Title: The Higher Algebra of Supersymmetry

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Collection: Octonions and the Standard Model

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Abstract: We have already met the octonionic Fierz identity satisfied by spinors in 10-dimensional spacetime. This identity makes super-Yang-Mills "super" and allows the Green-Schwarz string to be kappa symmetric. But it is also the defining equation of a "higher" algebraic structure: an L-infinity algebra extending the supersymmetry algebra. We introduce this L-infinity algebra in octonionic language, and describe its cousins in various dimensions. We then survey various consequences of its existence, such as the brane bouquet of Fiorenza-Sati-Schreiber.

The Higher Algebra of Supersymmetry

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Octonions and the Standard Model
Perimeter Institute (virtually)
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The Octonions

- $\mathbb{O}$ is **alternative**—the associator is alternating:

$$(x, y, z) = (xy)z - x(yz).$$

- Equivalently, any subalgebra with two generators is associative [Artin].
- This is just enough associativity so that

$$[\psi, \psi] \cdot \psi = 0, \quad \psi \in \mathbb{O}^2.$$

- But what is the meaning of this identity?

The spinor identity

- Vectors $V = \left\{ \begin{bmatrix} t+x & \bar{y} \\ y & t-x \end{bmatrix} : t, x \in \mathbb{R}, y \in \mathbb{O} \right\}$.
- Spinors $S = \mathbb{O}^2$.
- Two actions of vectors, depending on chirality:

$$A \cdot \psi = A\psi, \quad \psi \in S_+$$

$$A \cdot \phi = \widetilde{A}\phi, \quad \phi \in S_-$$

- Likewise, two spinor to vector pairings:

$$[\psi_1, \psi_2] = \psi_1 \psi_2^\dagger + \psi_2 \psi_1^\dagger, \quad \psi_1, \psi_2 \in S_+$$

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$$\tilde{A} = A - \mathrm{tr}(A)\,1$$

Proof of identity

Theorem

For $\psi \in S_\pm$, we have $[\psi, \psi] \cdot \psi = 0$.

Proof.

Due to Dray–Janesky–Manogue:

$$
\begin{aligned}
[\psi, \psi] \cdot \psi &= 2(\psi\psi^\dagger)\psi \\
&= 2((\psi\psi^\dagger)\psi - \psi(\psi^\dagger\psi)) \\
&= 0
\end{aligned}
$$

ψ involves two elts of $\mathbb{O}$. □

Physical consequences

- $D = 10, N = 1$ super-Yang–Mills theory depends on this identity for its supersymmetry.
- The Green–Schwarz action for the superstring in $\mathbb{R}^{9,1|IIA}$ or $\mathbb{R}^{9,1|IIB}$ depend on this identity for κ-symmetry.

Equivalent forms

- $[\psi, \psi] \cdot \psi = 0.$
- $g([\psi . \psi], [\psi, \psi]) = 0.$
- The expression

$$\gamma(A, \psi, \phi) = g(A, [\psi, \phi])$$

defines cocycle in Lie algebra cohomology on

$$\mathcal{T} = V \oplus S.$$

the supertranslation algebra.

$\mathcal{T}$ Lie superalg. Only nonzero bracket $[-,-] : S \otimes S \longrightarrow V$

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Part of the supersymmetry alg: $\mathfrak{so}(V) \ltimes \mathcal{T}$.

Lie algebra cohomology

Is def. for $\mathfrak{g}$ on the CE cpx:

$$C^\bullet(\mathfrak{g}) = \Lambda^\bullet \mathfrak{g}^*$$

w/ differential

$$d\alpha(X_1, \ldots, X_{p+1}) = \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], \ldots)$$

Compare w/ the usual coord free def'n of exterior derivative.

$$d^2 = 0$$

$$H^\bullet(\mathfrak{g}) = H^\bullet(C^\bullet(\mathfrak{g}))$$

L_∞-algebras

Theorem (Baez–H.)
There is an L_∞-algebra $\mathfrak{superstring}(\mathcal{T})$ extending $\mathcal{T}$ via γ.

L_∞-algebras: idea

A L_∞-algebra L is like a graded Lie algebra:

$$L = \bigoplus_{i=0}^{\infty} L_i.$$

with a differential:

$$d \colon L_i \to L_{i-1}.$$

There's a bracket:

$$[-.-] \colon L \otimes L \to L,$$

but Jacobi fails:

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = d[x,y,z].$$

Only holds up to an exact term.

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$[\ ,\ ,\ ] : L \otimes L \otimes L \to L$. Only holds up

satisfies an analogue of Jacobi to an exact term.

