Title: Can We Understand the Standard Model Using Octonions?

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Collection: Octonions and the Standard Model

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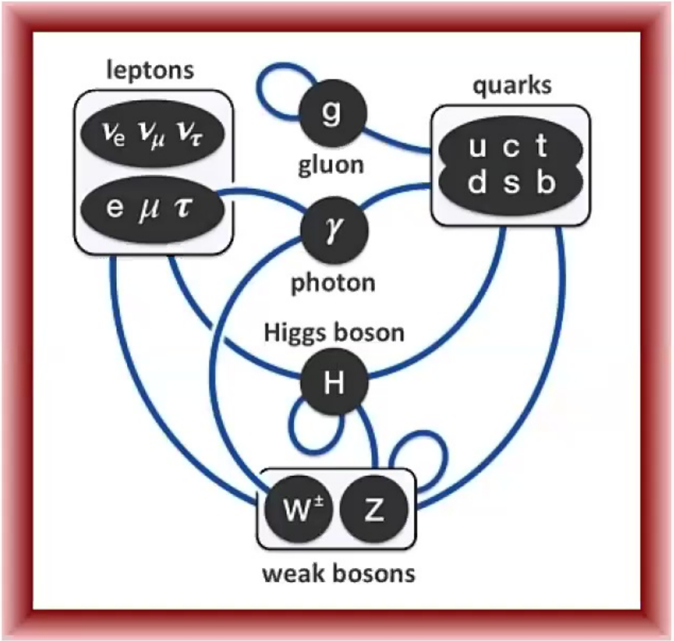
Abstract: Dubois-Violette and Todorov have shown that the Standard Model gauge group can be constructed using the exceptional Jordan algebra, consisting of 3Ã—3 self-adjoint matrices of octonions. After an introduction to the physics of Jordan algebras, we ponder the meaning of their construction. For example, it implies that the Standard Model gauge group consists of the symmetries of an octonionic qutrit that restrict to symmetries of an octonionic qubit and preserve all the structure arising from a choice of unit imaginary octonion. It also sheds light on why the Standard Model gauge group acts on 10d Euclidean space, or Minkowski spacetime, while preserving a 4+6 splitting.

CAN WE UNDERSTAND THE STANDARD MODEL USING OCTONIONS?



John Baez
Octonions and the Standard Model
5 April 2021

Can we derive the Standard Model — or something close — from reasonable principles?

The internal degrees of freedom — hypercharge, isospin, color — seem to be described by algebras of observables connected to representations of $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$. Why this particular group, and these representations?

Connes and others have tried to answer this using noncommutative geometry, for example:

- Ali Chamseddine and Alain Connes, Why the Standard Model?

I'll present some much more tentative thoughts involving octonions and Jordan algebras.

Jordan algebras are a framework for dealing with observables in quantum physics. The exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ plays a unique role. It's the algebra of observables of an "octonionic qutrit".

Following ideas of Dubois-Violette and Todorov, we'll see that the true gauge group of the Standard Model, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$, consists of the symmetries of an octonionic qutrit that

1. preserve all the structure arising from a choice of unit imaginary octonion $i \in \mathbb{O}$

and

2. restrict to give symmetries of an octonionic qubit.

But let's start at the beginning: what can we do with observables?

For example, suppose "observables" are self-adjoint complex matrices, $A \in \mathfrak{h}_n(\mathbb{C})$.

We can take real-linear combinations of them.

The product of two self-adjoint matrices is not self-adjoint, but the *square* of a self-adjoint matrix is self-adjoint. From squaring and linear combinations we can define the **Jordan product**

$$a \circ b = \frac{1}{2}((a+b)^2 - a^2 - b^2) = \frac{1}{2}(ab + ba).$$

This product is commutative. It is not associative, but it is **power-associative**: any way of parenthesizing a product of copies of the same observable gives the same result.

Jordan, von Neumann and Wigner turned these ideas into a definition:

A **Euclidean Jordan algebra** is a real vector space with a bilinear, commutative and power-associative product satisfying

$$a_1^2 + \cdots + a_n^2 = 0 \implies a_1 = \cdots = a_n = 0$$

for all n.

