Title: Supertranslation invariance of angular momentum

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Abstract: LIGO's successful detection of gravitational waves has revitalized the theoretical understanding of the angular momentum carried away by gravitational radiation. An infinite-dimensional supertranslation ambiguity has presented an essential difficulty for decades of study. Recent advances were made to quantify the supertranslation ambiguity in the context of binary coalescence. In this talk, we will present the first definition of angular momentum in general relativity that is completely free from supertranslation ambiguity. The new definition of angular momentum is derived from the limit of the quasilocal angular momentum we defined previously.

Supertranslation invariance of angular momentum

Po-Ning Chen

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April 1st, 2021
Perimeter Institute

Introduction

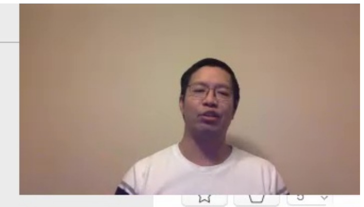
In today's talk, we will discuss the definition of angular momentum for a distant observer, namely, angular momentum at null infinity in general relativity.

In particular, how much angular momentum is carried away from a system during a gravitation radiation?

An essential difficulty is presented by the ambiguity of supertranslations, an infinite dimensional subgroup of the Bondi-Metzner-Sachs (BMS) group.

We will first quantize the supertranslation ambiguity of the classical definition of angular momentum and then describe a new definition of angular momentum which is supertranslation invariant.

Joint work with Jordan Keller, Mu-Tao Wang, Ye-Kai Wang and Shing-Tung Yau



Bondi–Sachs coordinate: The Ansatz

We consider a Bondi–Sachs coordinate system (u, r, x^2, x^3) in which the physical spacetime metric g takes the form

$$-UVdu^2 - 2Ududr + r^2 h_{AB}(dx^A + W^A du)(dx^B + W^B du), A, B = 2, 3.$$

where

$$V = 1 - \frac{2m}{r} + O(r^{-2}),$$

$$W^A = \frac{1}{2r^2}\nabla_B C^{AB} + \frac{1}{r^3}\left(\frac{2}{3}N^A - \frac{1}{16}\nabla^A|C|^2 - \frac{1}{2}C^{AB}\nabla^D C_{BD}\right) + O(r^{-4}),$$

$$h_{AB} = \sigma_{AB} + \frac{C_{AB}}{r} + O(r^{-2}).$$

Here $\sigma_{AB}(x)$ is a standard round metric on S^2 and ∇_A denotes the covariant derivative with respect to σ_{AB}.

Bondi–Sachs coordinate: The data

Bondi et al. observed this Ansatz greatly simplified the Einstein equation.

$$Ric(g) = 0$$

Namely, on a null cone $u = u_0$ (initial data), the Einstein equation becomes a system of nested ODE which determines the metric completely in terms of the free data:

h_{AB}, the metric on S^2, m, the mass aspect function, and N_A the angular momentum aspect function.

Moreover, the evolution of the Einstein equation is completely capture by the news tensor

$$N_{AB} = \partial_u C_{AB}$$

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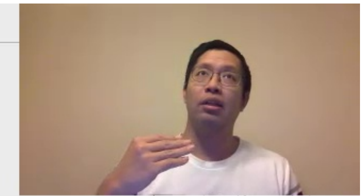

Bondi–Sachs coordinate: The evolution

Indeed, the news tensors capture all the information about the dynamics of a given spacetime. The Einstein equation implies that

$$
\begin{aligned}
\partial_u m &= -\frac{1}{8} N_{AB} N^{AB} + \frac{1}{4} \nabla^A \nabla^B N_{AB} \\
\partial_u N_A &= \nabla_A m - \frac{1}{4} \nabla^D (\nabla_D \nabla^E C_{EA} - \nabla_A \nabla^E C_{ED}) \\
&\quad + \frac{1}{4} \nabla_A (C_{BE} N^{BE}) - \frac{1}{4} \nabla_B (C^{BD} N_{DA}) + \frac{1}{2} C_{AB} \nabla_D N^{DB}.
\end{aligned}
$$

There is also a formulae for $\partial_u h_{AB}$ which we will omit here.

Energy and linear momentum at null infinity

The future null infinity $\mathscr{I}^+$ corresponds to the null hypersurface $r = \infty$.

It can be viewed $\mathscr{I}^+ = I \times S^2$ with coordinates (u, x) with a degenerate metric σ_{AB}.

In particular, any cross section $u = f(\theta, \phi)$ is round and indistinguishable from a level set of u.