L_∞-algebras: definition

An $\boldsymbol{L_\infty}$**-algebra** is a graded vector space:

$$L = \bigoplus_{i=0}^{\infty} L_i.$$

together with maps:

$$l_n \colon L^{\otimes n} \to L, \quad \text{of degree } n-2$$

∞ many operations

satisfying the **generalized Jacobi identities**:

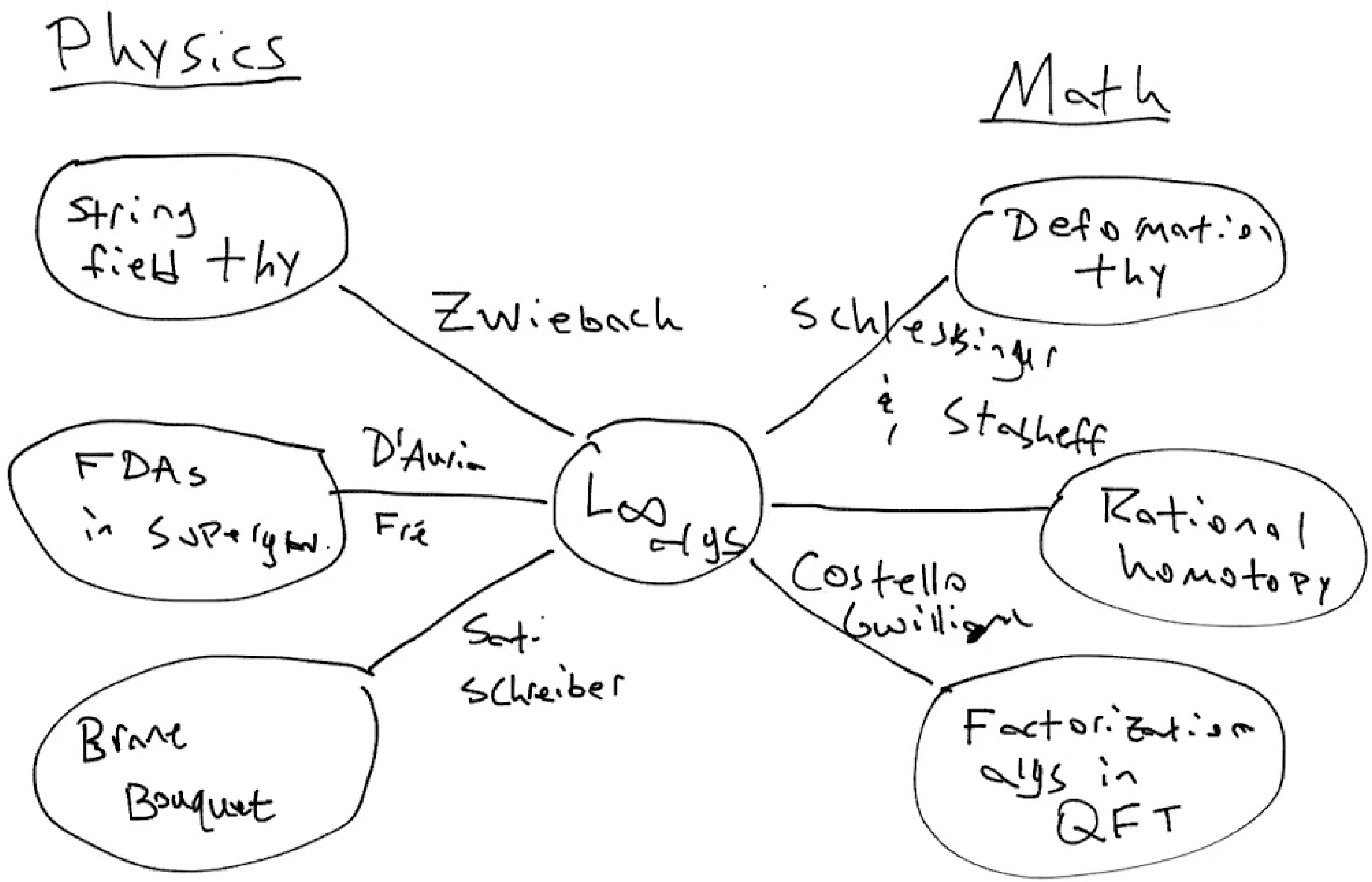
$$\sum_{i+j=n+1} \sum_{\sigma} \pm l_i(l_j(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0.$$

∞ many identities.

$l_1 = d$, deg $1-2=-1$, $d: L_i \to L_{i-1}$

$l_2 = [-,-]$ deg 0 $l_3 = [-,-,-]$ deg 1

L_∞-algebras: why?



L_∞-algebra: $\mathfrak{superstring}(\mathcal{T})$

$L_0 = \mathcal{T}$ $\qquad$ $L_i = 0, \ i \geq 0.$

$L_1 = \mathbb{R}$

$l_1 = d = 0$

$l_2 = [-,-]$ on $\mathcal{T}$.

$l_3 = \gamma$, $\quad l_3 : \mathcal{T}^{\otimes 3} \longrightarrow \mathbb{R}$

$\qquad\qquad\qquad\qquad L_0 \qquad\qquad L_1$

l_3 deg 1 ✓

L_∞-algebra: $\mathfrak{superstring}(\mathcal{T})$

$L_0 = \mathcal{T}$ ~ $\mathbb{Z}/2$ grading $\quad L_i = 0,\ i \geq 0.$

$L_1 = \mathbb{R}$ ~ $\mathbb{Z}/2$

Generalized Jacobi identities $\Leftrightarrow \gamma$ is a cocycle.

