Title: Characterising and bounding the set of quantum behaviours in contextuality scenarios

Speakers: Victoria Wright

Series: Quantum Foundations

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Abstract: The predictions of quantum theory resist generalised noncontextual explanations. In addition to the foundational relevance of this fact, the particular extent to which quantum theory violates noncontextuality limits available quantum advantage in communication and information processing. In the first part of this work, we formally define contextuality scenarios via prepare-and-measure experiments, along with the polytope of general contextual behaviours containing the set of quantum contextual behaviours. This framework allows us to recover several properties of set of quantum behaviours in these scenarios . Most surprisingly, we discover contextuality scenarios and associated noncontextuality inequalities that require for their violation the individual quantum preparation and measurement procedures to be mixed states and unsharp measurements. With the framework in place, we formulate novel semidefinite programming relaxations for bounding these sets of quantum contextual behaviours. Most significantly, to circumvent the inadequacy of pure states and projective measurements in contextuality scenarios, we present a novel unitary operator based semidefinite relaxation technique. We demonstrate the efficacy of these relaxations by obtaining tight upper bounds on the quantum violation of several noncontextuality inequalities and identifying novel maximally contextual quantum strategies. To further illustrate the versatility of these relaxations we demonstrate the monogamy of preparation contextuality in a tripartite setting, and present a secure semi-device independent quantum key distribution scheme powered by quantum advantage in parity oblivious random access codes.

Characterising and bounding the set of quantum behaviours in contextuality scenarios

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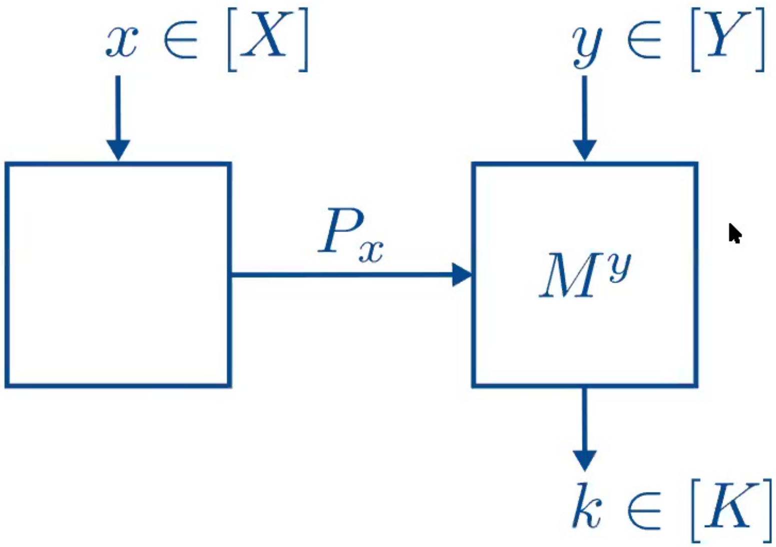
PI 05/03/2021 ICFO

