Title: Deformations of General Relativity, Geometrodynamics and reality conditions

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Abstract: A remarkable aspect of 4-dimensional complexified General Relativity (GR) is that it can be non-trivially deformed: there exists an infinite-parameter set of modifications with the same degree of freedom count. It is trivial to impose reality conditions that lead to real theories with Euclidean or split signature, but the situation is more complicated and not yet fully understood in the Lorentzian case, which is the subject of this talk. I will first show that the choice of potentially consistent reality conditions is essentially unique and boils down to the reality of the underlying 3-metric at the canonical level, as in the case of GR. For simplicity, I will focus on a subset of modified theories that correspond to a natural extension of Ashtekar's Hamiltonian constraint, namely, a linear combination of EEE, EEB, EBB and BBB. Interestingly, the evolution equations for the 3-metric and its first time-derivative take the same form as in GR, but with an effective stress tensor source which cannot be expressed in terms of these two fields. Modified theories therefore appear as essentially "non-metric" in that they do not admit a closed geometrodynamics form. In particular, this obstructs the conservation of the reality conditions, because the effective source remains complex. Alternatively, if we insist on reality, we obtain extra reality conditions which then leave no room for degrees of freedom. I will finally argue that this should be a generic feature of the Lorenzian modified theories, in stark contrast to their Euclidean and split-signature counterparts.

Deformations of GR, geometrodynamics and reality conditions

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- Complexified GR
- Reality conditions
- Deformations (Lorentzian signature problem)
- A natural subclass
- Geometrodynamics
- Obstruction to reality
- Take away

Plebanski SO(3,$\mathbb{C}$) action [Plebanski (1977)]

$$S = \int \left[B_i \wedge F^i - \frac{1}{2} \psi^{ij} B_i \wedge B_j - \phi \left(\lambda + \mu_{ij} \psi^{ij} \right) \right]$$

$$F^i := \mathrm{d}A^i + \frac{1}{2} \mu^{ij} \varepsilon_{jkl} A^k \wedge A^l$$

if <u>real</u> fields

$$B_i = \mu_{ij} e^0 \wedge e^j + \frac{1}{2} \varepsilon_{ijk} e^j \wedge e^k \,, \qquad \phi = e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

$\Rightarrow$ real Palatini-Holst action (real GR)

- $\mu_{ij} = \delta_{ij} \quad \Rightarrow \quad$ SO(3) $\simeq$ self-dual SO(4)
- $\mu_{ij} = \eta_{ij} \quad \Rightarrow \quad$ SO(1,2) $\simeq$ self-dual SO(2,2)

but $\mathfrak{so}(1,3) \simeq \mathfrak{sl}(2,\mathbb{C}) \quad \Rightarrow \quad$ Lorentzian self-dual is complex

$\Rightarrow$ complex fields $+$ non-linear reality conditions

$$\mathrm{Re}\left[\delta^{ij} B_i \wedge B_j\right] = 0 \qquad B_i \wedge \bar{B}_j = 0$$

underlying real metric [Urbantke (1984)]

$$g_{\mu\nu} \propto \varepsilon_{ijk} \varepsilon^{\alpha\beta\gamma\delta} B^i_{\mu\alpha} B^j_{\nu\beta} B^k_{\gamma\delta} \qquad \sqrt{-g} = \frac{\mathrm{i}}{4} \delta^{ij} \varepsilon^{\mu\nu\rho\sigma} B_{i\mu\nu} B_{j\rho\sigma}$$

$$B_i = \mathrm{i}\delta_{ij} e^0 \wedge e^j + \frac{1}{2} \varepsilon_{ijk} e^j \wedge e^k \,, \qquad \phi = e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

$\Rightarrow$ Lorentzian Palatini-Holst action

$\Rightarrow$ real Lorentzian GR

Back to the complex theory: deformation

$$S = \int \left[B_i \wedge F^i - \frac{1}{2}\psi^{ij} B_i \wedge B_j - \phi\left(\lambda + \mu_{ij}\psi^{ij}\right)\right]$$
$$\to \int \left[B_i \wedge F^i - \frac{1}{2}\psi^{ij} B_i \wedge B_j - \phi\mathcal{H}(\psi)\right]$$

[Capovilla ++ (1989), Bengtsson (1991), Krasnov (2006)]

- same degree of freedom count!
- dynamical internal metric

$$\frac{1}{2} B_i \wedge B_j = -\phi\mathcal{H}_{ij} \qquad \mathcal{H}_{ij} := \frac{\partial\mathcal{H}}{\partial\psi^{ij}} \qquad \mathcal{H}_{ij}^{\mathrm{GR}} := \mu_{ij}$$

- pure-connection form [Krasnov (2011)]

$$S = \int f\left(F^i \wedge F^j\right) \qquad S_{\mathrm{GR}} = \int \left[\mathrm{Tr}\sqrt{F^i \wedge F^j}\right]^2$$

