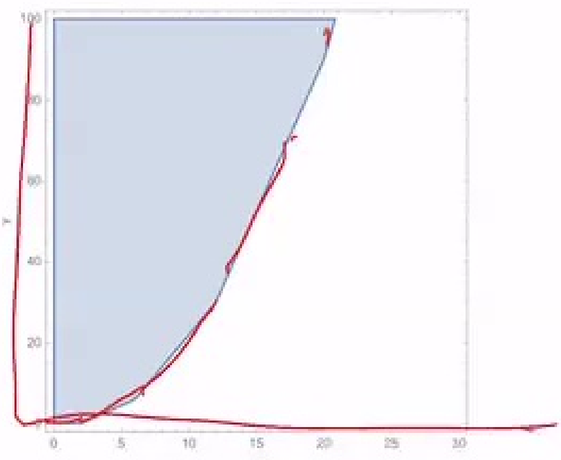
$$\langle X, \vec{v}_i, \vec{v}_{i+1}, \infty \rangle \geq 0 \Rightarrow$$



General Gegenbauer polytope

- Expand to higher order in Mandelstam variables, a coefficient vector $\in \mathbb{P}^p$ should lie within the higher dimensional Gegenbauer polytope. Boundary structure:
  $\langle X, \vec{v}_i, \vec{v}_{i+1}, \vec{v}_j, , \vec{v}_{j+1}, \cdots \rangle > 0$ for even p
  $\langle 0, X, \vec{v}_i, \vec{v}_{i+1}, \vec{v}_j, , \vec{v}_{j+1}, \cdots \rangle \geq 0$ and $\langle X, \vec{v}_i, \vec{v}_{i+1}, \cdots, \infty \rangle \geq 0$ for odd p.
- Example: $(1, x, y, y) \in \mathbb{P}^3$ that satisfies permutation invariance.

$$\langle X, \vec{v}_i, \vec{v}_{i+1}, \infty \rangle \geq 0 \Rightarrow$$



Spinning polytope

- When external particles have spins, the Legendre polynomials become Wigner d-matrices:

$$G_\ell^{h_1,h_2,h_3,h_4}(\cos\theta) = \langle \ell, h_1 - h_2 | e^{-i\theta J_y} | \ell, h_3 - h_4 \rangle .$$

- One can similarly expand amplitudes on this basis with proper helicity configuration:

$$\mathrm{Res}_{s=3} u^2 \left(\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1+u)} + \text{sym.} \right) = \frac{27}{28} G_4^{h_i} + \frac{9}{4} G_3^{h_i} + \frac{39}{14} G_2^{h_i},$$

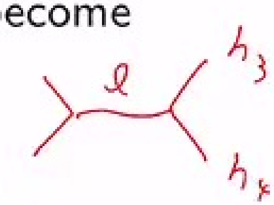
where $h_i = 1^+2^-3^+4^-$.

- The vectors formed by taking derivative of $G_\ell^{h_i}(1+x)$ also furnish a cyclic polytope for any h_i. On the other hand, for $h_i = 1^+2^-3^-4^+$ we should be expanding $G_\ell^{h_i}(1+\frac{2u}{s}) = G_\ell^{h_i}(-1-\frac{2t}{s})$, which translates to an alternating sign $(-1)^\ell$ when using $1+\frac{2t}{s}$ as argument.

Spinning polytope

- When external particles have spins, the Legendre polynomials become Wigner d-matrices:

$$\boxed{G_\ell^{h_1,h_2,h_3,h_4}(\cos\theta)} = \langle \ell, h_1 - h_2 | e^{-i\theta J_y} | \ell, h_3 - h_4 \rangle .$$



- One can similarly expand amplitudes on this basis with proper helicity configuration:

$$\mathrm{Res}_{s=3}\, u^2 \left(\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1+u)} + \text{sym.} \right) = \frac{27}{28} G_4^{h_i} + \frac{9}{4} G_3^{h_i} + \frac{39}{14} G_2^{h_i},$$

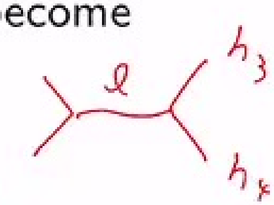
where $h_i = 1^+ 2^- 3^+ 4^-$.