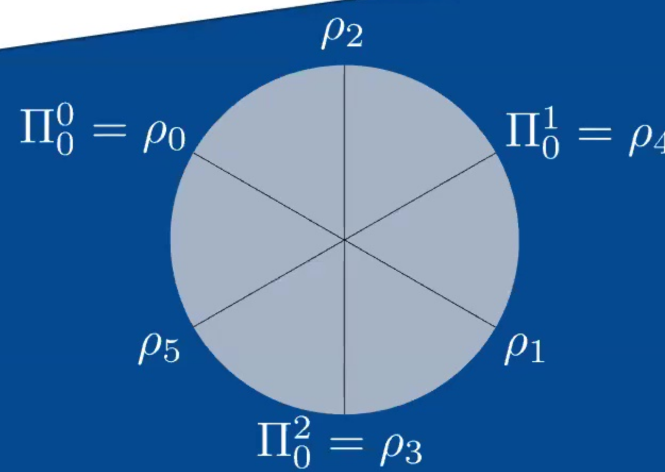


Contextuality scenario

$$T = (X, Y, K, \mathcal{OE}_P, \mathcal{OE}_M)$$

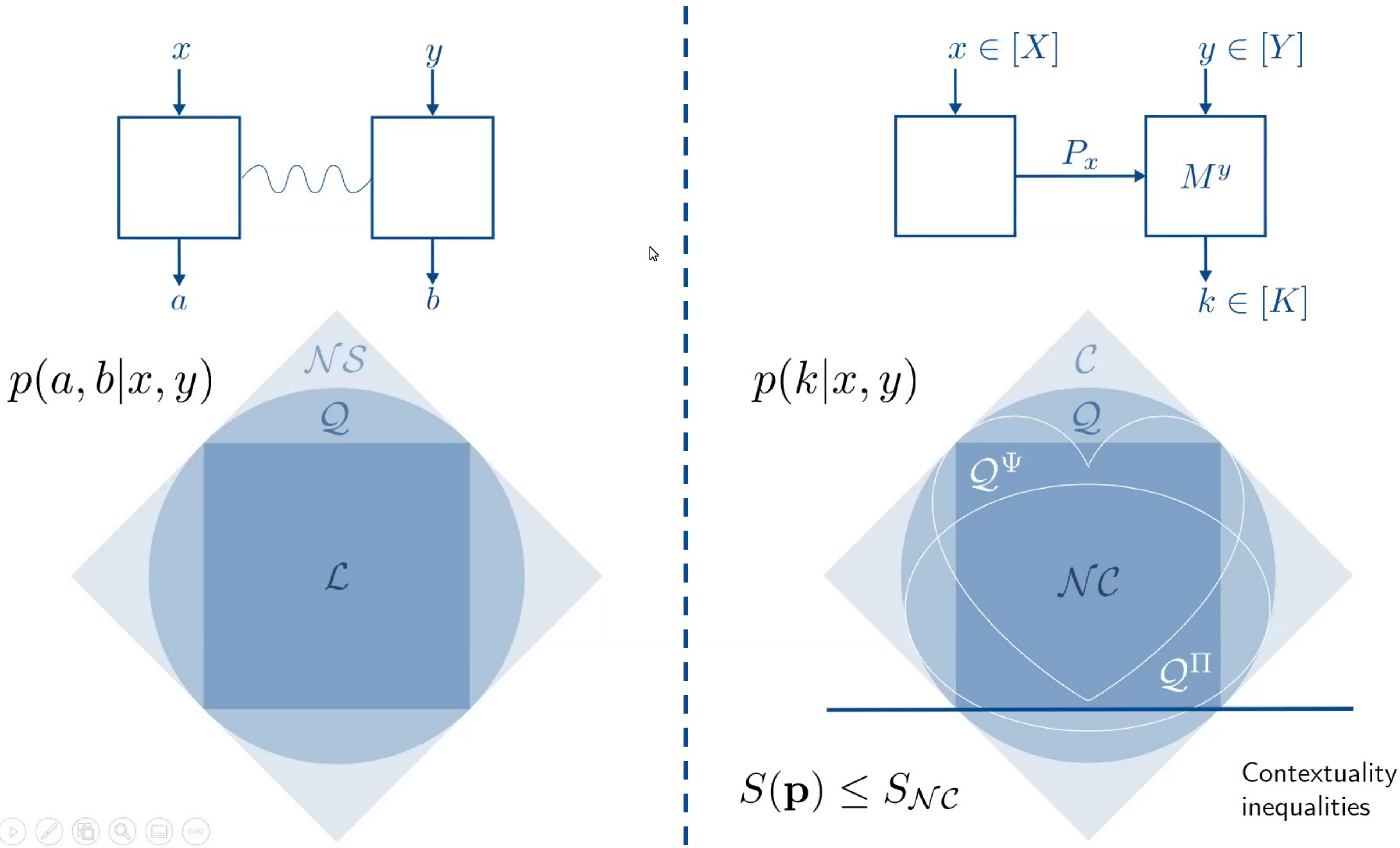
$$p(k|x,y) \qquad \mathbf{p} \in \mathbb{R}^{XYK}$$

Example



$$\frac{1}{2}(P_0 + P_1) \simeq \frac{1}{2}(P_2 + P_3) \simeq \frac{1}{2}(P_4 + P_5)$$

$$\frac{1}{3}([0|M_0] + [0|M_1] + [0|M_2]) \simeq \frac{1}{3}([1|M_0] + [1|M_1] + [1|M_2])$$



x
y
a
b
$p(a,b|x,y)$
$\mathcal{NS}$
$\mathcal{Q}$
$\mathcal{L}$
$x \in [X]$
$y \in [Y]$
P_x
M^y
$k \in [K]$
$p(k|x,y)$
$\mathcal{C}$
$\mathcal{Q}$
$\mathcal{Q}^\Psi$
$\mathcal{NC}$
$\mathcal{Q}^\Pi$
$S(\mathbf{p}) \leq S_{\mathcal{NC}}$
Contextuality inequalities

Generic operational theory: $\mathcal{P}$ $\mathcal{M}$ $\mathcal{E}$

$P, P' \in \mathcal{P}$ are equivalent: $P \simeq P'$

$$\mathrm{Prob}(k|P,M) = \mathrm{Prob}(k|P',M) \quad \text{for all } [k|M] \in \mathcal{E}$$

