Title: Lie Algebroids and BRST

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Series: Quantum Fields and Strings

Date: April 20, 2021 - 2:00 PM

URL: http://pirsa.org/21040027

Abstract: I will give an introduction to the geometric aspects of Lie algebroids and show how they give the correct framework to discuss gauge theories. The analysis is solely based on geometry, and thus applies to every gauge theory, independently of their specific features and dynamics. By thoroughly re-formulating the physical content of gauge theories on Atiyah Lie algebroids, we will show that the BRST construction is part of the formalism, indicating a fascinating interplay between classical geometry and quantum physics. Time permitting, we will show how our setup encompasses gravitational theories too, once the frame bundle and solder form are added to the picture.

Lie Algebroids and BRST

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Perimeter Institute, Waterloo, 20-04-2021

Based on 2101.03974 with Rob Leigh

Motivations

Geometric framework for gauge theories: Lie algebroids

Completely off-shell and gauge independent

BRST symmetry built-in

Classical geometry featuring quantum symmetries

Outline

- Lie algebroids in a nutshell
- Atiyah Lie algebroids
- Gauge theories and BRST
- Gravitational theories

Main ref: Mackenzie's books

Lie Algebroids In A Nutshell

Definition

Lie algebroids in a nutshell   Atiyah Lie algebroids   Gauge theories and BRST   Gravitational theories

TLA: vector bundle $A \to M$ with anchor map $\rho : A \to TM$

$\underline{\mathfrak{X}}, \underline{\mathfrak{Y}} \in \Gamma(A), f, g \in C^\infty(M)$, Lie bracket on A

$$\rho(\underline{X}) = X^\mu \partial_\mu$$

$$[f\underline{\mathfrak{X}}, g\underline{\mathfrak{Y}}] = fg\,[\underline{\mathfrak{X}}, \underline{\mathfrak{Y}}] + f\rho(\underline{\mathfrak{X}})(g)\,\underline{\mathfrak{Y}} - g\rho(\underline{\mathfrak{Y}})(f)\,\underline{\mathfrak{X}}$$

ρ is a morphism: $R^\rho(\underline{\mathfrak{X}}, \underline{\mathfrak{Y}}) = [\rho(\underline{\mathfrak{X}}), \rho(\underline{\mathfrak{Y}})] - \rho([\underline{\mathfrak{X}}, \underline{\mathfrak{Y}}]) = \underline{0}$

If $\underline{\mathfrak{X}}, \underline{\mathfrak{Y}} \in$ ker $\rho \Rightarrow$ Linearity

$$\rho(\underline{X}) = 0 = \rho(\underline{Y})$$

ker ρ = vertical sub-bundle V of A, unambiguously defined

Introduce the bundle $L \to M$ such that $\iota : L \to A$ is an inclusion map ($R^\iota = \underline{0}$)

$$0 \longrightarrow L \stackrel{\iota}{\longrightarrow} A \stackrel{\rho}{\longrightarrow} TM \longrightarrow 0$$

This short sequence is exact: $\rho \circ \iota = 0$ and $TM = A/V$

Lie bracket on L linear: L has a Lie algebra structure and $\iota(L) =$ ker $\rho = V$