Reality conditions for <u>deformed theories</u>?

$$\mathrm{GR}: \qquad \mathrm{Re}\left[\delta^{ij} B_i \wedge B_j\right] = 0 \qquad B_i \wedge \bar{B}_j = 0$$

<u>Scalar constraint:</u> generalization $\mathrm{Re}\left[f\left(B_i \wedge B_j\right)\right] = 0$

amounts to changing $\mathcal{H}(\psi)$ $\Rightarrow$ no new direction in theory space

<u>Tensor constraint:</u> too rigid

reality of conformal class of metrics $[g_{\mu\nu}]$

$$g_{\mu\nu} \propto \varepsilon_{ijk} \varepsilon^{\alpha\beta\gamma\delta} B^i_{\mu\alpha} B^j_{\nu\beta} B^k_{\gamma\delta} \qquad \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} B_{i\rho\sigma} \equiv \mathrm{i}\sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} B_{i\rho\sigma}$$

Reality conditions for deformed theories?

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$\Rightarrow$ unique reality constraints (up to redef. of $\mathcal{H}$)

$$\Rightarrow \qquad \mathrm{Re}\left[\mathcal{H}^{ij}(\psi)\, B_i \wedge B_j\right] = 0 \qquad \mathcal{H}_{ij} := \frac{\partial \mathcal{H}}{\partial \psi^{ij}} \qquad \mathcal{H}_{ik}\mathcal{H}^{kj} \equiv \delta_i^j$$

we implement at the action level: $\quad S = S_h + S_c$

$$\begin{aligned} S_h &= \int \left[\frac{1}{\mathrm{i}}\left(B_i \wedge F^i - \frac{1}{2}\,\psi^{ij} B_i \wedge B_j\right) - \phi\mathcal{H}(\psi)\right] \\ S_c &= \int \left[\chi\, \mathrm{Re}\left(\mathcal{H}^{ij} B_i \wedge B_j\right) + \chi^{ij} B_i \wedge \bar{B}_j\right] \end{aligned}$$

Important action structure: holomorphic + reality constraints

$\Rightarrow$ equations of conjugate fields $\bar{B}_i \quad \chi, \chi^{ij} = 0$

$\Rightarrow$ S_c does not affect holomorphic equations of motion

$$\boxed{S = \int \left[\frac{1}{\mathrm{i}}\left(B_i \wedge F^i - \frac{1}{2}\,\psi^{ij} B_i \wedge B_j\right) - \phi\mathcal{H}(\psi) + \chi^{ij} B_i \wedge \bar{B}_j\right] \qquad \phi \in \mathbb{R}}$$

compatibility? $\Rightarrow$ canonical formulation : $x^\mu \to \{t, x^\alpha\}$

$$E_i^\alpha := \psi_{ij} B^{j\alpha} \qquad B_i^\alpha := \frac{1}{2}\,\varepsilon^{\alpha\beta\gamma} F_{\beta\gamma}^i \qquad \psi^{ij} \equiv E_\alpha^i B^{j\alpha}$$

canonical action

$$S = \int \mathrm{d}^4 x \left[\frac{1}{\mathrm{i}}\left(E_i^\alpha \dot{A}_\alpha^i - \theta^i \mathcal{G}_i - N^\alpha \mathcal{D}_\alpha\right) - N\mathcal{H}(\psi) - \xi_{\alpha\beta}\,\mathrm{Im}[\tilde{q}^{\alpha\beta}]\right]$$

with $N, N^\alpha \in \mathbb{R}$ and

$$\tilde{q}^{\alpha\beta} := \mathcal{H}^{ij}(\psi)\, E_i^\alpha E_j^\beta$$

3-metric density from Urbantke 4-metric

$\mathcal{H}, \mathcal{D}_\alpha, \mathcal{G}_i$ first-class algebra (3-metric again $\tilde{q}^{\alpha\beta}$)

simple subset of theories

$$\mathcal{H} = \lambda_0\,(E,E,E) + \lambda_1\,(E,E,B) + \lambda_2\,(E,B,B) + \lambda_3\,(B,B,B)$$

$$(X,Y,Z) := \frac{\varepsilon_{\alpha\beta\gamma}\varepsilon^{ijk} X_i^\alpha Y_j^\beta Z_k^\gamma}{n_X!\,n_Y!\,n_Z!}$$

$$\Rightarrow \quad \tilde{q}^{\alpha\beta} = \lambda_E E_i^\alpha E_i^\beta + \lambda_S E_i^\alpha B^{i\beta} + \lambda_B B^{i\alpha} B^{i\beta}$$

$$\lambda_E := \lambda_1^2 - \lambda_0\lambda_2 \qquad \lambda_S := \lambda_1\lambda_2 - \lambda_0\lambda_3 \qquad \lambda_B := \lambda_2^2 - \lambda_1\lambda_3$$