- The vectors formed by taking derivative of $G_\ell^{h_i}(1+x)$ also furnish a cyclic polytope for any h_i. On the other hand, for $h_i = 1^+ 2^- 3^- 4^+$ we should be expanding $G_\ell^{h_i}(1+\frac{2u}{s}) = G_\ell^{h_i}(-1-\frac{2t}{s})$, which translates to an alternating sign $(-1)^\ell$ when using $1+\frac{2t}{s}$ as argument.

Spinning polytope

- When external particles have spins, the Legendre polynomials become Wigner d-matrices:

$$\boxed{G_\ell^{h_1,h_2,h_3,h_4}(\cos\theta)} = \langle \ell, h_1 - h_2 | e^{-i\theta J_y} | \ell, h_3 - h_4 \rangle .$$



- One can similarly expand amplitudes on this basis with proper helicity configuration:

$$\text{Res}_{s=3} u^2 \left(\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1+u)} + \text{sym.} \right) = \frac{27}{28} G_4^{h_i} + \frac{9}{4} G_3^{h_i} + \frac{39}{14} G_2^{h_i},$$

where $h_i = 1^+ 2^- 3^+ 4^-$.

- The vectors formed by taking derivative of $G_\ell^{h_i}(1+x)$ also furnish a cyclic polytope for any h_i. On the other hand, for $h_i = 1^+2^-3^-4^+$ we should be expanding $G_\ell^{h_i}(1+\frac{2u}{s}) = G_\ell^{h_i}(-1-\frac{2t}{s})$, which translates to an alternating sign $(-1)^\ell$ when using $1+\frac{2t}{s}$ as argument.

u-channel mixing in moment curves

- Consider

$$-\sum_i \frac{c_i G_{\ell_i}(1+\frac{2t}{m_i^2})}{s-m_i^2}+\frac{d_i G_{\ell_i'}(1+\frac{2t}{m_i^2})}{u-m_i^2}.$$

$$\frac{G_\ell\left(1+\frac{2t}{m^2}\right)}{m^2-u}=\frac{1}{m^2}\left(1-\frac{s+t}{m^2}+\frac{s^2+2st+t^2}{m^4}+\cdots\right)\left(1+2v_{\ell,1}\frac{t}{m^2}+\cdots\right).$$

- Fixed mass dimension coefficients in u-channel are now "deformed":

$$s^{2-i}t^i: \quad \frac{1}{m^6}\left(s^2+(2-2v_{\ell,1})\,st+(1-2v_{\ell,1}+4v_{\ell,2})\,t^2\right)$$

  but still cyclic.
- Not necessarily cyclic when combining vectors from both s- and u-channel.

u-channel mixing in moment curves

- Consider

$$-\sum_i \frac{c_i G_{\ell_i}(1+\frac{2t}{m_i^2})}{s-m_i^2} + \frac{d_i G_{\ell_i'}(1+\frac{2t}{m_i^2})}{u-m_i^2}$$

$$\frac{G_\ell\left(1+\frac{2t}{m^2}\right)}{m^2-u} = \frac{1}{m^2}\left(1-\frac{s+t}{m^2}+\frac{s^2+2st+t^2}{m^4}+\cdots\right)\left(1+2v_{\ell,1}\frac{t}{m^2}+\cdots\right).$$

- Fixed mass dimension coefficients in u-channel are now "deformed":