ker ρ = vertical sub-bundle V of A, unambiguously defined

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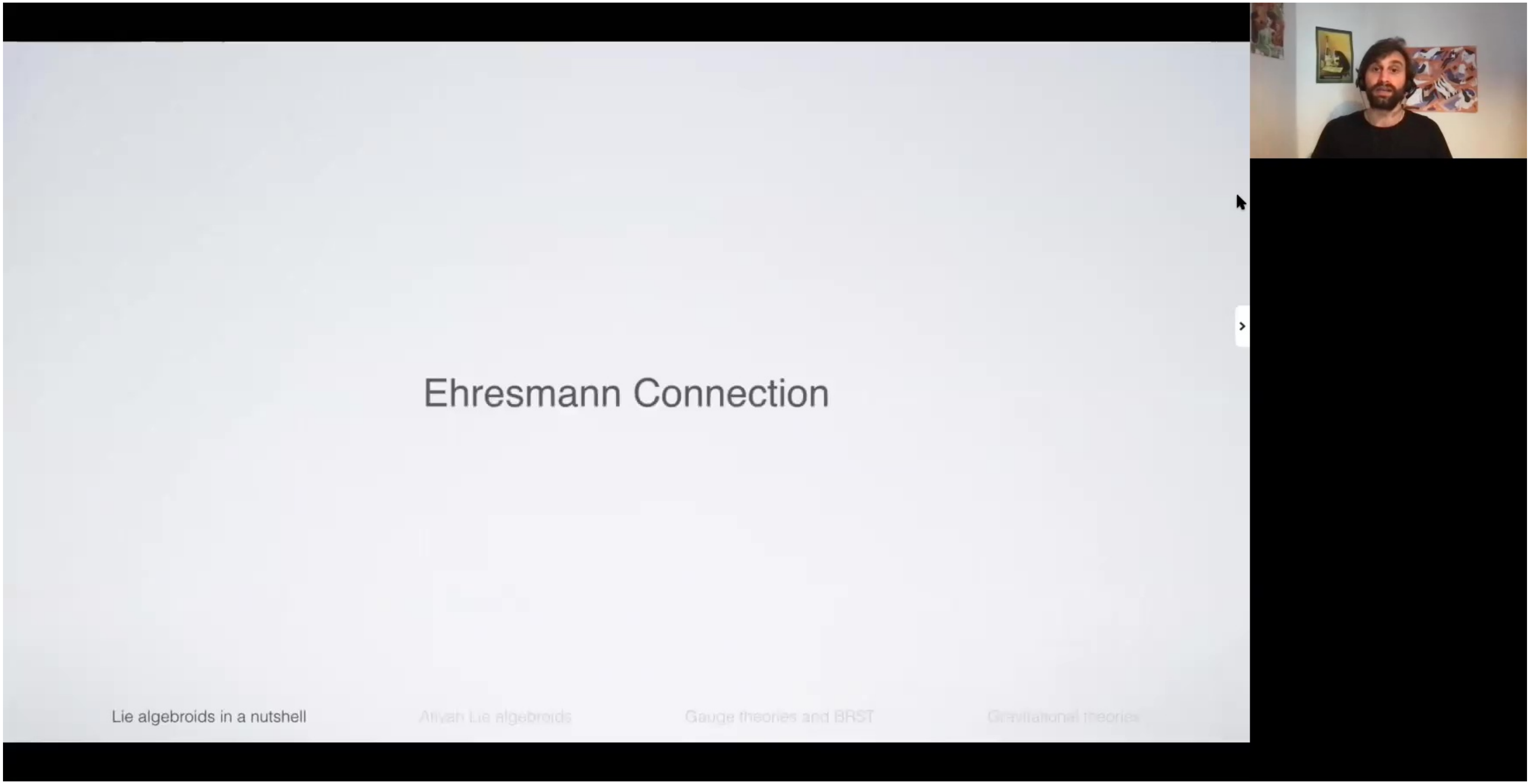
Ehresmann Connection
Lie algebroids in a nutshell
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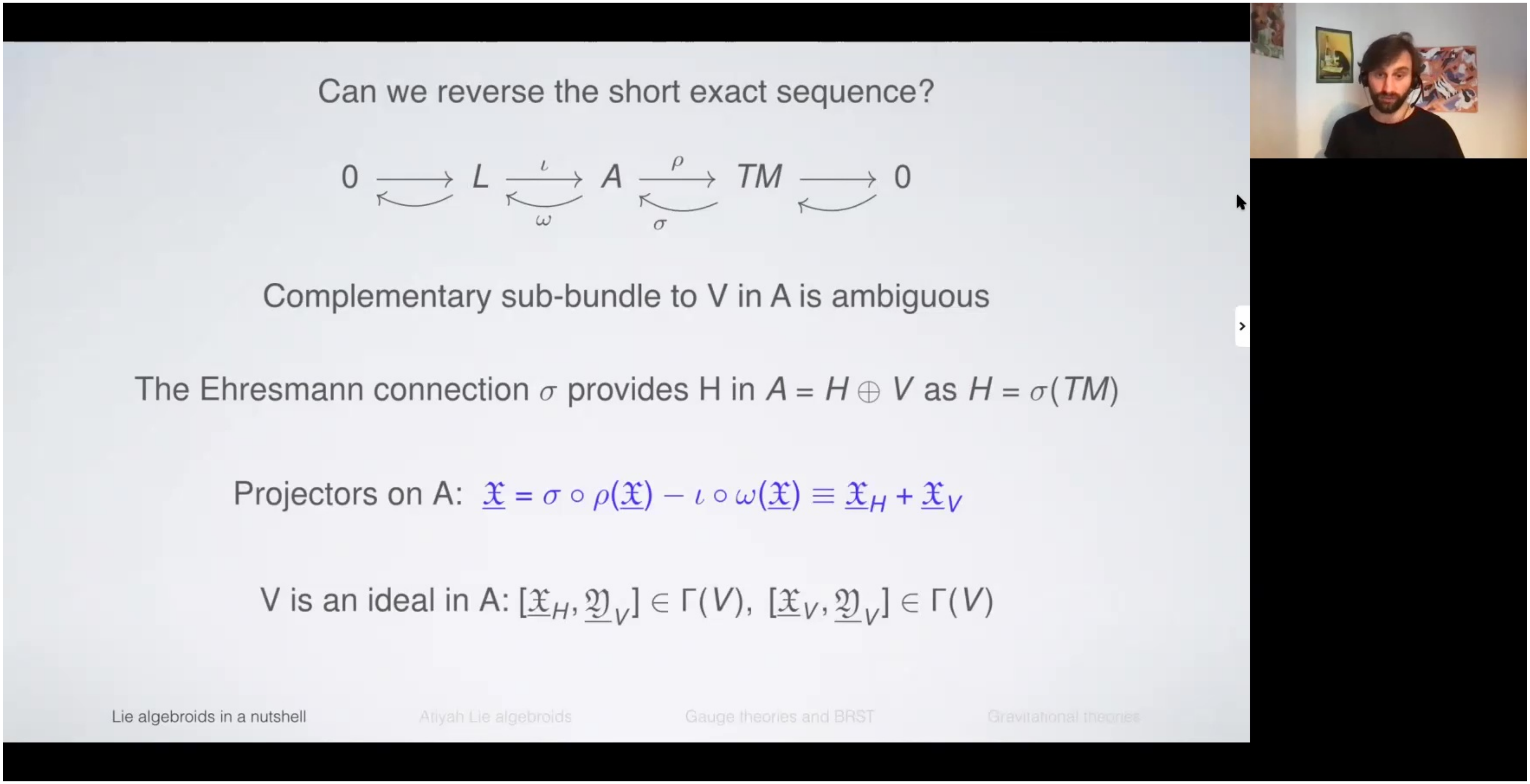
Can we reverse the short exact sequence?
$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} TM \longrightarrow 0$
ω
σ
Complementary sub-bundle to V in A is ambiguous
The Ehresmann connection σ provides H in $A = H \oplus V$ as $H = \sigma(TM)$
Projectors on A: $\underline{\mathfrak{X}} = \sigma \circ \rho(\underline{\mathfrak{X}}) - \iota \circ \omega(\underline{\mathfrak{X}}) \equiv \underline{\mathfrak{X}}_H + \underline{\mathfrak{X}}_V$
V is an ideal in A: $[\underline{\mathfrak{X}}_H, \underline{\mathfrak{Y}}_V] \in \Gamma(V)$, $[\underline{\mathfrak{X}}_V, \underline{\mathfrak{Y}}_V] \in \Gamma(V)$
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Derivations and $\hat{d}$