$[k|M], [k'|M'] \in \mathcal{E}$ are equivalent: $[k|M] \simeq [k'|M']$

$$\mathrm{Prob}(k|P,M) = \mathrm{Prob}(k'|P,M') \quad \text{for all } P \in \mathcal{P}$$

Noncontextual ontological model

Λ

ontic state space

$\mu_P : \Lambda \to [0,1]$

epistemic state

$\xi_M(k|\lambda)$

response function

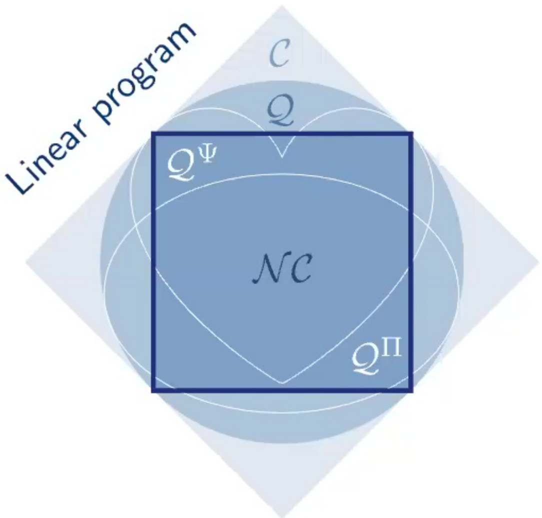
$$\text{Prob}(k|P,M) = \int_\Lambda \xi_M(k|\lambda)\mu_P(\lambda)d\lambda$$

ontological

noncontextual

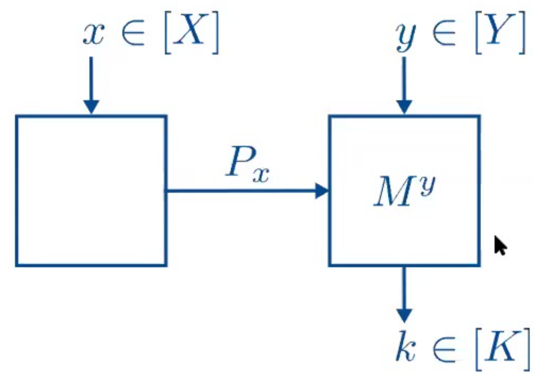
$$P \simeq P' \implies \mu_P = \mu_{P'}$$

$$[k|M] \simeq [k'|M'] \implies \xi_M(k|\cdot) = \xi_{M'}(k'|\cdot)$$



Noncontextual set

See Phys. Rev. A 97, 062103 (2018)



$$\tfrac{1}{2}(P_0 + P_1) \simeq \tfrac{1}{2}(P_2 + P_3) \simeq \tfrac{1}{2}(P_4 + P_5)$$

implies

$$\tfrac{1}{2}(\mu_{P_0} + \mu_{P_1}) = \tfrac{1}{2}(\mu_{P_2} + \mu_{P_3}) = \tfrac{1}{2}(\mu_{P_4} + \mu_{P_5})$$

$$S(\mathbf{p}) = p(0|0,0) + p(0|1,1) + p(0|4,2) \leq 2.5$$

Quantum contextuality

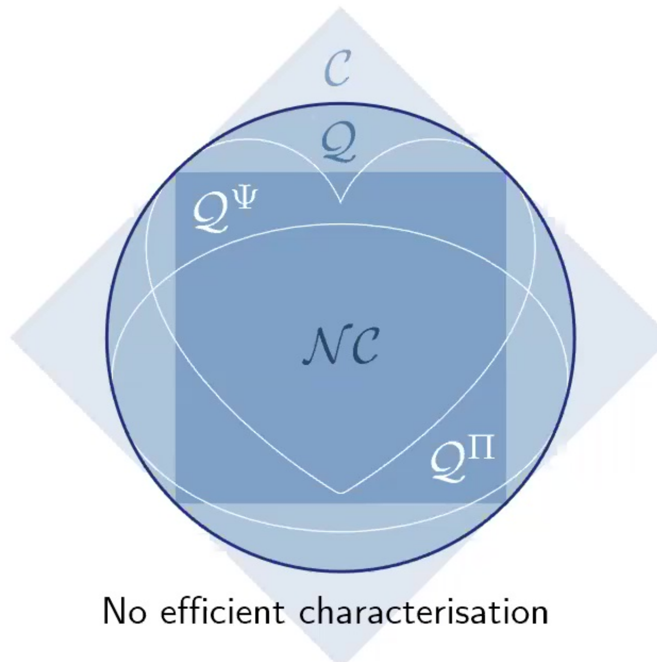
on some finite dimensional Hilbert space