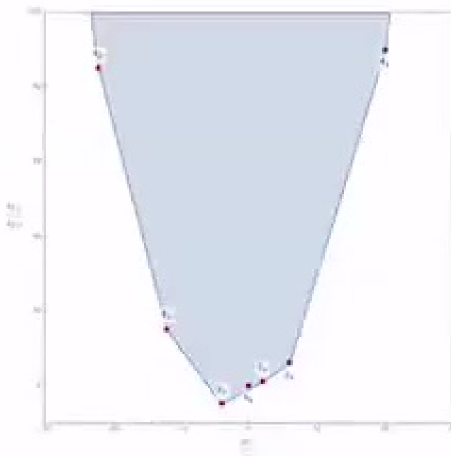
$$s^{2-i}t^i: \quad \frac{1}{m^6}\left(s^2+(2-2v_{\ell,1})\,st+(1-2v_{\ell,1}+4v_{\ell,2})\,t^2\right)$$

but still cyclic.

- Not necessarily cyclic when combining vectors from both s- and u-channel.

$s - u$ polytope

- For $\langle - - + + \rangle$ or $\langle - + - + \rangle$ configuration, we have to take Minkowski sum of two separate polytopes



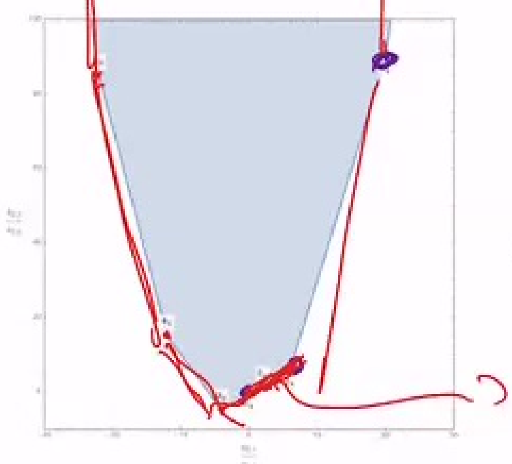
$$\Rightarrow \quad \langle X, \tilde{v}_{s,2}, \tilde{v}_{s,4} \rangle \geq 0, \langle X, \tilde{v}_{u,4}, \tilde{v}_{u,3} \rangle \geq 0 \cdots$$

- Observe that the resulting polytope still retains cyclic polytope boundaries. The only addition here are a few mixed boundaries between s- and u-vectors, e.g. $\langle X, \tilde{v}_{u,2}, \tilde{v}_{s,2} \rangle \geq 0$.

10:45 AM Tue Apr 20 68%

$s - u$ polytope

- For $\langle - - + + \rangle$ or $\langle - + - + \rangle$ configuration, we have to take Minkowski sum of two separate polytopes



$$\Rightarrow \quad \langle X, \tilde{v}_{s,2}, \tilde{v}_{s,4} \rangle \geq 0, \langle X, \tilde{v}_{u,4}, \tilde{v}_{u,3} \rangle \geq 0 \cdots$$

$\langle X, 2_u, 2_s \rangle \geq 0.$

- Observe that the resulting polytope still retains cyclic polytope boundaries. The only addition here are a few mixed boundaries between s- and u-vectors, e.g. $\langle X, \tilde{v}_{u,2}, \tilde{v}_{s,2} \rangle \geq 0$.

Combined constraints

- Two families of constraints: the moment problem in s-expansion, and the Gegenbauer/spinning polytope for fixed mass dimension coefficients

$$\underbrace{a_{k,q}\mathcal{W}_l^q \geq 0}_{\text{Polytope}} \quad \text{and} \quad \underbrace{K[\vec{a}_{*,q}] \geq 0, \tilde{K}[\vec{a}_{*,q}] \geq 0}_{\text{Moment}}$$

- s-channel EFThedron: $K[\overrightarrow{(a\cdot\mathcal{W})_l}] \geq 0, \tilde{K}[\overrightarrow{(a\cdot\mathcal{W})_l}] \geq 0$.
- The above constraints are always necessary for any scalar amplitudes that has physical spectrum in the s-channel.
- Sufficiency is not established, and in fact one might need extra constraints. Related to the fact that bivariate moment problems are more than tensor product of ordinary moment problems.