Can we reverse the short exact sequence?

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Can we reverse the short exact sequence?

$$R^{\iota}=0 \qquad R^{\rho}=0$$

$$0 \rightleftarrows L \underset{\omega}{\overset{\iota}{\rightleftarrows}} A \underset{\sigma}{\overset{\rho}{\rightleftarrows}} TM \rightleftarrows 0$$

$$R^{\omega}\neq 0 \qquad R^{\sigma}\neq 0$$

$\rho \circ \iota = 0$
$\omega \circ \sigma = 0$
$\mathrm{Ker}\,\omega = H$
$\mathrm{Im}\,\sigma$

Complementary sub-bundle to V in A is ambiguous

The Ehresmann connection σ provides H in $A = H \oplus V$ as $H = \sigma(TM)$

Projectors on A: $\underline{\mathfrak{X}} = \sigma \circ \rho(\underline{\mathfrak{X}}) - \iota \circ \omega(\underline{\mathfrak{X}}) \equiv \underline{\mathfrak{X}}_H + \underline{\mathfrak{X}}_V$

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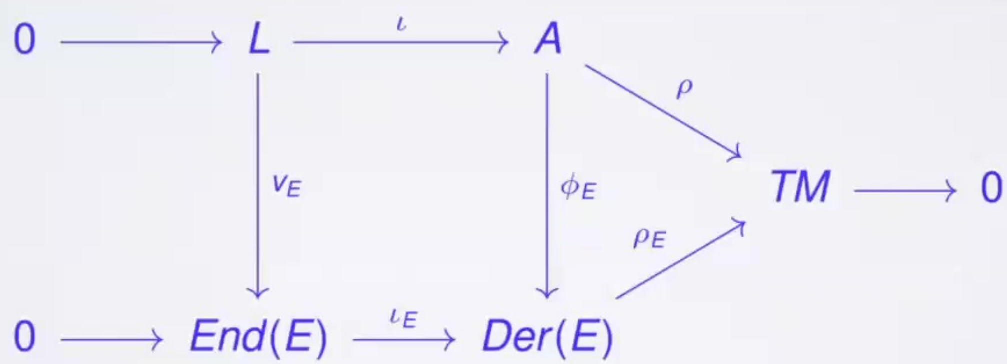
Derivations and $\hat{d}$

$E \to M$ bundle. $\mathfrak{D}$ is a section of $Der(E)$ with $\rho_E : Der(E) \to TM$ if

$$\mathfrak{D}(f\underline{\psi}) = f\mathfrak{D}(\underline{\psi}) + \rho_E(\mathfrak{D})(f)\ \underline{\psi}, \quad \underline{\psi} \in \Gamma(E), f \in C^\infty(M)$$

$\mathfrak{D} \in \ ker\rho_E$ is an endo $\Rightarrow$ $Der(E)$ is a Lie algebroid with prekernel $End(E)$

$Der(E)$ represents A supplying the morphisms $\phi_E : A \to Der(E)$ and $v_E : L \to End(E)$