The standard formulae for the Bondi-Sachs energy-momentum at a u cut is

$$E(u) = \int_{S^2} 2m(u, \cdot).$$

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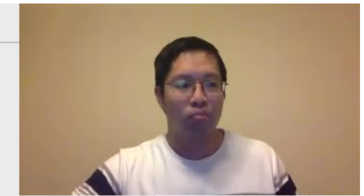
$$E(u) = \int_{S^2} 2m(u, \cdot).$$

$$P_k(u) = \int_{S^2} \tilde{x}^i m$$

$i = 1, 2, 3$, $\tilde{x}^i$ coordinate of unit sphere.



BMS group and supertranslation

The asymptotic symmetry of $\mathscr{I}^+$ is the Bondi-Metzner-Sachs (BMS) group.

These are the coordinate transformations at null infinity which keeps the Bondi–Sachs coordinate form.

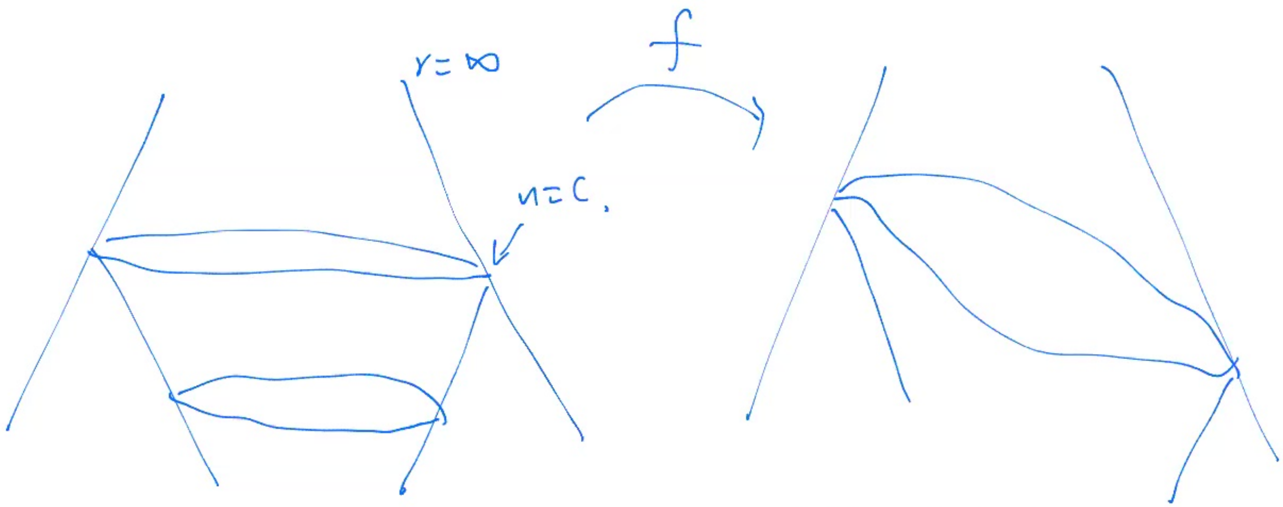
The BMS group consists of two type of transformation, the Poincare group and the supertranslation.

The Poincare group is the 10 dimensional, including time translation, space translations, rotation and boost.

A *supertranslation* is a change of Bondi-Sachs coordinates $(\bar{u}, \bar{x}) \to (u, x)$ such that

$$u = \bar{u} + f(\bar{x}), x = \bar{x} \tag{1}$$

on $\mathscr{I}^+$ for a function f that is defined on S^2.



(u, v, θ, ϕ)

$g_{uu}\ \ g_{uv}\ \cdots$

$C_{AB},\ N_{AB},\ m,\ N_A$

$(\bar{u}, \bar{v}, \theta, \phi)$

$g_{\bar{u}\bar{u}},\ g_{\bar{u},\bar{v}},\ \ldots$

$\bar{C}_{AB},\ \bar{N}_{AB},\ \bar{m},\ \bar{N}_A,$

The Bondi mass loss

We assume $\mathscr{I}^+$ extends from the spacelike infinity i^0 $(u = -\infty)$ to the future timelike infinity i^+ $(u = +\infty)$.

From the evolution of the mass aspect function, it follows that

$$\frac{d}{du}E(u) = -\frac{1}{4}\int_{S^2} |N|^2 \leq 0$$

This is often viewed as the first theoretical verification of gravitational radiation.

$$\delta E = E(\infty) - E(-\infty)$$

Total energy radiated away and its supertranslation invariance

From the Bondi mass loss formula, we can compute the total change of the energy

$$\delta E = E(\infty) - E(-\infty) = \int_{-\infty}^{\infty} \int_{S^2} |N|^2 dS^2 du$$

As mentioned before, there is no preferred Bondi–Sachs coordinate. What if we use a different coordinate $\bar{u}$ related to u by the supertranslation f?