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} & \cdots \\ m_{10} & m_{11} & m_{12} & \cdots \\ m_{20} & m_{21} & m_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \sum_i c_i \begin{pmatrix} 1 & x_i & x_i^2 & \cdots \\ y_i & x_i y_i & x_i^2 y_i & \cdots \\ y_i^2 & x_i y_i^2 & x_i^2 y_i^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Combined constraints

- Two families of constraints: the moment problem in s-expansion, and the Gegenbauer/spinning polytope for fixed mass dimension coefficients

$$\underbrace{a_{k,q}\mathcal{W}_l^q \geq 0}_{\text{Polytope}} \quad \text{and} \quad \underbrace{K[\vec{a}_{*,q}] \geq 0, \tilde{K}[\vec{a}_{*,q}] \geq 0}_{\text{Moment}}$$

- s-channel EFThedron: $K[\overrightarrow{(a\cdot\mathcal{W})_l}] \geq 0, \tilde{K}[\overrightarrow{(a\cdot\mathcal{W})_l}] \geq 0$.
- The above constraints are always necessary for any scalar amplitudes that has physical spectrum in the s-channel.
- Sufficiency is not established, and in fact one might need extra constraints. Related to the fact that bivariate moment problems are more than tensor product of ordinary moment problems.

$$\begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} & \cdots \\ m_{10} & m_{11} & m_{12} & \cdots \\ m_{20} & m_{21} & m_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \sum_i c_i \begin{pmatrix} 1 & x_i & x_i^2 & \cdots \\ y_i & x_i y_i & x_i^2 y_i & \cdots \\ y_i^2 & x_i y_i^2 & x_i^2 y_i^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Deformed moment curves and scalar EFThedron

- Example of a deformed moment problem:

$$(a_0, a_1, a_2, \cdots)_q = \sum_i c_i \left(u_{\ell_i,2,q}, u_{\ell_i,4,q}\frac{1}{m_i^2}, u_{\ell_i,6,q}\frac{1}{m_i^4}, \cdots\right).$$

The problem does not depend on q before because the scaling factor was universal. Now the factor $u_{\ell_i,2n+2,q}$ in front of $\frac{1}{m_i^{2n}}$ is spin-dependent and can be negative!

- Combine the whole family of moment problems parametrized by $q \in \mathbb{N}$:

$$(\vec{a}_0, \vec{a}_1, \vec{a}_2, \cdots) = \sum_i c_i \left(\vec{u}_{\ell_i,2}, \vec{u}_{\ell_i,4}\frac{1}{m_i^2}, \vec{u}_{\ell_i,6}\frac{1}{m_i^4}, \cdots\right).$$

- Recall definition of walls $\mathcal{W}_I$: every vector v^I inside the convex hull will have to satisfy $v^I \mathcal{W}_I \geq 0$. This includes vectors coming from a single spin! $\Rightarrow$ The outermost boundary is the Minkowski sum of $\{\text{Walls}(k) \text{ of } \vec{u}_{\ell_i,k}\}$

Deformed moment curves and scalar EFThedron

- Example of a deformed moment problem:

$$(a_0, a_1, a_2, \cdots)_q = \sum_i c_i \left(u_{\ell_i,2,q}, u_{\ell_i,4,q} \frac{1}{m_i^2}, u_{\ell_i,6,q} \frac{1}{m_i^4}, \cdots \right).$$

The problem does not depend on q before because the scaling factor was universal. Now the factor $u_{\ell_i,2n+2,q}$ in front of $\frac{1}{m_i^{2n}}$ is spin-dependent and can be negative!