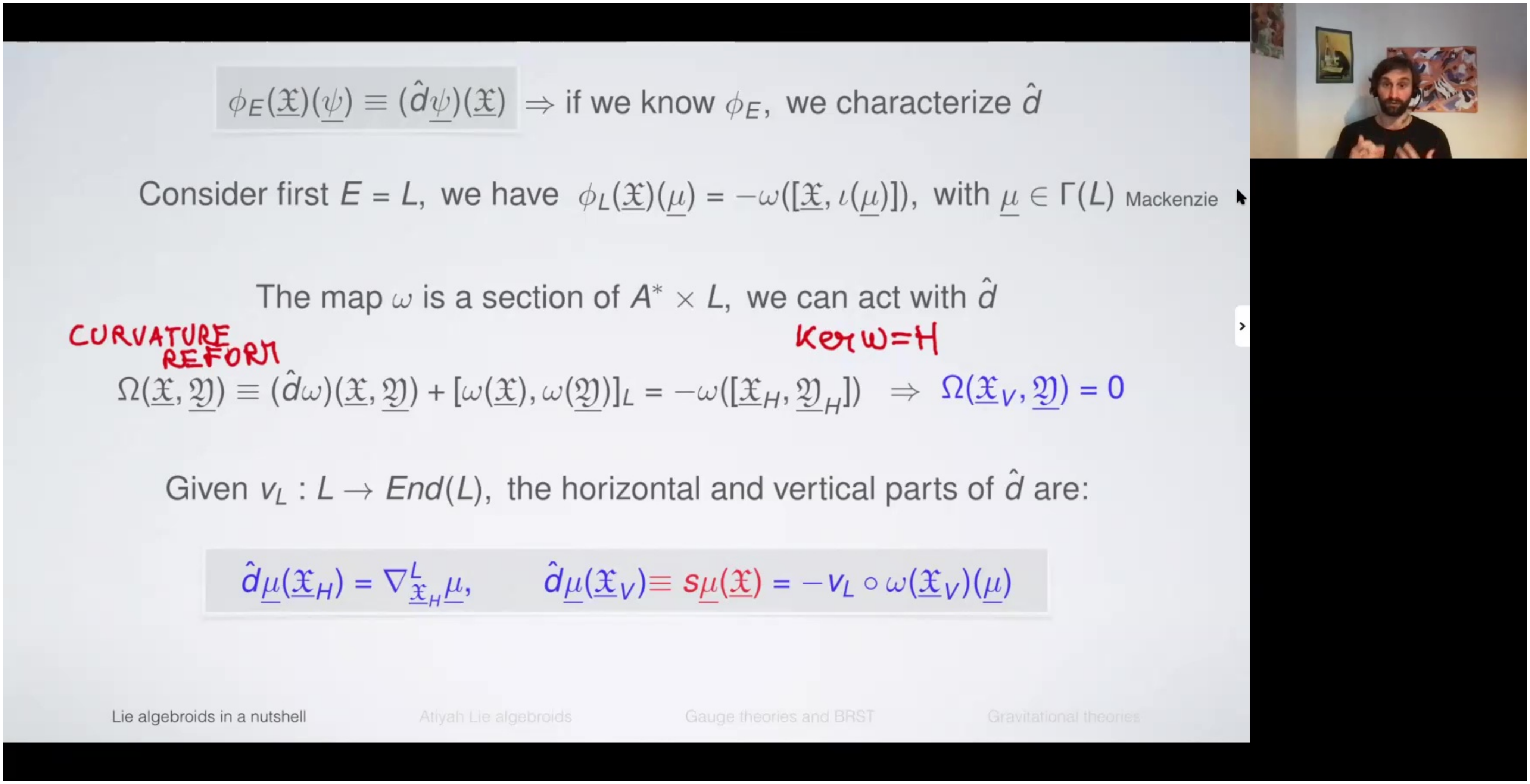
$\phi_E(\underline{\mathfrak{X}}) \in Der(E)$, given $\underline{\psi} \in \Gamma(E)$ we introduce $\hat{d}\underline{\psi} \in \Gamma(A^* \times E)$ as

$$\phi_E(\underline{\mathfrak{X}})(\underline{\psi}) \equiv (\hat{d}\underline{\psi})(\underline{\mathfrak{X}})$$

Define $\underline{\psi}_n \in \Gamma(\wedge^n A^* \times E) \equiv \Omega^n(A, E) \;\Rightarrow\; \Omega^\bullet(A, E) = \bigoplus \Omega^n(A, E),\; (\Omega^0(A, E) = \Gamma(E))$

$\hat{d}$ is generalized in $\Omega^\bullet(A, E)$ via the Koszul formula with ϕ_E morphism $\;\Rightarrow\; \hat{d}^2 = 0$

$$(\hat{d}\underline{\psi}_n)(\underline{\mathfrak{X}}_1, ..., \underline{\mathfrak{X}}_{n+1}) \equiv \sum_{r=1}^{n+1} (-1)^{r+1} \phi_E(\underline{\mathfrak{X}}_r)(\underline{\psi}_n(\underline{\mathfrak{X}}_1, ..., \widehat{\underline{\mathfrak{X}}_r}, ..., \underline{\mathfrak{X}}_{n+1}))$$
$$+ \sum_{r<s} (-1)^{r+s} \underline{\psi}_n([\underline{\mathfrak{X}}_r, \underline{\mathfrak{X}}_s]_A, \underline{\mathfrak{X}}_1, ..., \widehat{\underline{\mathfrak{X}}_r}, ..., \widehat{\underline{\mathfrak{X}}_s}, .., \underline{\mathfrak{X}}_{n+1})$$