Density operators ρ_x and POVMs $\{E_k^y\}_k$ satisfying,

$$\mathrm{Tr}(E_k^y \rho_x) = p(k|x,y)$$

$$\tfrac{1}{2}(\rho_0 + \rho_1) = \tfrac{1}{2}(\rho_2 + \rho_3) = \tfrac{1}{2}(\rho_4 + \rho_5)$$

$$\tfrac{1}{3}(E_0^0 + E_0^1 + E_0^2) = \tfrac{1}{3}(E_1^0 + E_1^1 + E_1^2)$$



$$S(\mathbf{p}) = p(0|0,0) + p(0|1,1) + p(0|4,2) \lessapprox 2.8660254$$

Quantum contextuality

on some finite dimensional Hilbert space

PVMs $\{\Pi_k^y\}_k$

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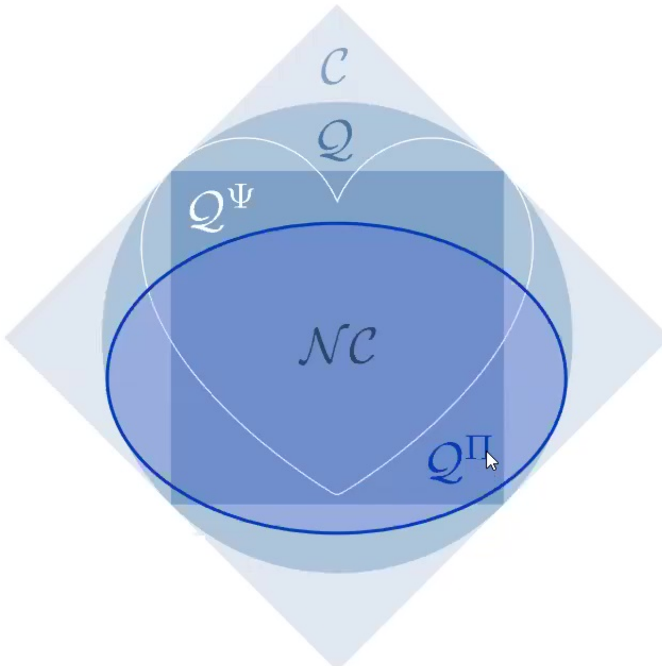

$$\tfrac{1}{2}(\rho_0+\rho_1) = \tfrac{1}{2}(\rho_2+\rho_3) = \tfrac{1}{2}(\rho_4+\rho_5)$$

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Quantum set

$X \leq 3$

or

$T = (X, Y, K, \emptyset, \mathcal{OE}_M)$

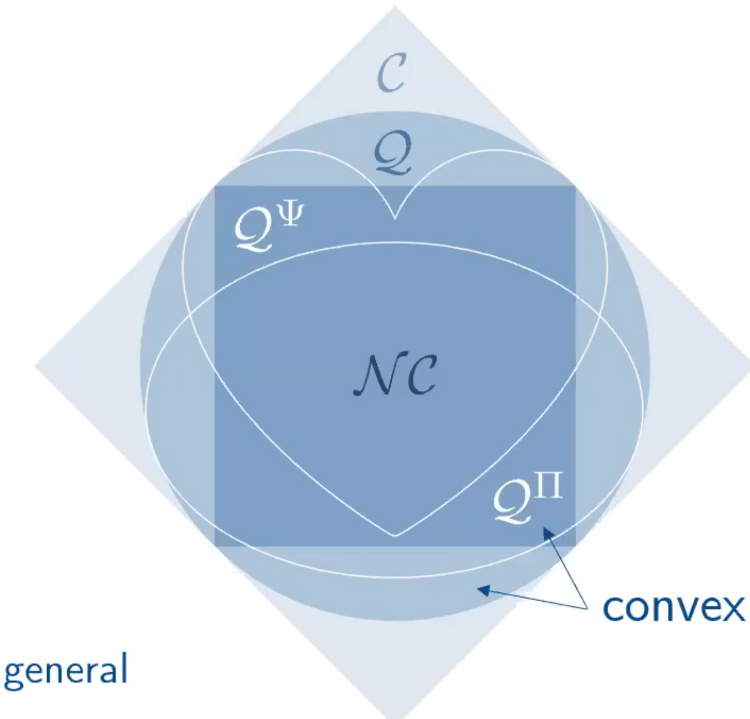
$\Big\}\ \mathcal{NC} = \mathcal{Q} = \mathcal{C}$

$T = (X, Y, K, \mathcal{OE}_P, \emptyset)$ $\qquad \mathcal{Q} = \mathcal{Q}^{\Pi}$

non-trivial preparation equivalences $\qquad \mathcal{Q}^{\Psi} \subsetneq \mathcal{Q}$

in general

non-trivial measurement equivalences $\qquad \mathcal{Q}^{\Pi} \subsetneq \mathcal{Q}$