If δE really captures the energy radiated away, we should have

$$\delta E = \delta E_f$$

Translation of the Bondi–Sachs data

It turns out the transformation of the mass aspect and shear tensors under supertranslation are rather complicated. However, the transformation of the news tensor is simple: For a translation $u = \bar{u} + f(x)$

$$
\begin{aligned}
\bar{N}_{AB}(\bar{u}, x) =& N_{AB}(\bar{u} + f, x) \\
\bar{C}_{AB}(\bar{u}, x) =& C_{AB}(\bar{u} + f, x) - 2\nabla_A \nabla_B f + \Delta f \sigma_{AB} \\
\bar{m}(\bar{u}, x) =& m(\bar{u} + f, x) + \frac{1}{2}(\nabla^B N_{BD})(\bar{u} + f, x)\nabla^D f \\
& + \frac{1}{4}(\partial_u N_{BD})(\bar{u} + f, x)\nabla^B f \nabla^D f + + \frac{1}{4} N_{BD}(\bar{u} + f, x)\nabla^B \nabla^D f
\end{aligned}
$$

The desired equality

$$\delta E = \delta E_f$$

follows from the "supertranslation invariance" of the news tensor.

Total energy radiated away and its supertranslation invariance

From the Bondi mass loss formula, we can compute the total change of the energy

$$\delta E = E(\infty) - E(-\infty) = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{S^2} |N|^2 dS^2 du$$

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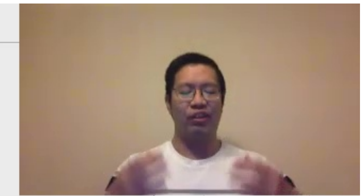
$$\delta E_f = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{S^2} |\bar{N}|^2 dS^2 d\bar{u}$$

Angular momentum at null infinity



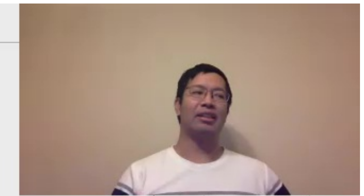
The definition of angular momentum turns out to be more subtle.

Different approaches (Hamiltonian, spinor-twistor, etc.) lead to different definition by Ashtekar–Hansen, Barnich–Troessaert, Bramson, Chrusciel–Jezierski–Kijowski, Dougan–Manson, Dray–Streubel, Hawking–Perry–Strominger, Ludvigsen–Vickers, Rizzi, Winicour-Tamburino, etc.

The Dray–Streubel angular momentum of a u cut is defined to be :

$$\tilde{J}(u, Y) = \int_{S^2} Y^A \left(N_A - \frac{1}{4} C_{AB} \nabla_D C^{DB} \right)(u, \cdot)$$

for a rotation Killing field Y. Fixing a rotation Killing field, denote this simply by $\tilde{J}(u)$.



Evolution and total change of the classical angular momentum

From the evolution of the angular momentum aspect, one can derive :

$$\frac{d}{du}\tilde{J}(u)$$
$$=\frac{1}{4}\int_{S^2}\left[\epsilon^{AE}\nabla_E\tilde{X}^k(C_{AB}\nabla_D N^{BD}-N_{AB}\nabla_D C^{BD})+\tilde{X}^k\epsilon^{AB}(C_A{}^D N_{DB})\right]$$

and similarly define

$$\delta\tilde{J}=\tilde{J}(\infty)-\tilde{J}(-\infty)$$

It is then given by

$$\frac{1}{4}\int_{-\infty}^{\infty}\int_{S^2}\left[\epsilon^{AE}\nabla_E\tilde{X}^k(C_{AB}\nabla_D N^{BD}-N_{AB}\nabla_D C^{BD})+\tilde{X}^k\epsilon^{AB}(C_A{}^D N_{DB})\right].$$

Supertransltion ambiguity of the angular momentum

According to Penrose (1982), the very concept of angular momentum gets shifted by these supertranslations and "it is hard to see in these circumstances how one can rigorously discuss such questions as the angular momentum carried away by gravitational radiation".

In other word, if one choose another Bondi-Sachs coordinates $(\bar{u}, \bar{x})$ related to (u, x) by a supertranslation f, Penrose worries that $\delta\tilde{J}$ may not be the same as $\delta\tilde{J}_f$.

We shall refer to the difference

$$\delta\tilde{J}_f - \delta\tilde{J}$$

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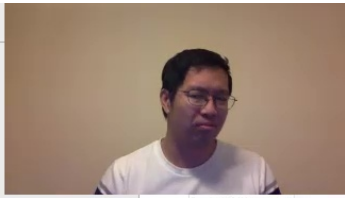

What exactly is the supertranslation ambiguity?

