Title: TBA

Speakers: Spyros Alexakis

Series: Strong Gravity

Date: March 25, 2021 - 1:00 PM

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Abstract: Abstract: TBD

Singularity formation in Black hole interiors.

Spyros Alexakis

Ioannina, March 2021.

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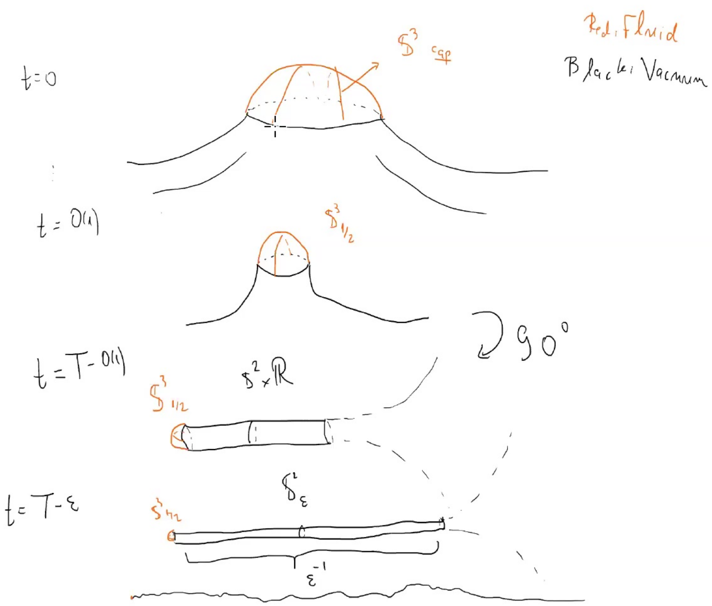

Figure: Time snapshots of spatial geometry in Oppenheimer-Snyder.

Examples of singularity formation: Black hole interiors.

(Kruskal 1960): Proper understanding of connection to the Schwarzschild solution (discovered in 1915). **Singularity forms in the black hole interior**.

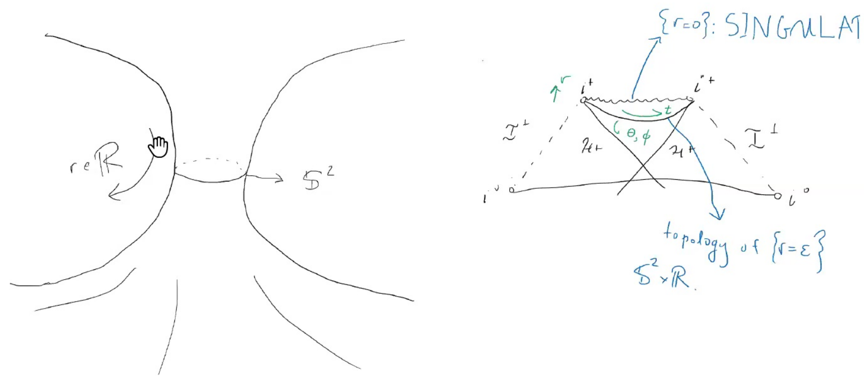

Figure: LEFT: The initial metric of (maximally extended) Schwarzschild: Two ended. RIGHT: Penrose diagram of Schwarzschild.

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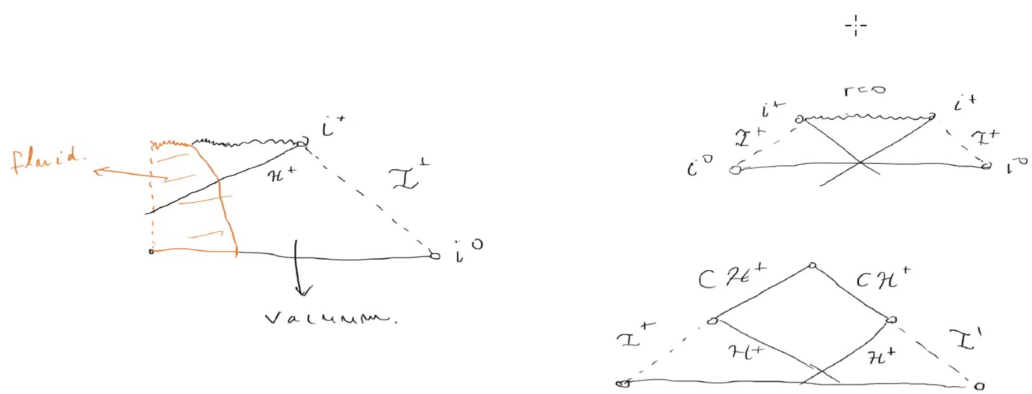

Figure: Penrose diagrams of Oppenheimer-Snyder, Schwarzschild and Kerr.