$\phi_E(\underline{\mathfrak{X}})(\underline{\psi}) \equiv (\hat{d}\underline{\psi})(\underline{\mathfrak{X}})$ $\Rightarrow$ if we know ϕ_E, we characterize $\hat{d}$
Consider first $E = L$, we have $\phi_L(\underline{\mathfrak{X}})(\underline{\mu}) = -\omega([\underline{\mathfrak{X}}, \iota(\underline{\mu})])$, with $\underline{\mu} \in \Gamma(L)$ Mackenzie
The map ω is a section of $A^* \times L$, we can act with $\hat{d}$
CURVATURE REFORM
Ker ω=H
$\Omega(\underline{\mathfrak{X}}, \underline{\mathfrak{Y}}) \equiv (\hat{d}\omega)(\underline{\mathfrak{X}}, \underline{\mathfrak{Y}}) + [\omega(\underline{\mathfrak{X}}), \omega(\underline{\mathfrak{Y}})]_L = -\omega([\underline{\mathfrak{X}}_H, \underline{\mathfrak{Y}}_H]) \Rightarrow \Omega(\underline{\mathfrak{X}}_V, \underline{\mathfrak{Y}}) = 0$
Given $v_L : L \to End(L)$, the horizontal and vertical parts of $\hat{d}$ are:
$\hat{d}\underline{\mu}(\underline{\mathfrak{X}}_H) = \nabla^L_{\underline{\mathfrak{X}}_H}\underline{\mu}$, $\hat{d}\underline{\mu}(\underline{\mathfrak{X}}_V) \equiv s\underline{\mu}(\underline{\mathfrak{X}}) = -v_L \circ \omega(\underline{\mathfrak{X}}_V)(\underline{\mu})$
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BRST
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$\underline{\psi}_0 \in \Gamma(E)$

Generalized to an arbitrary vector bundle E:

$$\phi_E(\underline{\mathfrak{X}})(\underline{\psi}_0) = (\hat{d}\underline{\psi}_0)(\underline{\mathfrak{X}}) = \nabla^E_{\underline{\mathfrak{X}}_H}\underline{\psi}_0 \boxed{- v_E \circ \omega(\underline{\mathfrak{X}}_V)(\underline{\psi}_0)} = \nabla^E_{\underline{\mathfrak{X}}_H}\underline{\psi}_0 + s\underline{\psi}_0(\underline{\mathfrak{X}}_V)$$

$$= -c_E(\underline{\psi}_0)(\underline{\mathfrak{X}}_V)$$

Grassmann algebra ⟷ Vertical exterior algebra


Atiyah Lie Algebroids
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Gauge theories and BRST

Principal bundle $\pi : P \to M$, fibres=Lie group G, locally $P = U \times G$

2 group actions

Left action → Gauge transformations

Right action → Redundancy

The tangent bundle TP has 2 disadvantages

1) Right action redundancy

2) Bundle over P

Atiyah Lie algebroid TP/G: mod TP by the group right action

TP/G

~~Right redundancy~~ ← TP/G → Bundle over M

$$0 \longrightarrow L_G \xrightarrow{\iota} TP/G \xrightarrow{\pi_*} TM \longrightarrow 0$$

with $L_G \equiv P \times_{Ad_G} \mathfrak{g}$

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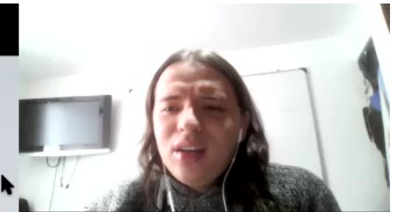

Atiyah Lie algebroid TP/G: mod TP by the group right action

TP/G

~~Right redundancy~~ ← TP/G → Bundle over M

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Equivariant Principal Connections ⟶ Ehresmann Connections

Gauge Theories and BRST

Physics vs Geometry

Physics	Geometry
Gauge group G	Principal bundle
Gauge connection	Ehresmann connection
Gauge curvature	Non integrability of H
Charged matter	Components of $\underline{\psi} \in \Gamma(E)$
Covariant derivative ∇^E	Horizontal part of $\hat{d}$ for $\Omega^\bullet(A, E)$
Fadeev-Popov ghost	Components of $c_E = v_E \circ \omega$
BRST operator s	Vertical part of $\hat{d}$ for $\Omega^\bullet(A, E)$

Indices and bases: $\underline{t}_A \in \Gamma(L)$, $\underline{E}_\alpha \in \Gamma(H)$, $\underline{E}_{\underline{A}} \in \Gamma(V)$, $\underline{\partial}_\mu \in \Gamma(TM)$, $\underline{e}_a \in \Gamma(E)$

$$0 \longrightarrow L \underset{\omega}{\overset{\iota}{\rightleftarrows}} A \underset{\sigma}{\overset{\rho}{\rightleftarrows}} TM \longrightarrow 0 \qquad A = TP/G$$

Consider ∇^L, the connection coefficients are $\nabla^L_{\underline{E}_\alpha} \underline{t}_A = A^B_{\alpha A} \underline{t}_B$

$L = P \times_{\mathrm{Ad}G} \mathfrak{g}$

Transition functions on L $\Rightarrow$ Adjoint group action $\underline{t}_A^{U_i} = g_{ij}{}^B{}_A \underline{t}_B^{U_j}$

$$(\nabla^L_{\underline{\mathfrak{X}}_H} \underline{\mu})_{U_i} \overset{!}{=} (\nabla^L_{\underline{\mathfrak{X}}_H} \underline{\mu})_{U_j} \text{ with } \underline{\mu} = \mu^A \underline{t}_A \in \Gamma(L) \Rightarrow A^B_{\alpha}{}_C = J^\beta{}_\alpha \left(g^{-1}\rho(\underline{E}_\beta)g + g^{-1}A_\beta g\right)^B{}_C$$