- Combine the whole family of moment problems parametrized by $q \in \mathbb{N}$:

$$(\vec{a}_0, \vec{a}_1, \vec{a}_2, \cdots) = \sum_i c_i \left(\vec{u}_{\ell_i,2}, \vec{u}_{\ell_i,4} \frac{1}{m_i^2}, \vec{u}_{\ell_i,6} \frac{1}{m_i^4}, \cdots \right).$$

- Recall definition of walls $\mathcal{W}_I$: every vector v^I inside the convex hull will have to satisfy $v^I \mathcal{W}_I \geq 0$. This includes vectors coming from a single spin! $\Rightarrow$ The outermost boundary is the Minkowski sum of $\{\text{Walls}(k) \text{ of } \vec{u}_{\ell_i,k}\}$

Deformed moment curves and scalar EFThedron

- For increasing k, the walls are actually contained within each other. Let the largest total mass dimension be K.

$$\Rightarrow \alpha_{\ell,k} = \mathcal{W}_l^K \cdot \vec{u}_{\ell,k} \geq 0.$$

- The deformed moment curve becomes

$$\sum_i c_i(\alpha_{\ell_i,2}, \frac{\alpha_{\ell_i,4}}{m_i^2}, \cdots)_l, \quad \alpha_{\ell_i,n} \geq 0.$$

- Taking the convex hull again, we arrive at "boundaries" of convex hulls of moment curves deformed by $\alpha_{\ell,k}$. For example, suppose we want to find the outermost wall for $\sum_i p_i(\alpha_{\ell_i,2}, \frac{\alpha_{\ell_i,4}}{m_i^2}, \frac{\alpha_{\ell_i,6}}{m_i^4})$, which can be written as the positive cone of $(1, x, \tilde{\alpha}x^2)$. We would simply find $\tilde{\alpha}$ such that

$$\begin{pmatrix} \alpha_{\ell,2} & \alpha_{\ell,4} \\ \alpha_{\ell,4} & \frac{\alpha_{\ell,6}}{\tilde{\alpha}} \end{pmatrix} \geq 0. \quad \forall \ell$$

- The full EFThedron for the deformed case is then $K[\vec{A}_l]_{\vec{\tilde{\alpha}}} \geq 0$.

Deformed moment curves and scalar EFThedron

- For increasing k, the walls are actually contained within each other. Let the largest total mass dimension be K.

$$\Rightarrow \alpha_{\ell,k} = \mathcal{W}_l^K \cdot \vec{u}_{\ell,k} \geq 0.$$

- The deformed moment curve becomes

$$\sum_i p_i\left(\alpha_{\ell_i,2}, \frac{\alpha_{\ell_i,4}}{m_i^2}, \cdots\right)_l, \quad \alpha_{\ell_i,n} \geq 0.$$

- Taking the convex hull again, we arrive at "boundaries" of convex hulls of moment curves deformed by $\alpha_{\ell,k}$. For example, suppose we want to find the outermost wall for $\sum_i p_i(\alpha_{\ell_i,2}, \frac{\alpha_{\ell_i,4}}{m_i^2}, \frac{\alpha_{\ell_i,6}}{m_i^4})$, which can be written as the positive cone of $(1, x, \tilde{\alpha}x^2)$. We would simply find $\tilde{\alpha}$ such that

$$\begin{pmatrix} \alpha_{\ell,2} & \alpha_{\ell,4} \\ \alpha_{\ell,4} & \frac{\alpha_{\ell,6}}{\tilde{\alpha}} \end{pmatrix} \geq 0. \quad \forall \ell$$

- The full EFThedron for the deformed case is then $K[\vec{A}_l]_{\vec{\tilde{\alpha}}} \geq 0$.

Caveat & Example

- For some spin ℓ there must exist a piece of wall such that $\mathcal{W} \cdot \vec{u}_{\ell,k} = 0$. This occurs for the largest value of k. The moment problem then becomes

$$\sum_i c_i(1, x, \tilde{\alpha} x^2) = d_\ell(\alpha_{\ell,2}, \frac{\alpha_{\ell,4}}{m_i^2}, 0) \Rightarrow \tilde{\alpha} = 0.$$

- Fix: scan over all sets of walls that are outside of the largest k one to find the strongest bound.
- Consider

$$\begin{pmatrix} a_{4,0} & a_{4,2} \\ a_{6,0} & a_{6,2} \\ a_{8,0} & a_{8,2} \end{pmatrix} = \sum_i c_i \begin{pmatrix} x_i^4 u_{\ell_i,4,0} & x_i^4 u_{\ell_i,4,2} \\ x_i^6 u_{\ell_i,6,0} & x_i^6 u_{\ell_i,6,2} \\ x_i^8 u_{\ell_i,8,0} & x_i^8 u_{\ell_i,8,2} \end{pmatrix}.$$

Furthermore let $\beta_n := \frac{a_{n,2}}{a_{n,0}}$.