Examples of singularity formation: *Big Bang singularities*.

Examples of *initial* singularities obtained in the 1920s: Friedmann-Lemaitre-Robertson-Walker. Constructed examples in the presence of matter: (Fluids, of different types). These explicit solutions are entirely *isotropic*. One example is where the metric is defined for all $t > 0$:

$$\mathbf{g} = -dt^2 + t^{2/3}(dx^1)^2 + t^{2/3}(dx^2)^2 + t^{2/3}(dx^3)^2.$$

(Note there is a fluid of some time-dependent density which we do not write out here). *Space shrinks down to nothing as* $t \to 0^+$.

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Note in particular that one of the q_1's must be negative for *vacuum* solutions. The FLRW solutions have a *fluid* present and can admit contractions in all directions–below to generalized Kasner family.

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Take a cosmic journey with the world's most famous physicist.

Prof Hawking on Desert Island Discs

First broadcast on Christmas Day in 1992.

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Melvyn Bragg discusses black holes, the dead collapsed ghosts of massive stars.

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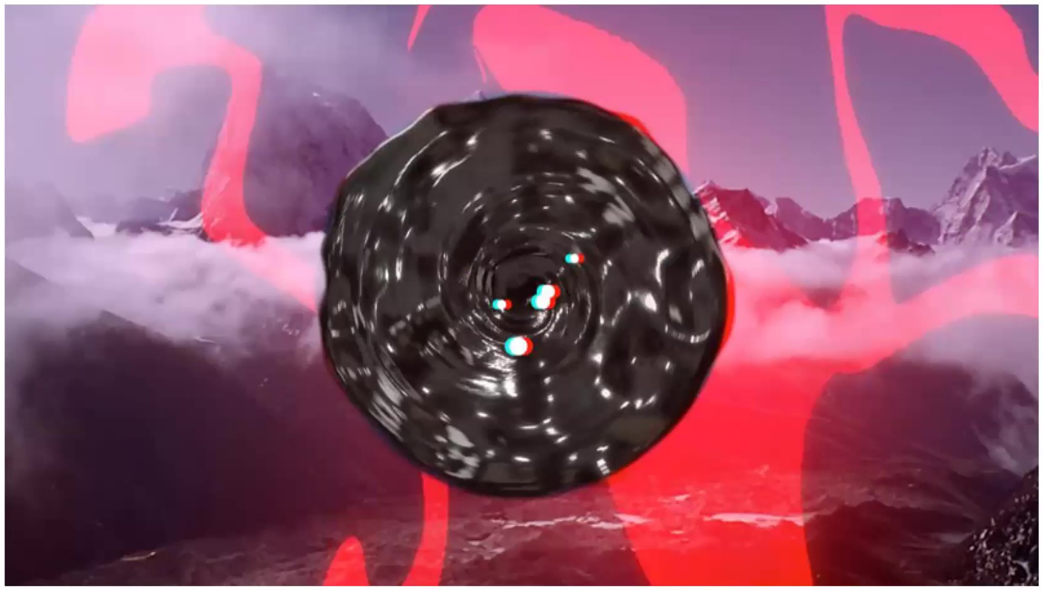

Question 3 of 9

When a star collapses into a black hole all its mass gets squeezed into:

- The singularity
- The event horizon
- Another dimension

Nature of the singularity–Mixmaster chaotic behaviour.

Conjecture (Belinskii-Khalatnikov-Lifshitz)

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Special situation in BKL broad picture: No oscillations, and Asymptotically Velocity-term dominated (**AVTD**) behaviour of the metric: In synchronized time function τ:

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Velocity component of $\partial\mathbf{g}$ dominates all spatial components.