Jordan, von Neumann and Wigner proved:

Theorem. Every finite-dimensional Euclidean Jordan algebra is isomorphic to a direct sum of ones on this list:

- $\mathfrak{h}_n(\mathbb{R})$: $n \times n$ self-adjoint real matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{C})$: $n \times n$ self-adjoint complex matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{H})$: $n \times n$ self-adjoint quaternionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- $\mathfrak{h}_n(\mathbb{O})$: $n \times n$ self-adjoint octonionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$, where $n \leq 3$.
- The **spin factor** $\mathbb{R} \oplus \mathbb{R}^n$, with

$$(t, \vec{x}) \circ (t', \vec{x}') = (tt' + \vec{x} \cdot \vec{x}', t\vec{x}' + t'\vec{x}).$$

What about the spin factors?

Every Euclidean Jordan algebra J comes with a cone of **nonnegative** elements:

$$J_+ = \{a_1^2 + \cdots + a_n^2 : \ a_i \in J\}$$

For the spin factor $\mathbb{R} \oplus \mathbb{R}^n$ this cone is isomorphic to the future cone in $(n+1)$-dimensional Minkowski spacetime!

Every Euclidean Jordan algebra automatically comes with a **determinant** function $\det\colon J \to \mathbb{R}$ that vanishes on the boundary of J_+. For the spin factor this is the Minkowski metric!

$$\det(t, \vec{x}) = t^2 - \vec{x} \cdot \vec{x}$$

So, spin factors are not only algebras of observables. They are also Minkowski spacetimes!

Jordan algebras of 2×2 self-adjoint matrices are isomorphic to spin factors:

$$\begin{array}{lcccr} \mathfrak{h}_2(\mathbb{R}) & \cong & \mathbb{R} \oplus \mathbb{R}^2 & \cong & \text{3d Minkowski spacetime} \\ \mathfrak{h}_2(\mathbb{C}) & \cong & \mathbb{R} \oplus \mathbb{R}^3 & \cong & \text{4d Minkowski spacetime} \\ \mathfrak{h}_2(\mathbb{H}) & \cong & \mathbb{R} \oplus \mathbb{R}^5 & \cong & \text{6d Minkowski spacetime} \\ \mathfrak{h}_2(\mathbb{O}) & \cong & \mathbb{R} \oplus \mathbb{R}^9 & \cong & \text{10d Minkowski spacetime} \end{array}$$

$$\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = t^2 - x^2 - |y|^2$$

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\mathfrak{h}_2(\mathbb{H}) & \cong & \mathbb{R} \oplus \mathbb{R}^5 & \cong & \text{6d Minkowski spacetime} \\
\mathfrak{h}_2(\mathbb{O}) & \cong & \mathbb{R} \oplus \mathbb{R}^9 & \cong & \text{10d Minkowski spacetime}
\end{array}
$$

$$
\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = t^2 - x^2 - |y|^2
$$

How can we understand this?

A Euclidean Jordan algebra does not merely describe observables. It also describes states.

Any Euclidean Jordan algebra automatically comes with a **trace** $\mathrm{tr}: J \to \mathbb{R}$. An element $s \in J_+$ with $\mathrm{tr}(s) = 1$ is called a **state**.

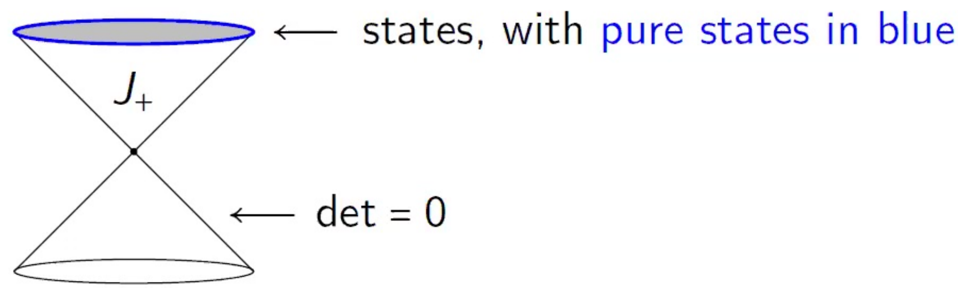
Given a state s and an observable a, the **expected value** of a in the state s is $\mathrm{tr}(s \circ a)$.

