Title: General conditions for universality of Quantum Hamiltonians

Speakers: Tamara Kohler

Series: Perimeter Institute Quantum Discussions

Date: April 14, 2021 - 4:00 PM

URL: http://pirsa.org/21040020

Abstract: Recent work has defined what it means for one quantum system to simulate the full physics of another, and demonstrated thatâ€”within a very demanding definition of simulation â€”there exist families of local Hamiltonians that are universal, in the sense that they can simulate all other quantum Hamiltonians. This rigorous mathematical framework of Hamiltonian simulation not only gave a theoretical foundation for describing analogue Hamiltonian simulation. It also unified many previous Hamiltonian complexity results, and implied new ones. It has even found applications in constructing the first rigorous holographic dualities between local Hamiltonians, providing richer toy models of AdS/CFT duality in quantum gravity.

All previous constructions of universal Hamiltonians have relied heavily on using perturbation gadgets, and constructing complicated â€˜chainsâ€™ of simulations to prove that simple models are indeed universal. In recent work we developed a new method for proving universality. Unlike perturbation- gadget approaches, this directly leverages the ability to encode computation into the ground states of QMA-hard Hamiltonians. With this technique we are able to derive necessary and sufficient complexity-theoretic conditions characterising universal Hamiltonians. We also use our new simulation method to provide a simple construction of two new universal models. Both of these are translationally invariant systems in 1D, and we show that one of these constructions is efficient in terms of the number of spins in the universal construction (but not in terms of the norm of the simulating Hamiltonian). This is the first translationally invariant universal model which is efficient in terms of system size overhead.

Based on joint work with Stephen Piddock, Johannes Bausch and Toby Cubitt (arXiv:2003.13753, arXiv:2101.12319)

General conditions for universality of Quantum Hamiltonians

Tamara Kohler

Joint work with Stephen Piddock, Johannes Bausch and Toby Cubitt

Analogue Hamiltonian simulation

- Analogue Hamiltonian simulation - directly engineer the Hamiltonian of interest, and study its properties experimentally.
- It's known that there exist simple families of Hamiltonian which are 'universal'.

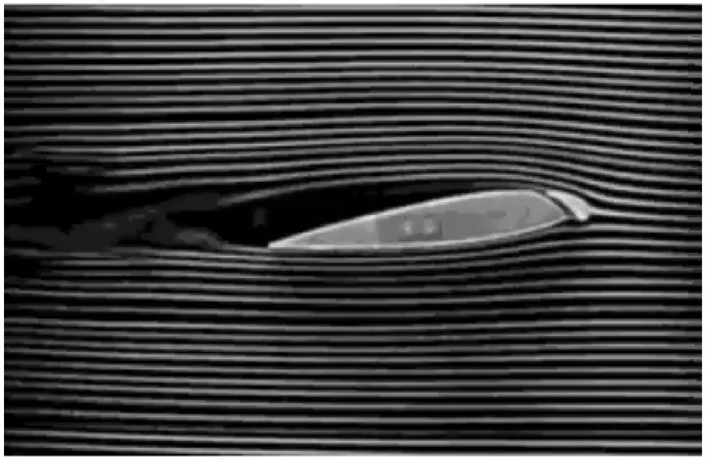


Figure: An aerofoil in a wind tunnel

Complexity and universality

In the classical case it's known that universality is closely linked to complexity [Cuevas and Cubitt 2016].

Type of interaction	*Universal for...*	*Complexity classification*
all classical	all classical	NP-complete (plus...)
2-qubit	all classical	NP-complete
2-qubit	stoquastic	StoqMA-complete
2-qubit	all	QMA-complete

Is there a general link in the quantum case?

Yes! But deriving it requires a new method for proving universality

General conditions for universality of quantum Hamiltonians

Theorem

A family of Hamiltonians, $\mathcal{M}$, is an efficient universal model iff its ground state energy problem is QMA-complete under faithful reductions, and $\mathcal{M}$ is closed.

- *Faithfulness*: requires that the reduction maps the subspace picked out by the verification circuit, to the low energy subspace of the Hamiltonian
- *Closure*: given $H_A^{(1)}, H_B^{(2)} \in \mathcal{M}$, there exists $H^{(3)} \in \mathcal{M}$ that can simulate $H_A^{(1)} + H_B^{(2)}$

Other key results

- A recipe for modifying history state Hamiltonians so the canonical reduction is faithful
- Complexity theoretic classifications of universal models which are not efficient, but still interesting
- Provide a construction of two new universal models - both translationally invariant systems in 1D
- One of these is efficient in terms of system size overhead

Outline

- Hamiltonian simulation & previous universality results
- Sketch of proof for general conditions for universality
- Sketch of proof for translationally invariant universal models in 1D
- Conclusions

6/ 22

Rigorous definition of (approximate) simulation

Consider two maps:

$$\mathcal{E}(M) = V(M \otimes P + \overline{M} \otimes Q)V^\dagger \text{ and } \tilde{\mathcal{E}}(M) = \tilde{V}(M \otimes P + \overline{M} \otimes Q)\tilde{V}^\dagger \quad (1)$$

where $\mathcal{E}(M)$ is 'local' (i.e. preserves tensor product structure) and P, Q are projectors, and $\|V - \tilde{V}\| \leqslant \eta$.