Theorem (C.–Keller–Wang–Wang–Yau)

Suppose the news tensor decays as

$$N_{AB}(u,x) = O(|u|^{-1-\varepsilon}) \text{ as } u \to \pm\infty$$

and two Bondi-Sachs coordinate systems are related by a supertranslation f. Then

$$\delta\tilde{J}_f - \delta\tilde{J} = -\int_{S^2} \left(2fY^A \nabla_A (m(+) - m(-))\right)$$

Here

$$m(\pm) = \lim_{u\to\pm\infty} m(u,x)$$

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Remark about the ambiguity

Remark about the ambiguity

In particular, if $Y^A \nabla_A (m(+) - m(-))$ has any $l \geq 2$ harmonic mode, then the total flux of the classical angular momentum can assume any value after applying a supertrnalstions.

How does the proof work?

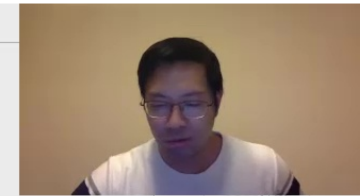
The angular momentum aspect transformed extremely complicated under a supertranslation. We use the evolution formula and how mass aspect, shear tensor and news tensor are transformed under a supertranslation.

The decay rates of the news tensor is used to justify the change of variable dealing with the integral in u (Fubini's theorem) as well as integration by parts in u.

A new definition of angular momentum at null infinity

We give a new definition of angular momentum at null infinity as follows:

1. Quasi-local angular momentum (C.–Wang–Yau 2015) based on the theory of quasi-local mass (Wang–Yau 2009).
2. Hamiltonian approach at the quasi-local level, using isometric embedding into the Minkowski space as ground state refernece
3. On each 2-surface Σ in spacetime, take the physical data $(\sigma, \mathbf{H})$
4. Solve the optimal embedding equation of Σ into the Minkowski space to obtain the reference term in the Hamiltonian theory.
5. Pull back the rotation Killing field of the Minkowski space to defines angular momentum on Σ
6. In a Bondi–Sachs coordinate, on a fixed u-slice, consider the 2-surfaces $r = r_0$ and let r_0 to ∞
7. Use the limit of the quasi-local angular momentum as the angular momentum at null infinity (Evaluated by Keller–Wang–Yau 2019)



The formula for CWY angular momentum at null infinity

In order to evaluate the angular momentum, we consider the decomposition of C_{AB} into

$$C_{AB} = \nabla_A \nabla_B c - \frac{1}{2}\sigma_{AB}\Delta c + \frac{1}{2}(\epsilon_A{}^E \nabla_E \nabla_B \underline{c} + \epsilon_B{}^E \nabla_E \nabla_A \underline{c}) \qquad (2)$$

where ϵ_{AB} denotes the volume form of σ_{AB}.

$c = c(u,x)$ and $\underline{c} = \underline{c}(u,x)$ are the closed and co-closed potentials of $C_{AB}(u,x)$.

The CWY angular momentum associated to a rotation Killing field is

$$J(u,Y) = \int_{S^2} Y^A \left(N_A - \frac{1}{4} C_{AB} \nabla_D C^{DB} - c \nabla_A m \right)(u,\cdot)$$

One should note that c has never occurred in any previous definition of angular momentum. It is a "non-local term"

Theorem (C.–Keller–Wang–Wang–Yau)

Suppose the news tensor decays as

$$N_{AB}(u,x) = O(|u|^{-1-\varepsilon}) \text{ as } u \to \pm\infty.$$

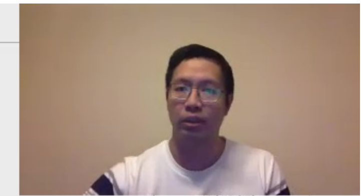
Then the total flux of J^k is supertranslation invariant.

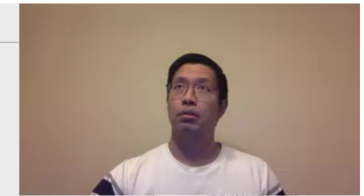
Remark

In the above statement, supertranslation invariant means that it is equivariant under ordinary ($l = 1$) translation and is invariant under higher mode ($l \geq 2$) of the supertranslation.

Remark about the result

More precisely, we showed that





Remark about the result

More precisely, we showed that

$$\delta J_f - \delta J = -\int_{S^2} \left(2 f_{l \leq 1} Y^A \nabla_A (m(+) - m(-)) \right)$$

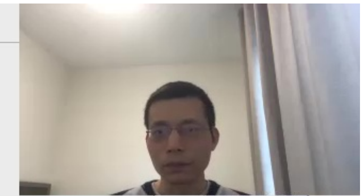
Here we decompose f into harmonic modes

$$f = f_{l \leq 1} + f_{l \geq 2}.$$

We can see that δJ is invariant under any higher mode ($l \geq 2$) of the supertranslation.

For an ordinary translation $f = \alpha_0 + \alpha_i \tilde{X}^i$, it follows that

$$\delta J_f - \delta J = \alpha_i \epsilon^{ik}{}_j \delta P^j$$



Thank you!

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