A **projection** $p \in J$ is an element with $p^2 = p$. A projection p with $\mathrm{tr}(p) = 1$ is a state called a **pure state**.

For $J = \mathfrak{h}_n(\mathbb{C})$, all this is familiar. Here a state is just a density matrix: a non-negative self-adjoint matrix with trace 1.

- The space of pure states for $\mathfrak{h}_n(\mathbb{R})$ is $\mathbb{R}\mathrm{P}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{C})$ is $\mathbb{C}\mathrm{P}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{H})$ is $\mathbb{H}\mathrm{P}^{n-1}$.
- The space of pure states for $\mathfrak{h}_n(\mathbb{O})$ is $\mathbb{O}\mathrm{P}^{n-1}$ (for $n \leq 3$).
- The space of pure states for $\mathbb{R} \oplus \mathbb{R}^n$ is S^{n-1}.

A picture of the spin factor $\mathbb{R} \oplus \mathbb{R}^n$ for $n = 2$:



In physics, *observables should generate symmetries.*

Of the Euclidean Jordan algebras on the list, only $\mathfrak{h}_n(\mathbb{C})$ can be made into a Lie algebra that acts nontrivially as derivations of the Jordan product:

$$a, b \in \mathfrak{h}_n(\mathbb{C}) \implies \{a, b\} := i(ab - ba) \in \mathfrak{h}_n(\mathbb{C})$$

$$\{a, b \circ c\} = \{a, b\} \circ c + b \circ \{a, c\}$$

- John Baez, Getting to the bottom of Noether's theorem.

Thus $\mathfrak{h}_n(\mathbb{C})$ is favored. But $\mathfrak{h}_n(\mathbb{R})$ and $\mathfrak{h}_n(\mathbb{H})$ actually *do* play a role in ordinary quantum mechanics:

- John Baez, Division algebras and quantum theory.

Remember from my last talk: choosing a unit imaginary octonion $i \in \mathbb{O}$ gives an inclusion

$$\mathbb{C} \hookrightarrow \mathbb{O}$$

and thus a splitting

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^\perp$$

and a complex structure on $\mathbb{C}^\perp$, from left multiplication by i.

It also gives an inclusion

$$\mathfrak{h}_2(\mathbb{C}) \hookrightarrow \mathfrak{h}_2(\mathbb{O})$$

and a splitting

$$\underbrace{\mathfrak{h}_2(\mathbb{O})}_{\text{10d}} \quad = \quad \underbrace{\mathfrak{h}_2(\mathbb{C})}_{\text{4d}} \quad \oplus \quad \underbrace{\mathbb{C}^\perp}_{\text{6d}}$$

$\mathfrak{h}_2(\mathbb{O})$ naturally has the structure of *both* a 10d Minkowski spacetime:

$$\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = t^2 - x^2 - |y|^2$$

and a 10d Euclidean space:

$$\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \circ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \right) = t^2 + x^2 + |y|^2$$

det and tr can both be defined in terms of $\circ$. Thus, the automorphism group of the Jordan algebra $\mathfrak{h}_2(\mathbb{O})$ must be contained in

$$\mathrm{O}(9,1) \cap \mathrm{O}(10) = \mathrm{O}(9)$$

This resolves the "Euclidean or Minkowskian?" puzzle from last time.

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More simply, since $\mathfrak{h}_2(\mathbb{O}) \cong \mathbb{R} \oplus \mathbb{R}^9$ with Jordan product

$$(t, \vec{x}) \circ (t', \vec{x}') = (tt' + \vec{x} \cdot \vec{x}', t\vec{x}' + t'\vec{x})$$

the automorphism group of $\mathfrak{h}_2(\mathbb{O})$ is exactly $\mathrm{O}(9)$.

The double cover of the identity component of $\mathrm{O}(9)$ is $\mathrm{Spin}(9)$.