- natural extension of Ashtekar formulation [Ashtekar (1986)]
- arise from 7D 3-form theories [Krasnov ++ (2017-18)]
- redundancy: canonical transformation $E_i^\alpha \to E_i^\alpha + cB_i^\alpha$
  invariant coupling: $\Delta := \lambda_S^2 - 4\lambda_E\lambda_B$

reality conditions? $\quad q_{\alpha\beta} \in \mathbb{R} \quad \Rightarrow \quad K_{\alpha\beta} \in \mathbb{R}$

$$(\partial_t - \mathcal{L}_N)\, K_{\alpha\beta} = \left[2K_{\alpha\gamma}K^{\gamma}_{\beta} - KK_{\alpha\beta} - R_{\alpha\beta} + \nabla_\alpha\nabla_\beta + q_{\alpha\beta}V + \Delta S_{\alpha\gamma\delta}S_{\beta}{}^{\gamma\delta}\right] N$$

in general $V, S_{\alpha\beta\gamma} \in \mathbb{C} \quad \Rightarrow \quad$ not conserved

imposing $T_{\alpha\beta} \in \mathbb{R} \quad \Rightarrow \quad$ zero degrees of freedom

exception: $\quad$ if $\Delta = 0 \quad \Rightarrow \quad V =$ constant

$\Rightarrow \quad$ closed <u>GR</u> geometrodynamics, and also GR constraints

$\Rightarrow \quad$ conserved reality constraints (if $V \in \mathbb{R}$)

what theories have $\Delta = 0$?

$\Delta := \lambda_S^2 - 4\lambda_E\lambda_B$ is discriminant of 3-metric

$$\tilde{q}^{\alpha\beta} = \lambda_E E_i^\alpha E_i^\beta + \lambda_S E_i^\alpha B^{i\beta} + \lambda_B B^{i\alpha} B^{i\beta}$$

$\Rightarrow \quad \Delta = 0$ are exact square 3-metrics

$$\tilde{q}^{\alpha\beta} = E_i^\alpha E_i^\beta \quad \Rightarrow \quad \mathcal{H} = \lambda + \delta_{ij}\psi^{ij} \quad \Rightarrow \quad \text{GR}$$

or

$$\tilde{q}^{\alpha\beta} = B_i^\alpha B_i^\beta \quad \Rightarrow \quad \mathcal{H} = \lambda + \delta^{ij}\psi_{ij} \quad \Rightarrow \quad \text{Self-Dual GR}$$

$$S_{\mathrm{SDG}} = \int \psi_{ij} F^i \wedge F^j \; + \text{ reality conditions}$$

[Krasnov (2017)]

what theories have $\Delta = 0$?

$\Delta := \lambda_S^2 - 4\lambda_E \lambda_B$ is discriminant of 3-metric

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$\Rightarrow$ $\Delta = 0$ are exact square 3-metrics

$$\tilde{q}^{\alpha\beta} = E_i^\alpha E_i^\beta \quad \Rightarrow \quad \mathcal{H} = \lambda + \delta_{ij}\psi^{ij} \quad \Rightarrow \quad \text{GR}$$

or

$$\tilde{q}^{\alpha\beta} = B_i^\alpha B_i^\beta \quad \Rightarrow \quad \mathcal{H} = \lambda + \delta^{ij}\psi_{ij} \quad \Rightarrow \quad \text{Self-Dual GR}$$

$$S_{\mathrm{SDG}} = \int \psi_{ij} F^i \wedge F^j \; + \text{ reality conditions} \qquad \text{[Krasnov (2017)]}$$

(A)dS connection + linear chiral field ψ unbounded energy what about the generic theory?

obstructing term

$$S_{\gamma}{}^{\alpha\beta} = -\frac{1}{2\Delta}\,\varepsilon_{\gamma\delta\epsilon}\,\{q^{\alpha\delta}, q^{\beta\epsilon}\}$$

generic 3-metric

$$\tilde{q}^{\alpha\beta} = f_E(\psi)\,E_i^\alpha E_i^\beta + f_S(\psi)\,E_i^\alpha B^{i\beta} + f_B(\psi)\,B^{i\alpha}B^{i\beta} \qquad \{q, q\} \neq 0$$

$\Rightarrow$ 3-metric cannot be a phase space coordinate

$\Rightarrow$ no canonical 3-metric formulation

$\Rightarrow$ no closed geometrodynamics

$\Rightarrow$ no conserved reality conditions

Take away

- complexified GR admits an infinite-parameter family of deformations
- trivial real versions in Euclidean and split signature
- Lorentzian signature requires non-linear reality conditions
- generically not compatible with dynamics

  $\Rightarrow$ no deformed Lorentzian theories ☹

- better understanding of geometrodynamics for Euclidean/split

Thank you!