A Hamiltonian H' is a (Δ, η, ϵ)-simulation of the Hamiltonian H if:

- $S_{\tilde{\mathcal{E}}} = S_{\leqslant \Delta(H')}$
- $\|H'_{\leqslant \Delta} - \tilde{\mathcal{E}}(H)\| \leqslant \epsilon$

In [Cubitt, Montanaro, and Piddock 2018] it's shown that *approximate* simulation *approximately* preserves all physical quantities.

Previous universality results

A family of Hamiltonians, $\mathcal{M}$, is said to be universal if for any Hamiltonian H, there exists $H' \in \mathcal{M}$ which can (approximately) simulate H.

- In [Cubitt, Montanaro, and Piddock 2018] the authors prove that certain simple spin lattice models, including the 2D Heisenberg interaction, are universal simulators
- In [Piddock and Bausch 2020] the authors construct a translationally invariant universal Hamiltonian acting on a 2D lattice

All previous universality results use 'perturbation gadgets'.

Outline

- Hamiltonian simulation & previous universality results
- Sketch of proof for general conditions for universality
- Sketch of proof for translationally invariant universal models in 1D
- Conclusions

New method for proving universality

Consider a computational history state $|\Phi\rangle_{CQ} \in \mathcal{H}_C \otimes \mathcal{H}_Q$ of the form:

$$|\Phi\rangle_{CQ} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left|\phi^{(t)}\right\rangle |t\rangle ,$$

where $\{|t\rangle\}$ is an orthonormal basis for $\mathcal{H}_C$ and $\left|\phi^{(t)}\right\rangle = \Pi_{i=1}^{t} U_i |\phi_0\rangle$ for some initial state $|\phi_0\rangle \in \mathcal{H}_Q$ and set of unitaries $U_i \in \mathcal{B}(\mathcal{H}_Q)$.

Use a history state Hamiltonian to 'compute' the energy of a target Hamiltonian H_{target} in the form $E = a\sqrt{2} - b$ using the phase estimation algorithm:

$$H_{\mathrm{univ}} = \Delta H_{\mathrm{PE}} + T \sum_{i=0}^{N-1} \left(\sqrt{2}\Pi_a - \Pi_b\right) \tag{2}$$

where $\Delta \gg ||H_{\mathrm{target}}||$, T is the number of time steps in the computation

New method for proving universality

Before carrying out the computation the system 'idles' in its initial state for L time steps, so the ground state subspace of H_{PE} is:

$$|\Phi(\psi)\rangle = \frac{1}{\sqrt{T}}\left(|\psi\rangle \otimes \sum_{t=1}^{L'} |t\rangle + \sum_{t=L'+1}^{T} \left|\phi^{(t)}\right\rangle |t\rangle\right)$$

With high probability the simulator system is in the 'physical state' so the measurement statistics are simulated correctly.

More rigorously, the local encoding V and non-local encoding $\tilde{V}$ are given by:

$$V = \sum_i |i\rangle \otimes \sum_{t=1}^{L'} |t\rangle \langle i| \text{ and } \tilde{V} = \sum_i |\Phi(i)\rangle \langle i|$$

Idling to enhance to coherence has been used before - see [Aharonov et al. 2009] and [Aharonov and Zhou 2018].

Faithfulness condition

The acceptance operator, $Q(U)$, of a verification circuit, U, is:

$$Q(U) = \langle 0|^{\otimes m} U^{\dagger} \Pi_{\text{out}} U |0\rangle^{\otimes m} \quad (3)$$

We say a reduction from a promise problem to the GSE of a family of Hamiltonians, $\mathcal{M}$, is faithful with respect to the verification circuit U if for every YES instance there exists $H \in \mathcal{M}$ such that:

- $\|\Pi_{\mathcal{S}_0} - \Pi_{\mathcal{E}(\mathcal{L})}\| \leqslant \eta$ where S_0 is the low energy subspace of H, L is the subspace of $Q(U)$ with eigenvalue less than c, and $\mathcal{E}$ is a local encoding
- The gap in the spectrum of H above S_0 is $poly(g)$, where g is the gap in the spectrum of $Q(U)$ above $\mathcal{L}$

12/ 22

Proof that the conditions are necessary for universality

- *Closure* is clearly necessary.
- *GSE is QMA-complete under faithful reductions*. To show this is necessary we construct a family of Hamiltonian which is QMA-complete under faithful reductions, and use that any family of Hamiltonian which simulates this must itself be QMA-complete under faithful reductions
  - The family of Hamiltonians is a 'small-penalty' version of the canonical Kitaev history state Hamiltonian
  - The faithful reductions are the usual ones, with idling to enhance coherence.
  - Proof is via the Schrieffer-Wolf perturbative expansion

13/ 22

Proof that the conditions are sufficient for universality

Consider the problem, YES-Hamiltonian:

YES-Hamiltonian
Input: A k-local Hamiltonian, H
Output: YES

This problem is (clearly!) trivial, but we can choose a non-trivial verification circuit for it, that picks out a subspace which allows us to prove universality.