Semidefinite programming relaxations

Consider density operators ρ_x and POVMs $\{E_k^y\}_k$ such that $\mathrm{Tr}(E_k^y \rho_x) = p(k|x,y)$

$$(\Gamma_x)_{j,k} = \mathrm{Tr}({O_j}^\dagger O_k \rho_x)$$

any sequence of operators $O_1, O_2, \ldots$

	$\mathbb{I}$	E_0^0	E_0^1	E_0^2
$\mathbb{I}$	1	$\mathrm{Tr}(E_0^0\rho_x)$	$\mathrm{Tr}(E_0^1\rho_x)$	$\mathrm{Tr}(E_0^2\rho_x)$
E_0^0	$\mathrm{Tr}(E_0^0\rho_x)$	$\mathrm{Tr}({E_0^0}^2\rho_x)$		
E_0^1	$\mathrm{Tr}(E_0^1\rho_x)$	$\mathrm{Tr}(E_0^0 E_0^1\rho_x)$	$\mathrm{Tr}({E_0^1}^2\rho_x)$	
E_0^2	$\mathrm{Tr}(E_0^2\rho_x)$			$\mathrm{Tr}({E_0^2}^2\rho_x)$

Positive semidefinite   Satisfy operational equivalences $(\Gamma)_{\mathbb{I},E_k^y} = (\Gamma)_{E_k^y,\mathbb{I}} = p(k|x,y)$

$\Downarrow$

Necessary condition given by semidefinite feasibility problem

HOWEVER, currently very bad!

Quantum set

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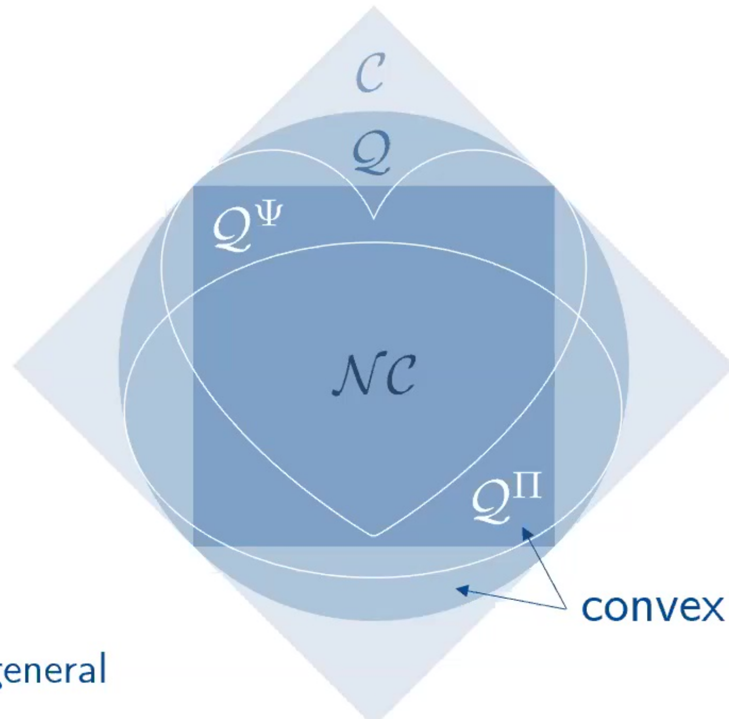

Semidefinite programming relaxations

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E_0^0	$\mathrm{Tr}(E_0^0 \rho_x)$	$\mathrm{Tr}({E_0^0}^2 \rho_x)$		
E_0^1	$\mathrm{Tr}(E_0^1 \rho_x)$	$\mathrm{Tr}(E_0^0 E_0^1 \rho_x)$	$\mathrm{Tr}({E_0^1}^2 \rho_x)$	
E_0^2	$\mathrm{Tr}(E_0^2 \rho_x)$			$\mathrm{Tr}({E_0^2}^2 \rho_x)$