The subgroup of $\mathrm{Spin}(9)$ preserving $\mathfrak{h}_2(\mathbb{C}) \subset \mathfrak{h}_2(\mathbb{O})$ is

$$(\mathrm{Spin}(3) \times \mathrm{Spin}(6))/\mathbb{Z}_2 \quad \cong \quad (\mathrm{SU}(2) \times \mathrm{SU}(4))/\mathbb{Z}_2$$

This contains a copy of the true gauge group of the Standard Model!

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There is thus a 6-1 homomorphism

$$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \to (\mathrm{SU}(2) \times \mathrm{SU}(4))/\mathbb{Z}_2$$

So: $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ acts as Jordan algebra automorphisms of $\mathfrak{h}_2(\mathbb{O})$ preserving $\mathfrak{h}_2(\mathbb{C})$.

Put more dramatically: the true gauge group of the Standard Model acts as symmetries of an octonionic qubit, and preserves the subalgebra of observables of a complex qubit.

That sounds impressive, but it leaves open two big questions:

A. While $\mathrm{Spin}(9)$ acts on $\mathfrak{h}_2(\mathbb{O})$, the automorphism group of $\mathfrak{h}_2(\mathbb{O})$ is actually $\mathrm{O}(9)$. Why work with $\mathrm{Spin}(9)$?

B. $(\mathrm{Spin}(3) \times \mathrm{Spin}(6))/\mathbb{Z}_2$ is the subgroup of $\mathrm{Spin}(9)$ preserving $\mathfrak{h}_2(\mathbb{C})$. What picks out the smaller subgroup $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$?

$\mathfrak{h}_3(\mathbb{O})$ is the Jordan algebra of observables of an "octonionic qutrit":

$$\mathfrak{h}_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R}, \ x, y, z \in \mathbb{O} \right\}$$

The automorphism group of $\mathfrak{h}_3(\mathbb{O})$ is the 52-dimensional compact Lie group F_4.

F_4 cannot act on $\mathbb{O}^3$ in any nontrivial way: its smallest nontrivial representation is 26-dimensional. There is thus no "Hilbert space" picture of the octonionic qutrit.

Pick any copy of $\mathfrak{h}_2(\mathbb{O})$ sitting inside $\mathfrak{h}_3(\mathbb{O})$ as a Jordan subalgebra, e.g.:

$$\mathfrak{h}_2(\mathbb{O}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & x \\ 0 & x^* & \gamma \end{pmatrix} : \beta, \gamma \in \mathbb{R},\ x \in \mathbb{O} \right\}$$

The subgroup of F_4 preserving this is $\mathrm{Spin}(9)$.

This answers question A: "why $\mathrm{Spin}(9)$ instead of $\mathrm{O}(9)$?"

Don't work with automorphisms of $\mathfrak{h}_2(\mathbb{O})$, which form the group $\mathrm{O}(9)$. Work with automorphisms of $\mathfrak{h}_3(\mathbb{O})$ that map $\mathfrak{h}_2(\mathbb{O}) \subset \mathfrak{h}_3(\mathbb{O})$ to itself. These form the group $\mathrm{Spin}(9)$.

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Why do we get $\mathrm{Spin}(9)$?

As representations of $\mathrm{Spin}(9)$ we have

$$\mathfrak{h}_3(\mathbb{O}) \cong \mathbb{R} \oplus \mathfrak{h}_2(\mathbb{O}) \oplus \mathbb{O}^2$$

$$\left(\begin{array}{c|cc} \alpha & z & y^* \\ \hline z^* & \beta & x \\ y & x^* & \gamma \end{array}\right) = \left(\begin{array}{cc} \alpha & \psi^\dagger \\ \psi & v \end{array}\right) \longmapsto (\alpha, v, \psi)$$

Here $\mathrm{Spin}(9)$ acts on $\mathbb{R}$ trivially, on $\mathbb{O}^2$ via the real spinor representation, and on $\mathfrak{h}_2(\mathbb{O})$ as before: it's (9+1)d spacetime, or 10d space.

The Jordan product on $\mathfrak{h}_3(\mathbb{O})$ can be described using $\mathrm{Spin}(9)$-invariant operations on $\mathbb{R}$, $\mathbb{O}^2$ and $\mathfrak{h}_2(\mathbb{O})$. Only $\mathrm{Spin}(9)$ preserves all these operations.