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$A = V \oplus H$

Trivialization: $\tau : A^U \to L^U \oplus TU$, with $\tau(\underline{\mathfrak{X}}_H) = X^\mu(\underline{\partial}_\mu + b_\mu^A \underline{t}_A)$, $\tau(\underline{\mathfrak{X}}_V) = X^A \underline{t}_A$

Gauge connection

τ is a morphism $\Leftrightarrow$ $b_\mu^A \leftrightarrow A_{\alpha}{}^{C}{}_{B}$

$$[\tau(\underline{\mathfrak{X}}_H), \tau(\underline{\mathfrak{Y}}_H)]_{L^U \oplus TU} = [\underline{X}, \underline{Y}]^\mu(\underline{\partial}_\mu + b_\mu^A \underline{t}_A) + X^\mu Y^\nu F_{\mu\nu}^A \underline{t}_A$$

With $F_{\mu\nu}^A = \partial_\mu b_\nu^A - \partial_\nu b_\mu^A + f_{BC}{}^A b_\mu^B b_\nu^C$ gauge curvature

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Gauge connection

τ is a morphism $\Leftrightarrow$ $b^A_\mu \leftrightarrow A^C_{\alpha B}$

$$[\tau(\underline{\mathfrak{X}}_H), \tau(\underline{\mathfrak{Y}}_H)]_{L\oplus TU} = [\underline{X}, \underline{Y}]^\mu(\underline{\partial}_\mu + b^A_\mu \underline{t}_A) + X^\mu Y^\nu F^A_{\mu\nu} \underline{t}_A$$

With $F^A_{\mu\nu} = \partial_\mu b^A_\nu - \partial_\nu b^A_\mu + f_{BC}{}^A b^B_\mu b^C_\nu$ gauge curvature

$F^A_{\mu\nu}$ controls integrability of Ehresmann connection

$$\omega = \omega^A{}_{\underline{A}} E^{\underline{A}} \otimes \underline{t}_A$$

$E^{\underline{A}}$ basis of V^*.

Contains the Fadeev-Popov ghost
$v_E \circ \omega = c^a{}_b \underline{e}_a \otimes f^b = c_E$

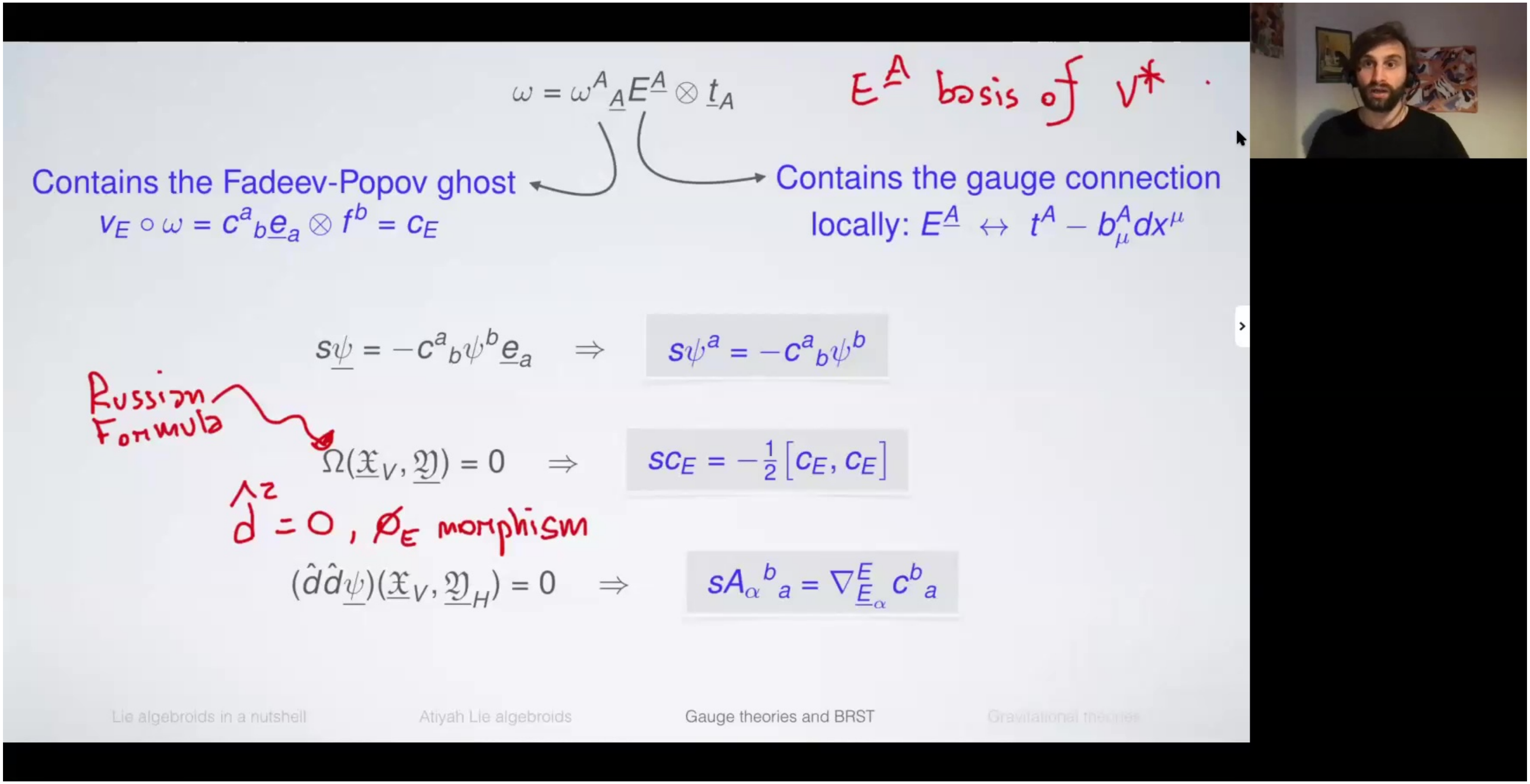
Contains the gauge connection
locally: $E^{\underline{A}} \leftrightarrow t^A - b^A_\mu dx^\mu$