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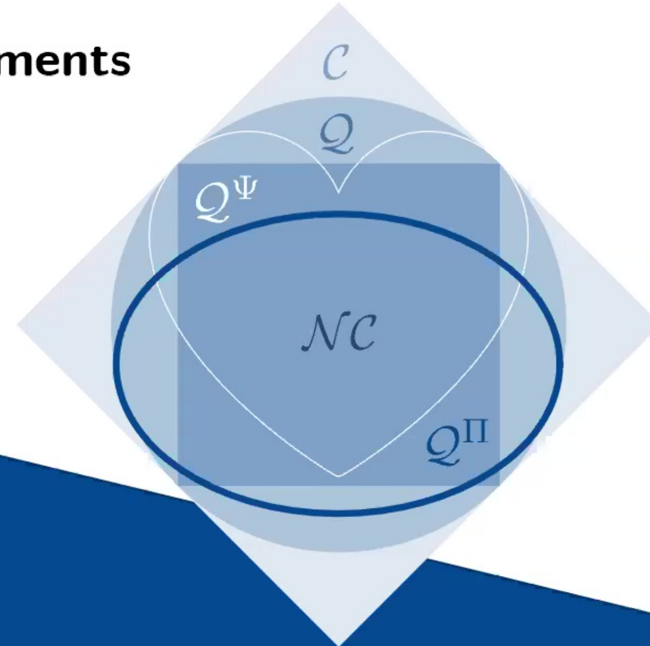
$\Downarrow$

Necessary condition given by semidefinite feasibility problem

HOWEVER, currently very bad!

PROBLEM: cannot assume projective measurements



SOLUTION: a helpful lemma!

$$E = \frac{\mathbb{I}}{2} + \frac{U + U^\dagger}{4}$$

Instead use the operators U_k^y and $U_k^{y\dagger}$ to index the moment matrix.

$$(\Gamma_x)_{\mathbb{I}, U_k^y} + (\Gamma_x)_{\mathbb{I}, U_k^{y\dagger}} = 4\left(p(k|x,y) - \frac{1}{2}\right)$$

Hurray!