Now we can answer question B: "what picks out the Standard Model gauge group as a subgroup of $\mathrm{Spin}(9)$?"

- First, choose a copy of $\mathfrak{h}_2(\mathbb{O})$ in $\mathfrak{h}_3(\mathbb{O})$. The subgroup of F_4 preserving this is $\mathrm{Spin}(9)$.
- Next, choose a unit imaginary octonion $i \in \mathbb{O}$. The subgroup of F_4 preserving all the structure this puts on $\mathfrak{h}_3(\mathbb{O})$ is

$$\frac{\mathrm{SU}(3) \times \mathrm{SU}(3)}{\mathbb{Z}_3}$$

- The subgroup of F_4 preserving *all* the above structure is the true gauge group of the Standard Model:

$$\frac{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(3)}{\mathbb{Z}_6} = \frac{\mathrm{SU}(3) \times \mathrm{SU}(3)}{\mathbb{Z}_3} \cap \mathrm{Spin}(9)$$

In short, the true gauge group of the Standard Model consists of precisely the symmetries of an octonionic qutrit that

1. preserve all the structure arising from a choice of unit imaginary octonion $i \in \mathbb{O}$

and

2. restrict to give symmetries of an octonionic qubit.

But let's see how this works in more detail.

If we choose a unit imaginary octonion $i \in \mathbb{O}$, we get an inclusion $\mathbb{C} \hookrightarrow \mathbb{O}$ and thus an inclusion

$$\mathfrak{h}_3(\mathbb{C}) \hookrightarrow \mathfrak{h}_3(\mathbb{O})$$

and a splitting

$$\mathfrak{h}_3(\mathbb{O}) \quad = \quad \mathfrak{h}_3(\mathbb{C}) \quad \oplus \quad \mathfrak{h}_3(\mathbb{C})^\perp$$

where

$$\begin{aligned} \mathfrak{h}_3(\mathbb{C})^\perp &= \{a \in \mathfrak{h}_3(\mathbb{O}) : \mathrm{tr}(a \circ x) = 0 \text{ for all } x \in \mathfrak{h}_3(\mathbb{C})\} \\ &= \left\{ \begin{pmatrix} 0 & z & y^* \\ z^* & 0 & x \\ y & x^* & 0 \end{pmatrix} : \; x, y, z \in \mathbb{C}^\perp \subset \mathbb{O} \right\} \end{aligned}$$

gets a complex structure from left multiplication by i.

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gets a complex structure from left multiplication by i.

Theorem. For any choice of unit imaginary octonion $i \in \mathbb{O}$, the subgroup of F_4 that preserves the resulting splitting

$$\mathfrak{h}_3(\mathbb{O}) \quad = \quad \mathfrak{h}_3(\mathbb{C}) \quad \oplus \quad \mathfrak{h}_3(\mathbb{C})^\perp$$

and complex structure on $\mathfrak{h}_3(\mathbb{C})^\perp$ is isomorphic to

$$\frac{\mathrm{SU}(3) \times \mathrm{SU}(3)}{\mathbb{Z}_3}$$

Proof. This follows, with some work, from Theorem 2.12.2 in

- Ichiro Yokota, Exceptional Lie groups.

But let's see how $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ acts.

$\mathbb{C}^\perp \subset \mathbb{O}$ is a 3d complex vector space. Choosing an isomorphism $\mathbb{C}^\perp \cong \mathbb{C}^3$ we get

$$\begin{aligned}
\mathfrak{h}_3(\mathbb{C})^\perp &= \left\{ \begin{pmatrix} 0 & z & y^* \\ z^* & 0 & x \\ y & x^* & 0 \end{pmatrix} : \; x, y, z \in \mathbb{C}^\perp \subset \mathbb{O} \right\} \\
&\cong \{(x, y, z) : \; x, y, z \in \mathbb{C}^\perp\} \\
&\cong \mathrm{M}_3(\mathbb{C})
\end{aligned}$$

where $\mathrm{M}_3(\mathbb{C})$ is the space of 3×3 complex matrices.