$$s\underline{\psi} = -c^a{}_b \psi^b \underline{e}_a \quad \Rightarrow \quad \boxed{s\psi^a = -c^a{}_b \psi^b}$$

$$\Omega(\underline{\mathfrak{X}}_V, \underline{\mathfrak{Y}}) = 0 \quad \Rightarrow \quad \boxed{sc_E = -\tfrac{1}{2}\left[c_E, c_E\right]}$$

$\hat{d}^2 = 0$, ϕ_E morphism

$$(\hat{d}\hat{d}\underline{\psi})(\underline{\mathfrak{X}}_V, \underline{\mathfrak{Y}}_H) = 0 \quad \Rightarrow \quad \boxed{sA_\alpha{}^b{}_a = \nabla^E_{\underline{E}_\alpha} c^b{}_a}$$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locally: $E^{\underline{A}} \leftrightarrow t^A - b^A_\mu dx^\mu$
$s\underline{\psi} = -c^a{}_b \psi^b \underline{e}_a \Rightarrow s\psi^a = -c^a{}_b \psi^b$
Russian Formula
$\Omega(\underline{\mathfrak{X}}_V, \underline{\mathfrak{Y}}) = 0 \Rightarrow sc_E = -\frac{1}{2}[c_E, c_E]$
$\hat{d}^2 = 0$, ϕ_E morphism
$(\hat{d}\hat{d}\underline{\psi})(\underline{\mathfrak{X}}_V, \underline{\mathfrak{Y}}_H) = 0 \Rightarrow sA_\alpha{}^b{}_a = \nabla^E_{\underline{E}_\alpha} c^b{}_a$
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$$\omega = \omega^{A}{}_{\underline{A}} E^{\underline{A}} \otimes \underline{t}_A$$

$E^{\underline{A}}$ basis of V^*

Contains the Fadeev-Popov ghost

$$v_E \circ \omega = c^a{}_b \underline{e}_a \otimes f^b = c_E$$

Contains the gauge connection locally: $E^{\underline{A}} \leftrightarrow t^A - b^A_\mu dx^\mu$

$$s\underline{\psi} = -c^a{}_b \psi^b \underline{e}_a \quad \Rightarrow \quad s\psi^a = -c^a{}_b \psi^b$$

Russian Formula

$$\Omega(\underline{\mathfrak{X}}_V, \underline{\mathfrak{Y}}) = 0 \quad \Rightarrow \quad sc_E = -\tfrac{1}{2}\left[c_E, c_E\right]$$

$\hat{d}^2 = 0$, ϕ_E morphism

$$(\hat{d}\hat{d}\underline{\psi})(\underline{\mathfrak{X}}_V, \underline{\mathfrak{Y}}_H) = 0 \quad \Rightarrow \quad sA_\alpha{}^b{}_a = \nabla^E_{\underline{E}_\alpha} c^b{}_a$$