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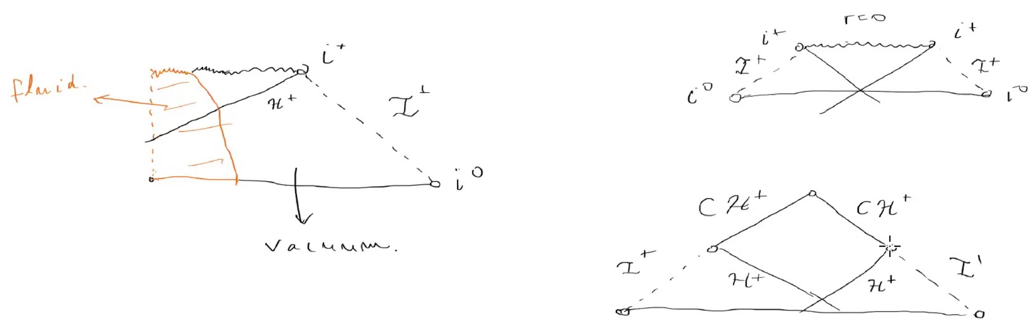

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Singularities, Some examples.

Theorem (Dafermos-Luk, 2017,+)

In vacuum: For generic perturbations of a Kerr solution exterior, $\exists$ a **portion** *of weak null singularity inside black hole.*

Theorem (Rodnianski-Speck, 2014, 2018, Fournodavlos-Rodnianski-Speck, 2020.)

In topology $\mathbb{T}^N$, all (generalized) Kasner solutions in the sub-critical regime in Einstein-scalar field/high-dimensional vacuum/$(3+1)$ polarized vacuum, are stable, for the backwards-in-time problem: Perturbations of the data at $\{\tau = 1\}$ lead to space-like singularity formation at $\{\tau = 0\}$: AVTD dynamics, and convergence (in some sense) to a different Kasner solution at each "point" on $\{\tau = 0\}$.

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$$\mathbf{g} = -d\tau^2 + \sum_{i=1}^{D} g_{ii}(x^1, \dots x^D)\tau^{2q_i(x^1,\dots,x^D)}.$$

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$\mathbf{g} = -d\tau^2 + \sum_{i=1}^{D} g_{ii}(x^1, \ldots x^D)\tau^{2q_i(x^1,\ldots,x^D)}$.Curvature blows up like $|R(\mathbf{g})| \sim \tau^{-2}$.

Our result: Stability under polarized axial symmetry.

$$\mathbf{g}_{\mathrm{Schw}} = (1 - \frac{2M}{r})^{-1} dr^2 - (1 - \frac{2M}{r}) dt^2 + r^2 (d\theta^2 + sin^2\theta d\phi^2).$$

∂_ϕ is Killing. *Polarized* Killing because $\partial_\phi \perp \partial_t, \partial_\theta, \partial_r$.

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Consider **axi-symmetric**, **polarized** *perturbations of the Schwarzschild data on $r = M, t \in [0, M]$. Then the solution $\mathbf{g}_{\mathrm{perturb}}$ of the vacuum Einstein equations develops a space-like singularity, with (gauge-normalized) asymptotics of the form:*

$$\begin{aligned} \mathbf{g}_{\mathrm{p}} \sim (1 - \frac{2M}{r})^{-1} dr^2 + g_{tt}(t,\theta) r^{\beta(t,\theta)} dt^2 + g_{\theta\theta} r^{2\delta(t,\theta)} d\theta^2 \\ + r^{\epsilon(t,\theta)} dr d\theta + g_{\phi\phi}(t,\theta) r^{2\alpha(t,\theta)} sin^2\theta d\phi^2. \end{aligned} \tag{1}$$

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In fact $\alpha(t,\theta) \sim 1, \delta = \delta(\alpha) \sim 1, \beta = \beta(\alpha) \sim -\frac{1}{2}, \epsilon = \epsilon(\alpha) \sim \frac{1}{2}$.

Key structure in axial symmetry.

The reduced Einstein equations:
Polarized axi-symmetry:

$$\mathbf{g}^{3+1} = \mathbf{g}_{\phi\phi} d\phi^2 + \mathbf{h}^{2+1}$$

EVE: $\mathrm{Ric}(\mathbf{g}) = 0$. Let $\gamma = \log(\mathbf{g}_{\phi\phi})$.

$$\Box_{\mathbf{g}}\gamma = 0, \mathrm{Ric}_{ij}(\mathbf{h}^{2+1}) = \nabla^2_{ij}\gamma + \nabla_i\gamma\nabla_j\gamma.$$

2nd equations can be expressed as **ODE**s:

Choose a geodesic gauge $\nabla_{e_0} e_0 = 0$, $\nabla_{e_0} e_i = 0$. The "main variables" become the e_0-derivatives of the metric $\mathbf{h}^{2+1}$:

$$K_{ij} = < \nabla_{e_i} e_0, e_j > .$$

Wave-ODE system. Formal Asymptotics at singularity.