We thus get an isomorphism

$$\begin{aligned}
\mathfrak{h}_3(\mathbb{O}) &= \mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{h}_3(\mathbb{C})^\perp \\
&\cong \mathfrak{h}_3(\mathbb{C}) \oplus \mathrm{M}_3(\mathbb{C})
\end{aligned}$$

The two $\mathrm{SU}(3)$'s in $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$ act very differently on $\mathfrak{h}_3(\mathbb{O})$.

The second $\mathrm{SU}(3)$ becomes the strong force $\mathrm{SU}(3)$: it acts *separately* on each matrix entry

$$\begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix}$$

as octonion automorphisms that preserve $i \in \mathbb{O}$.

The first $\mathrm{SU}(3)$ acts to *mix up* the matrix entries, and only the electroweak group $(\mathrm{U}(1) \times \mathrm{SU}(2))/\mathbb{Z}_2 \subset \mathrm{SU}(3)$ preserves

$$\mathfrak{h}_2(\mathbb{O}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & x \\ 0 & x^* & \gamma \end{pmatrix} : \ \beta, \gamma \in \mathbb{R}, \ x \in \mathbb{O} \right\} \subset \mathfrak{h}_3(\mathbb{O})$$

Using this idea one can show:

Theorem. Choose a unit imaginary octonion $i \in \mathbb{O}$, giving a Jordan subalgebra

$$\mathfrak{h}_3(\mathbb{C}) \subset \mathfrak{h}_3(\mathbb{O})$$

Also choose a Jordan subalgebra

$$\mathfrak{h}_2(\mathbb{O}) \subset \mathfrak{h}_3(\mathbb{O})$$

The group of automorphisms of the Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ that preserve

- the splitting $\mathfrak{h}_3(\mathbb{O}) = \mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{h}_3(\mathbb{C})^\perp$
- the complex structure on $\mathfrak{h}_3(\mathbb{C})^\perp$
- the Jordan subalgebra $\mathfrak{h}_2(\mathbb{O})$

is isomorphic to the true gauge group of the Standard Model, $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$.

Summary and Speculations

The true gauge group of the Standard Model consists of the automorphisms of $\mathfrak{h}_3(\mathbb{O})$ that

1. preserve all the structure coming from a unit imaginary octonion $i \in \mathbb{O}$

and

2. preserve a copy of $\mathfrak{h}_2(\mathbb{O})$ in $\mathfrak{h}_3(\mathbb{O})$.

These symmetries simultaneously act as symmetries of:

- an octonionic qutrit: $\mathfrak{h}_3(\mathbb{O})$
- an octonionic qubit: $\mathfrak{h}_2(\mathbb{O})$
- a complex qutrit: $\mathfrak{h}_3(\mathbb{C})$
- a complex qubit: $\mathfrak{h}_2(\mathbb{C})$.

Maybe this is all just a coincidence. Maybe not!

$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ is precisely the subgroup of $\mathrm{Spin}(9)$ whose action on $\mathbb{O}^2$ commutes with right multiplication by $i \in \mathbb{O}$:

- Kirill Krasnov, SO(9) characterisation of the Standard Model gauge group.

$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$ acts on $\mathbb{O}^2$ with this complex structure precisely as it does on the *left-handed* fermions in one generation.

If an "octonionic qutrit" is relevant to physics, what is it? $\mathfrak{h}_3(\mathbb{O})$ acts as operators on $\mathbb{O}^3$. But F_4 does not act on $\mathbb{O}^3$, only on $\mathfrak{h}_3(\mathbb{O})$ (observables) and $\mathbb{O}\mathrm{P}^2$ (pure states).

The "octonionic qubit" is less mysterious. The Standard Model gauge group

$$\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(3)) \quad \subset \quad \mathrm{Spin}(9) \quad \subset \quad \mathrm{F}_4$$

acts on $\mathfrak{h}_3(\mathbb{O})$, but also on $\mathfrak{h}_2(\mathbb{O})$. It also acts on $\mathbb{O}^2$ via the spinor representation of $\mathrm{Spin}(9)$. This is our octonionic qubit.

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So, the *left-handed* fermions in one generation can be seen as an octonionic qubit with a certain complex structure — but the octonionic qutrit remains mysterious.