- Moment: $a_{4,0}a_{8,0} - a_{6,0}^2 \geq 0$. Gegenbauer: $\beta_4 \geq -\frac{3}{2}, \beta_6 \geq -\frac{21}{4}, \beta_8 \geq -8$.
- Wall is at -8, but let's go outside of it by 0.01 and we'd obtain

$$\frac{(\beta_4 + 8 + 0.01)(\beta_8 + 8 + 0.01)}{\tilde{\alpha}_{\max}(0.01)} - (\beta_6 + 8 + 0.01)^2 \geq 0, \quad \tilde{\alpha}_{\max}(0.01) \approx 0.0085.$$

Caveat & Example

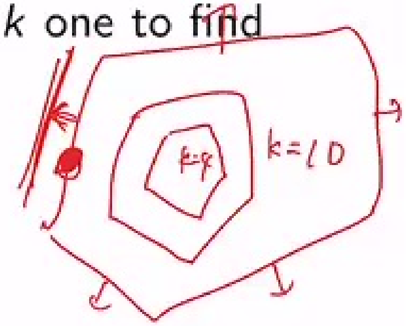
$\oint \frac{ds}{s^{n+1}} M(s,t^*) \to \oint \frac{ds}{(s^2+\mu^4)^{n+1}} M(s,t^*)$

- For some spin ℓ there must exist a piece of wall such that $\vec{W} \cdot \vec{u}_{\ell,k} = 0$. This occurs for the largest value of k. The moment problem then becomes

$$\sum_i c_i(1, x, \tilde{\alpha}x^2) = d_\ell(\alpha_{\ell,2}, \frac{\alpha_{\ell,4}}{m_i^2}, 0) \Rightarrow \tilde{\alpha} = 0.$$

$s^n t^x$

- Fix: scan over all sets of walls that are outside of the largest k one to find the strongest bound.



- Consider

$$\begin{pmatrix} a_{4,0} & a_{4,2} \\ a_{6,0} & a_{6,2} \\ a_{8,0} & a_{8,2} \end{pmatrix} = \sum_i c_i \begin{pmatrix} x_i^4 u_{\ell_i,4,0} & x_i^4 u_{\ell_i,4,2} \\ x_i^6 u_{\ell_i,6,0} & x_i^6 u_{\ell_i,6,2} \\ x_i^8 u_{\ell_i,8,0} & x_i^8 u_{\ell_i,8,2} \end{pmatrix}.$$

Furthermore let $\beta_n := \frac{a_{n,2}}{a_{n,0}}$.

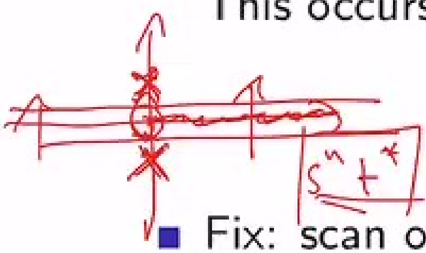
- Moment: $a_{4,0}a_{8,0} - a_{6,0}^2 \geq 0$. Gegenbauer: $\beta_4 \geq -\frac{3}{2}, \beta_6 \geq -\frac{21}{4}, \beta_8 \geq -8$.
- Wall is at -8, but let's go outside of it by 0.01 and we'd obtain

$$\frac{(\beta_4 + 8 + 0.01)(\beta_8 + 8 + 0.01)}{\tilde{\alpha}_{\max}(0.01)} - (\beta_6 + 8 + 0.01)^2 \geq 0, \quad \tilde{\alpha}_{\max}(0.01) \approx 0.0085.$$