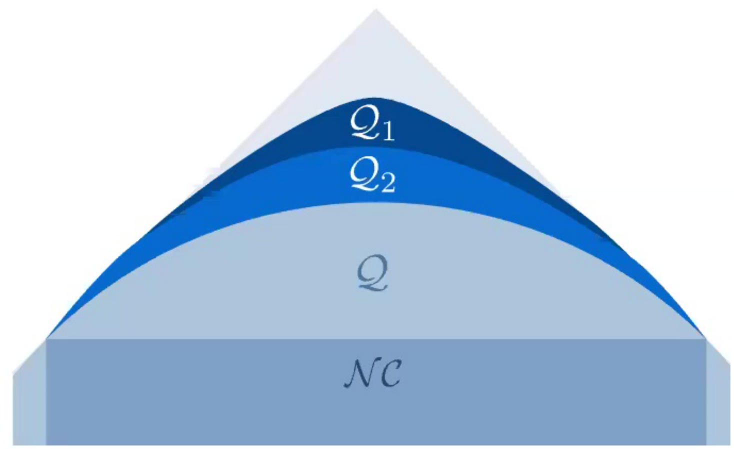
$$(\Gamma_x)_{U_k^y, U_k^y} = 1$$

Relaxations of the quantum set

Maximising success metrics over the relaxations is an SDP

"see-saw" + relaxation ⟶ tight upper bounds in examples considered

↳ up to machine precision



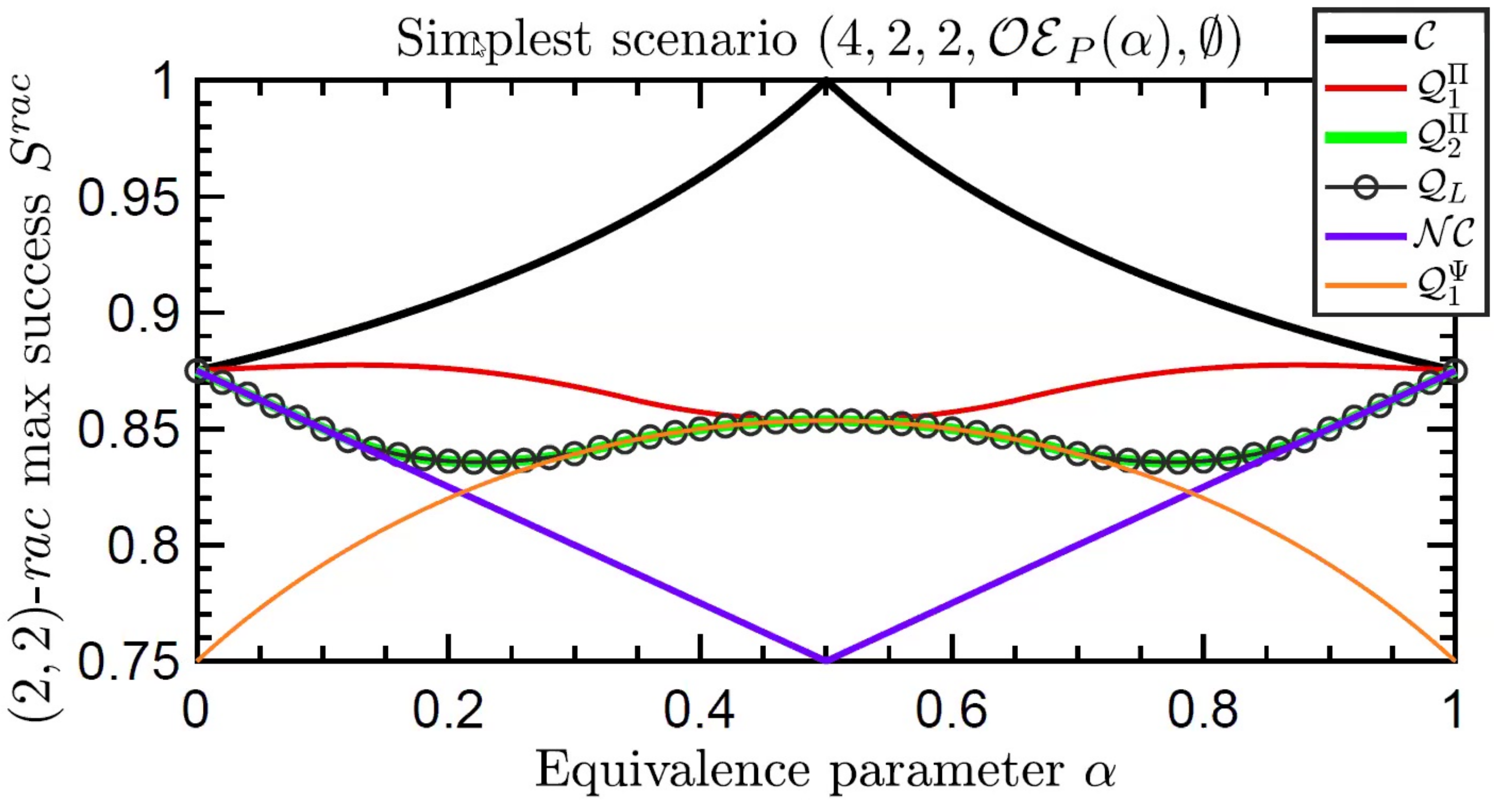
$S(\mathbf{p})$	$S_{\mathcal{C}}$	$S_{\mathcal{NC}}$	$S_{\mathcal{Q}_L}$	$S_{\mathcal{Q}_1}$	$S_{\mathcal{Q}_2}$
$p_{00}+p_{12}+p_{24}$	3	2.5	3	3	3
$p_{00}+p_{11}+p_{24}$	3	2.5	2.8660254	2.8660254	2.8660254
$p_{00}-p_{02}-2p_{02}-2p_{11}+2p_{12}+2p_{24}$	4.5	3	3.9209518	3.9209518	3.9209518
$2p_{00}-p_{11}+2p_{12}$	3.5	3	3.3660254	3.3660254	3.3660254
$p_{00}-p_{04}+p_{11}+p_{12}+2p_{24}$	5	4	4.6889010	4.6889010	4.6889010
$p_{00}-p_{04}+2p_{11}+2p_{24}$	5	4	4.6457513	4.6457513	4.6457513
$p_{00}-p_{03}-2p_{02}-2p_{11}+2p_{12}+2p_{24}$	4.5	3.5	3.5	3.5552760	3.5

Phys. Rev. A 97, 062103 (2018)

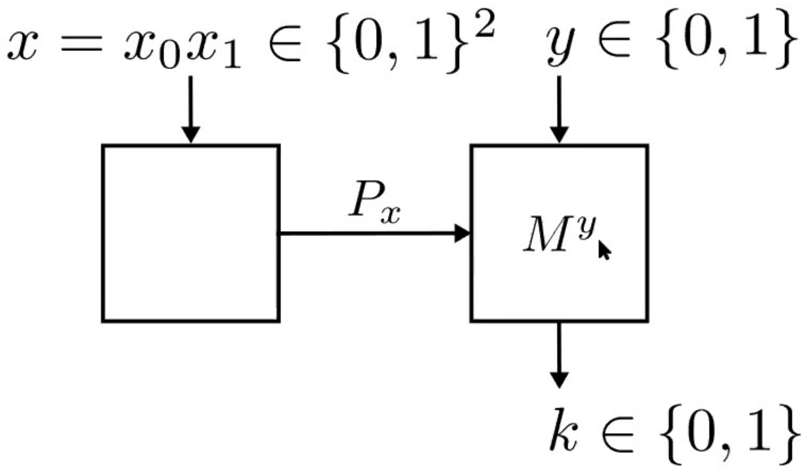
n	$S_{\mathcal{C}}^{ks}$	$S_{\mathcal{NC}}^{ks}$	$S_{\mathcal{Q}_L}^{ks}$	$S_{\mathcal{Q}_1}^{ks}$
5	6	5	5.2360679	5.2360679
7	7	6	6.3176672	6.3176672
9	8	7	7.3600895	7.3600895
11	9	8	8.3863029	8.3863029