The subspace is:

$$\mathcal{W} = span\left\{ |w_\mu\rangle = \frac{1}{\sqrt{a^2+1}} |\psi_\mu\rangle \left(a|\#\rangle + |E_\mu\rangle \right) : H|\psi_\mu\rangle = E_\mu |\psi_\mu\rangle \right\}. \tag{4}$$

UCL

Proof that the conditions are sufficient for universality

Verifier circuit acts on witness registers (A and A'), and ancilla registers (B and B'):

$$\mathcal{W} = span\left\{ |w_\mu\rangle = \frac{1}{\sqrt{a^2+1}} |\psi_\mu\rangle \left(a|\#\rangle + |E_\mu\rangle \right) : H|\psi_\mu\rangle = E_\mu |\psi_\mu\rangle \right\}. \tag{5}$$

1. Apply a unitary rotation $P_a : |0\rangle \rightarrow \frac{1}{\sqrt{a^2+1}}(a|\#\rangle + |1\rangle)$ to the B' register.
2. Controlled on the B' register - use phase-estimation on A with respect to the unitary generated by H_{target} to estimate E_μ, store the result in B register
3. Carry out a SWAP test between registers A' and B

The entire procedure takes time $T = O(poly(n, d^k, \|H\|)/\epsilon)$.

Proof that the conditions are sufficient for universality

By the faithfulness condition, for any target Hamiltonian H, $\exists H_{yes} \in \mathcal{M}$ with low energy subspace S_0 such $\|\Pi_{\mathcal{S}_0} - \Pi_{\mathcal{E}(\mathcal{W})}\| \leqslant \eta$ for some local encoding $\mathcal{E}$.

Using this, along with closure, we can show that:

$$H_{sim} = \Delta H_{yes} + a \sum_{i=n'+1}^{N} 2^{i-(n'+1)} H_i^{(1)} \tag{6}$$

Is a simulation of $H' = \sum_{\mu} E_{\mu} |w_{\mu}\rangle \langle w_{\mu}|$.

We then explicitly construct encodings to show that H' simulates H.

Outline

- Hamiltonian simulation & previous universality results
- Sketch of proof for general conditions for universality
- Sketch of proof for translationally invariant universal models in 1D
- Conclusions

How do we encode a description of the target Hamiltonian in the simulator system?

Universal Hamiltonian:

$$H_{\mathrm{univ}}(x) = \Delta H_{\mathrm{PE}}(x) + T \sum_{i=0}^{N-1} \left(\sqrt{2}\Pi_a - \Pi_b \right) \tag{7}$$

Do we need a different H_{PE} for every Hamiltonian we want to simulate?

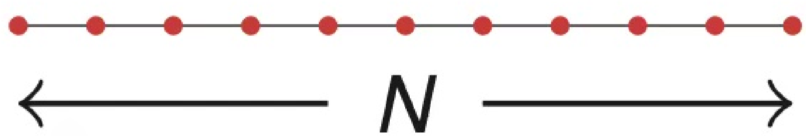
No!

We can encode a description of H_{target} into a property of the simulator system, and the computation can 'extract' this information prior to carrying out the phase estimation algorithm.

How do we encode a description of the target Hamiltonian in the simulator system?

Method 1: Binary counter

Key idea: Encode the description of the target Hamiltonian in the binary expansion of the length of the simulator spin chain [Gottesman and Irani 2009].



- Encode a binary counter Turing machine into the Hamiltonian H_{PE} before the phase estimation algorithm
- The output is the length of the spin chain, written in binary - a description of H_{target}
- The phase estimation takes this as input

How do we encode a description of the target Hamiltonian in the simulator system?

Method 2: Phase estimation (again!)

Key idea: Encode the description of the target Hamiltonian into the binary expansion of a phase in the Hamiltonian $e^{i\pi\phi(H_{\mathrm{target}})}A \in H_{\mathrm{PE}}$ [Cubitt, Perez-Garcia, and Wolf 2015]

- Encode a earlier phase estimation algorithm which computes the phase ϕ before the phase estimation algorithm to compute energy levels
- The output is ϕ, written in binary - a description of H_{target}
- The phase estimation takes this as input
- This model is efficient in terms of system size overhead

UCL

Outline

- Hamiltonian simulation & previous universality results
- Previous universality results
- Sketch of proof for general conditions for universality
- Sketch of proof for translationally invariant universal models in 1D
- Conclusions

Conclusions

- We've developed a new method to prove universality of quantum Hamiltonians, and used it to derive a rigorous connection between complexity and universality
- The new method has demonstrated universality of two new models, including the first translationally invariant model which is efficient in terms of system size overhead
- Can we use these techniques to develop physically realisable universal simulators?