Caveat & Example

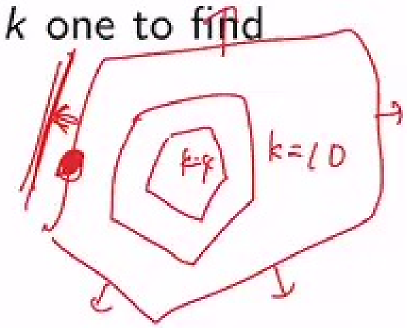
$\oint \frac{ds}{s^{n+1}} M(s,t^*) \to \oint \frac{ds}{(s^2+\mu^4)^{n+1}} M(s,t^*)$

- For some spin ℓ there must exist a piece of wall such that $\vec{W} \cdot \vec{u}_{\ell,k} = 0$. This occurs for the largest value of k. The moment problem then becomes



$$\sum_i c_i(1, x, \tilde{\alpha}x^2) = d_\ell(\alpha_{\ell,2}, \frac{\alpha_{\ell,4}}{m_i^2}, 0) \Rightarrow \tilde{\alpha} = 0.$$

- Fix: scan over all sets of walls that are outside of the largest k one to find the strongest bound.



- Consider

$$\begin{pmatrix} a_{4,0} & a_{4,2} \\ a_{6,0} & a_{6,2} \\ a_{8,0} & a_{8,2} \end{pmatrix} = \sum_i c_i \begin{pmatrix} x_i^4 u_{\ell_i,4,0} & x_i^4 u_{\ell_i,4,2} \\ x_i^6 u_{\ell_i,6,0} & x_i^6 u_{\ell_i,6,2} \\ x_i^8 u_{\ell_i,8,0} & x_i^8 u_{\ell_i,8,2} \end{pmatrix}.$$

Furthermore let $\beta_n := \frac{a_{n,2}}{a_{n,0}}$.

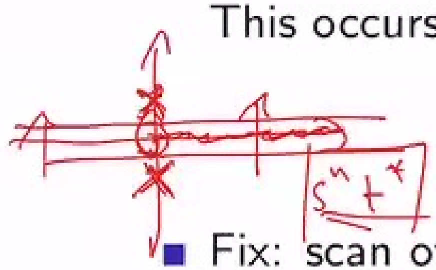
- Moment: $a_{4,0}a_{8,0} - a_{6,0}^2 \geq 0$. Gegenbauer: $\beta_4 \geq -\frac{3}{2}, \beta_6 \geq -\frac{21}{4}, \beta_8 \geq -8$.
- Wall is at -8, but let's go outside of it by 0.01 and we'd obtain

$$\frac{(\beta_4 + 8 + 0.01)(\beta_8 + 8 + 0.01)}{\tilde{\alpha}_{\max}(0.01)} - (\beta_6 + 8 + 0.01)^2 \geq 0, \quad \tilde{\alpha}_{\max}(0.01) \approx 0.0085.$$

Caveat & Example

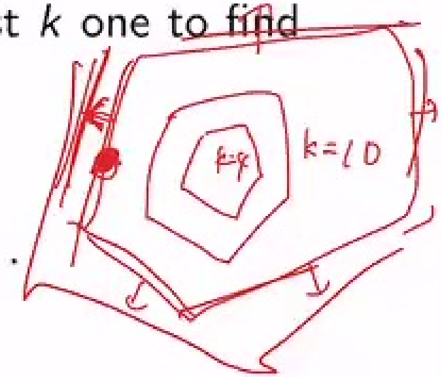
$\oint \frac{ds}{s^{n+1}} M(s,t^*) \to \oint \frac{ds}{(s^2+M^4)^{n+1}} M(s,t^*)$

- For some spin ℓ there must exist a piece of wall such that $\vec{W}\cdot\vec{u}_{\ell,k} = 0$. This occurs for the largest value of k. The moment problem then becomes



$$\sum_i c_i(1, x, \tilde{\alpha}x^2) = d_\ell(\alpha_{\ell,2}, \frac{\alpha_{\ell,4}}{m_i^2}, 0) \Rightarrow \tilde{\alpha} = 0.$$

- Fix: scan over all sets of walls that are outside of the largest k one to find the strongest bound.