Phys. Rep. 506, 1 (2011)
Phys. Rev. A94, 062103 (2016)

$$\frac{1}{2}P_{00} + \frac{1}{2}P_{11} \simeq \alpha P_{01} + (1-\alpha)P_{10}$$



Parity-oblivious random access codes



$$\frac{1}{2}(P_{00} + P_{11}) \simeq \frac{1}{2}(P_{01} + P_{10})$$

Win when:

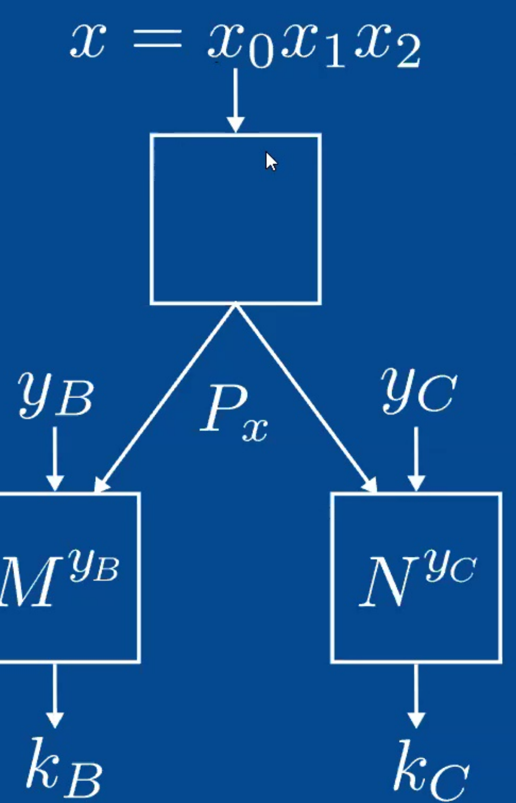
$$k = x_y$$

Average success probability:

$$S^{rac}(\mathbf{p}) = \frac{1}{8} \sum_{x,y} p(x_y | x, y)$$

$$S^{rac}_{\mathcal{NC}} = \frac{3}{4} < S^{rac}_{\mathcal{Q}} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

Monogamy of contextuality



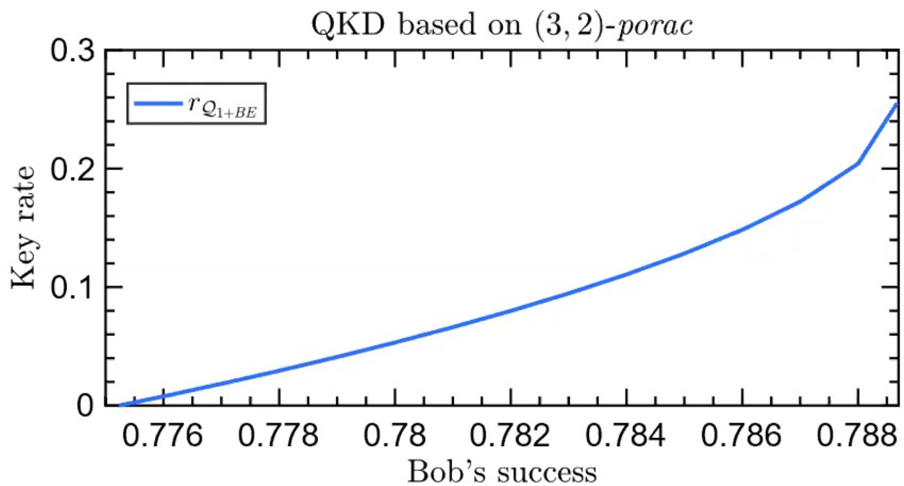
$$S_B^{rac} + S_C^{rac} \leq 1.392$$

SDIQKD

Assumptions:

- rounds i.i.d.
- free will
- no quantum memory for Eve
- preparation equivalences

$$r \geq -\log S_E^{rac} - h(S_B^{rac})$$



Summary

- Introduced a general framework for contextuality scenarios $T = (X, Y, K, \mathcal{OE}_P, \mathcal{OE}_M)$
- Semidefinite relaxations of the quantum set for ANY scenario $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$
- SDP for upper bounding quantum violation of generic non-contextuality inequalities
- Demonstrated monogamy of contextuality and used for an SDIQKD protocol
- Properties of quantum contextuality, including: necessity of mixed states and unsharp measurement

Open questions:

- Hierarchy converges?
- Infinite dimensions?