Our system reduces to:

$$\begin{aligned} \Box_{g(\gamma)}\gamma &= 0, \\ e^0 K &= K * K + \nabla^2\gamma + \nabla\gamma\nabla\gamma, \\ e^0 a &= K * a + \nabla^2\gamma + \nabla\gamma\nabla\gamma. \end{aligned} \tag{2}$$

(where *spatial* part of $\mathbf{g}$ given by $\mathbf{g}_{\mathrm{spatial}} = a \cdot a$). **Formal asymptotics:** $\gamma \sim \alpha(t,\theta) logr + B(t,\theta) + O(r)$. Assuming this for γ we have in principal frame for K:

$$\begin{aligned} K_{11} &= \beta(t,\theta) r^{-3/2} + O(r^{-1/2}) + \overline{y}(t,\theta) r^{\epsilon(t,\theta)}, \\ K_{22} &= \delta(t,\theta) r^{-3/2} + O(r^{-1/2}), K_{12} = O(r^{-\frac{1}{2}}). \end{aligned} \tag{3}$$

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$\beta(t,\theta) = \beta(\alpha(t,\theta)), \delta(t,\theta) = \delta(\alpha(t,\theta))$ are explicit and it turns out:

$$trK(r = \rho) = \frac{3}{2}\rho^{-3/2} + O(\rho^{-1/2}).$$

Math: Energy estimates, Gauge choice, singular branch.

Ricatti equation for K_{22} sees the (unique!) *collapsing* direction ∂_θ.
Can well blow up *before* $r = 0$ for gauge reasons. Want to solve *backwards* to avoid this: Forced to solve the above by *iteration*.
Forced to use energy estimates to produce *real* solution.
Asymptotically CMC of r-level sets is *crucial* for wave equation.
For Ricatti *danger* in differentiated equations: $\partial = \partial_t, \partial_\theta$.

$$e^0 \partial K_{11} + 2K_{11}\partial K_{11} + 2K_{12}\partial K_{12} = \partial[\nabla^2\gamma + \nabla\gamma\nabla\gamma]$$

admits *free branch* like $r^{\epsilon(t,\theta)}$ (consistent with *undifferentiated Ricatti*).

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admits free branch $r^{-3+\epsilon(t,\theta)}$.

Iteration, and location of initial data.

Set up an iteration to solve this system: $\mathbf{g}^m = (e^{2\gamma^m} d\phi^2, \mathbf{h}_m^{2+1})$:

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admits free branch: $r^{-3+\epsilon(t,\theta)}$. **If** this is present then *no* possibility of deriving any estimates for the system. (In the iteration estimates would be getting exponentially worse at each step).

Iteration, and location of initial data.

Set up an iteration to solve this system: $\mathbf{g}^m = (e^{2\gamma^m} d\phi^2, \mathbf{h}^{2+1}_m)$:

- Initial data induced on a hypersurface surface $\Sigma_{r_*^m(t,\theta)} := \{r = r_*^m(t,\theta)\}$.
- Solve $\Box_{\mathbf{g}^{m-1}} \gamma^m = 0$ *forwards, starting at* $\{r = r_*^{m-1}(t,\theta)\}$.
- Solve Ricatti equations for K^m_{22}, K^m_{12} *backwards, starting at the singularity*. [No free branch for these ODEs–for ∂K^m_{22} set "artificial" free branch to zero].
- Identify hypersurface $\{r = r_*^m(t,\theta)\}$ where prescribed initial data will be induced.
- Prescribe remaining initial data for K^m_{11}, a^m **on** $\{r = r_*^m(t,\theta)\}$ and solve these towards the singularity again.
- Iterate.

Outlook.

Open questions and future directions.

- A method is introduced to study Einstein's equations in (polarized) axial symmetry. Key structure is a free-wave ODE system. (Outside polarized wave-map and ODE system).
- Easier way to study EE than other gauges?
- E.g., perturbations of all $(3+1)$-solutions with 2 degrees of symmetry to just one degree of symmetry?
- On black hole singularities: One-ended initial data?
- Other settings: Black hole exteriors, other cosmological constants?