- Consider

$$\begin{pmatrix} a_{4,0} & a_{4,2} \\ a_{6,0} & a_{6,2} \\ a_{8,0} & a_{8,2} \end{pmatrix} = \sum_i c_i \begin{pmatrix} x_i^4 u_{\ell_i,4,0} & x_i^4 u_{\ell_i,4,2} \\ x_i^6 u_{\ell_i,6,0} & x_i^6 u_{\ell_i,6,2} \\ x_i^8 u_{\ell_i,8,0} & x_i^8 u_{\ell_i,8,2} \end{pmatrix}.$$

Furthermore let $\beta_n := \frac{a_{n,2}}{a_{n,0}}$.

- Moment: $a_{4,0}a_{8,0} - a_{6,0}^2 \geq 0$. Gegenbauer: $\beta_4 \geq -\frac{3}{2}, \beta_6 \geq -\frac{21}{4}, \beta_8 \geq -8$.
- Wall is at -8, but let's go outside of it by 0.01 and we'd obtain

$$\frac{(\beta_4 + 8 + 0.01)(\beta_8 + 8 + 0.01)}{\tilde{\alpha}_{\max}(0.01)} - (\beta_6 + 8 + 0.01)^2 \geq 0, \quad \tilde{\alpha}_{\max}(0.01) \approx 0.0085.$$

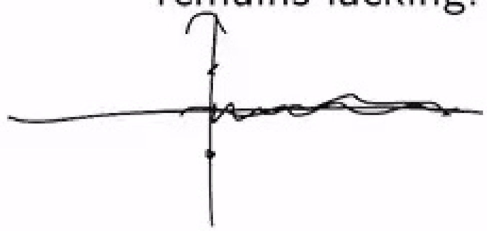
Summary

- Legendre/spinning polytopes naturally arise when considering forward limit expansion of residue
- Positivity of spectral density can be rephrased in terms of moment problems, and they come with rigorous criteria for determining feasibility
- The union of constraints from Stieltjes moment problem and Gegenbauer polytopes are probably not sufficient for the full EFThedron, which calls for further investigation.
- u-channel contribution can deform both the polytopes and the moment curves, and the analytic proof of boundary structure including all spins remains lacking.

Thank you!

11:13 AM Tue Apr 20 61%

- The union of constraints from Stieltjes moment problem and Gegenbauer polytopes are probably not sufficient for the full EFThedron, which calls for further investigation.
- u-channel contribution can deform both the polytopes and the moment curves, and the analytic proof of boundary structure including all spins remains lacking.

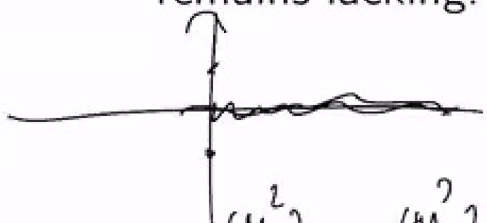


Thank you!

$$a^{(m^2)}_{n,0} = \frac{1}{2\pi i}\oint \frac{ds}{(s^2+\mu^4)^{n+1}} M(s,0)$$

problems, and they come with rigorous criteria for determining feasibility

- The union of constraints from Stieltjes moment problem and Gegenbauer polytopes are probably not sufficient for the full EFThedron, which calls for further investigation.
- u-channel contribution can deform both the polytopes and the moment curves, and the analytic proof of boundary structure including all spins remains lacking.



$(a^{(\mu^2)}_{2,0}, a^{(\mu^2)}_{4,0}, \cdots)$

Thank you!

$a^{(\mu^2)}_{n,0} = \frac{1}{2\pi i}\oint \frac{ds}{(s^2+\mu^4)^{n+1}} M(s,0)$

$\oint \frac{dt}{(t-\epsilon)}$