'80

$\hat{A} = A + c$

$\hat{F} = \hat{d}\hat{A}$

$\hat{d}^2 = 0$

$\hat{d} = d + s$

$\hat{F} = F$

Gravitational Theories

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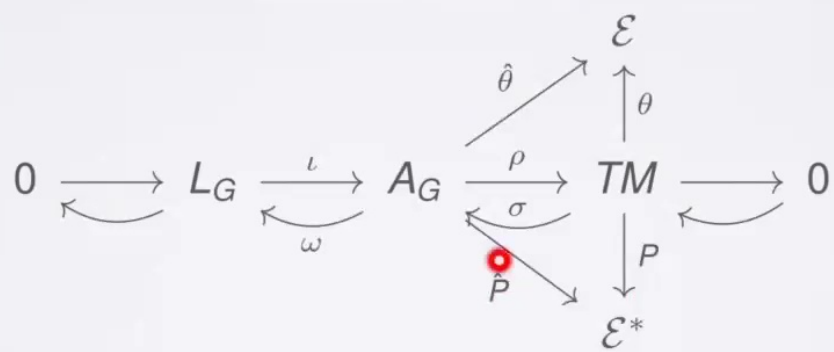
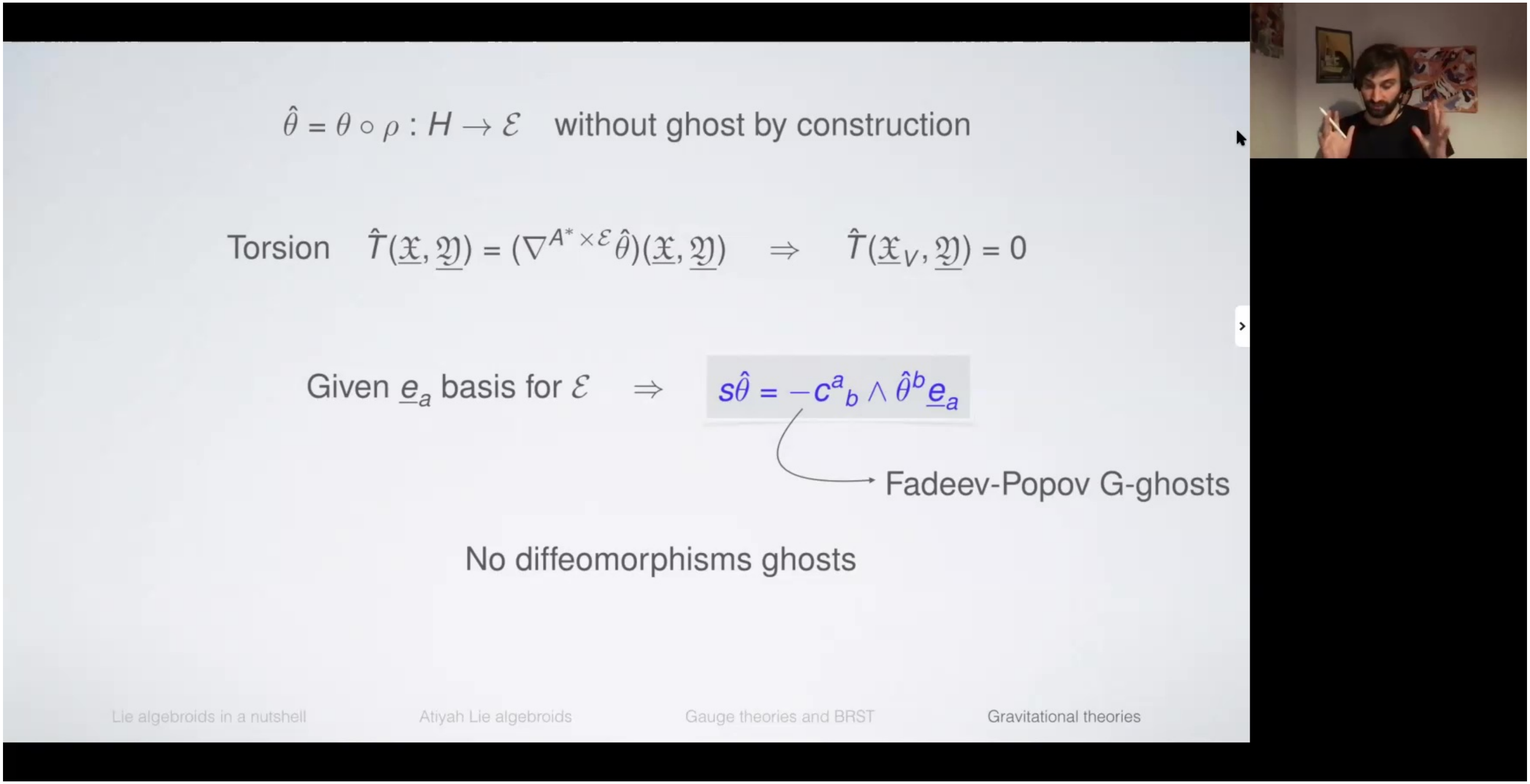
$A_G = TF_G/G, \quad F_G$ Principal Frame Bundle with Structure Group G

$G \subset GL(d, \mathbb{R})$, typically $SO(1, d-1)$ but it can be a non-Lorentzian G-structure

Solder and Schouten form $\theta : TM \to \mathcal{E},\ P : TM \to \mathcal{E}^*,\ \ \mathcal{E} = F_G \times_{R_G} \mathbb{V}$




$\hat{\theta} = \theta \circ \rho : H \to \mathcal{E}$ without ghost by construction
Torsion $\hat{T}(\underline{\mathfrak{X}}, \underline{\mathfrak{Y}}) = (\nabla^{A^* \times \mathcal{E}} \hat{\theta})(\underline{\mathfrak{X}}, \underline{\mathfrak{Y}}) \Rightarrow \hat{T}(\underline{\mathfrak{X}}_V, \underline{\mathfrak{Y}}) = 0$
Given $\underline{e}_a$ basis for $\mathcal{E} \Rightarrow$ $s\hat{\theta} = -c^a{}_b \wedge \hat{\theta}^b \underline{e}_a$
Fadeev-Popov G-ghosts
No diffeomorphisms ghosts
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Conclusions

Lie algebroids as off-shell geometric framework

No external assumptions, all built-in and gauge independent

Classical geometry and quantum symmetries

Outlooks

- Include dynamics
- Boundary and corners
- Geometric quantization of gravity
- Subregions and Entanglement