$l_1 = d = 0$

$l_2 = [-,-]$ on $\mathcal{T}$. $\Leftrightarrow [\psi,\psi]\cdot\psi = 0$ ✓

$l_3 = \gamma, \quad l_3 : \mathcal{T}^{\otimes 3} \longrightarrow \mathbb{R}$

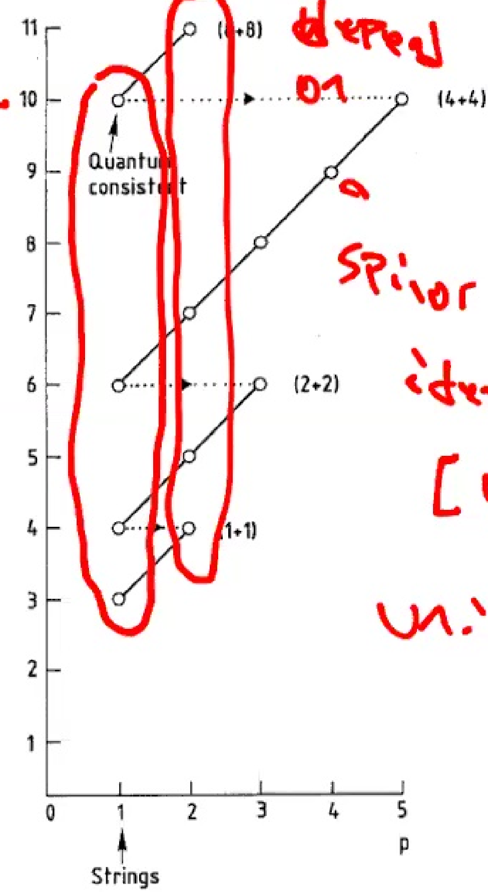
$l_n = 0,\ n \geq 4.$ $\quad l_3$ deg 1 ✓ ($L_0 \to L_1$)

The brane scan

superstring($\mathbb{T}$)

also depend on a spinor identity

each dot represents a Green-Schwarz p-brane action.



uniform in div algs

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$

$$[\psi, [\psi, \psi] \cdot \psi] = 0$$

uniform in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$

Figure by Mike Duff.

$\mathfrak{superstring}(\mathcal{T})$ and 2-$\mathfrak{brane}(\mathcal{T})$

Let $\mathbb{K}$ be a normed division algebra of dimension k.

Theorem (Baez–H.)

There is a supertranslation algebra $\mathcal{T}_K$ in dimensions $k+2$ and an L-infinity algebra extension $\mathfrak{superstring}(\mathcal{T}_K)$.

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There is a supertranslation algebra $\mathcal{T}_K$ in dimensions $k+3$ and an L-infinity algebra extension 2-$\mathfrak{brane}(\mathcal{T}_K)$.

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Theorem (Baez–H.)

$S = \mathbb{K}^4$

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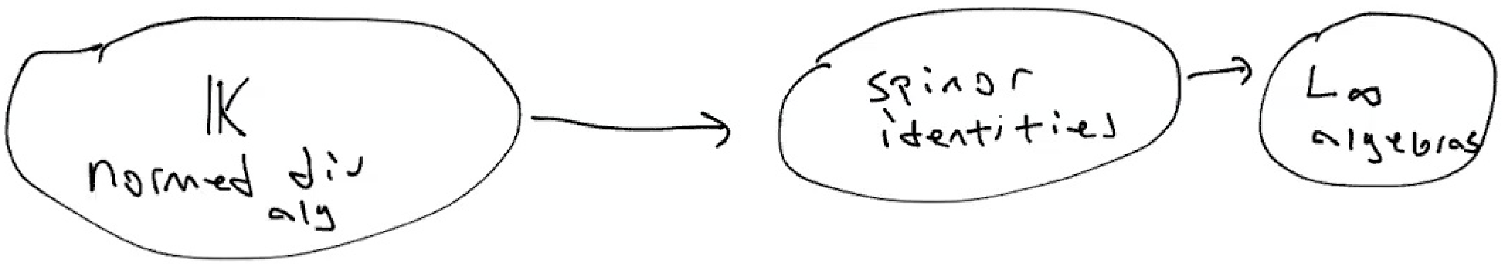
Div Algs &

Super Symmetry II.

$S = \mathbb{O}^4$.

The pipeline



$\mathfrak{superstring}(\mathcal{T})$ and 2-$\mathfrak{brane}(\mathcal{T})$

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$V = \mathbb{R}^{k+2,1}$ $S = \mathbb{K}^4$

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Div Algs &

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$S = \mathbb{O}^4$.

"L_∞-groups"

Idea An L_∞-alg is like a Lie alg.

Theorem
There is an "L_∞-group" with Lie algebra $\mathfrak{superstring}(\mathcal{T}_K)$.

Theorem
There is an "L_∞-group" with Lie algebra 2-$\mathfrak{brane}(\mathcal{T}_K)$.

The brane bouquet

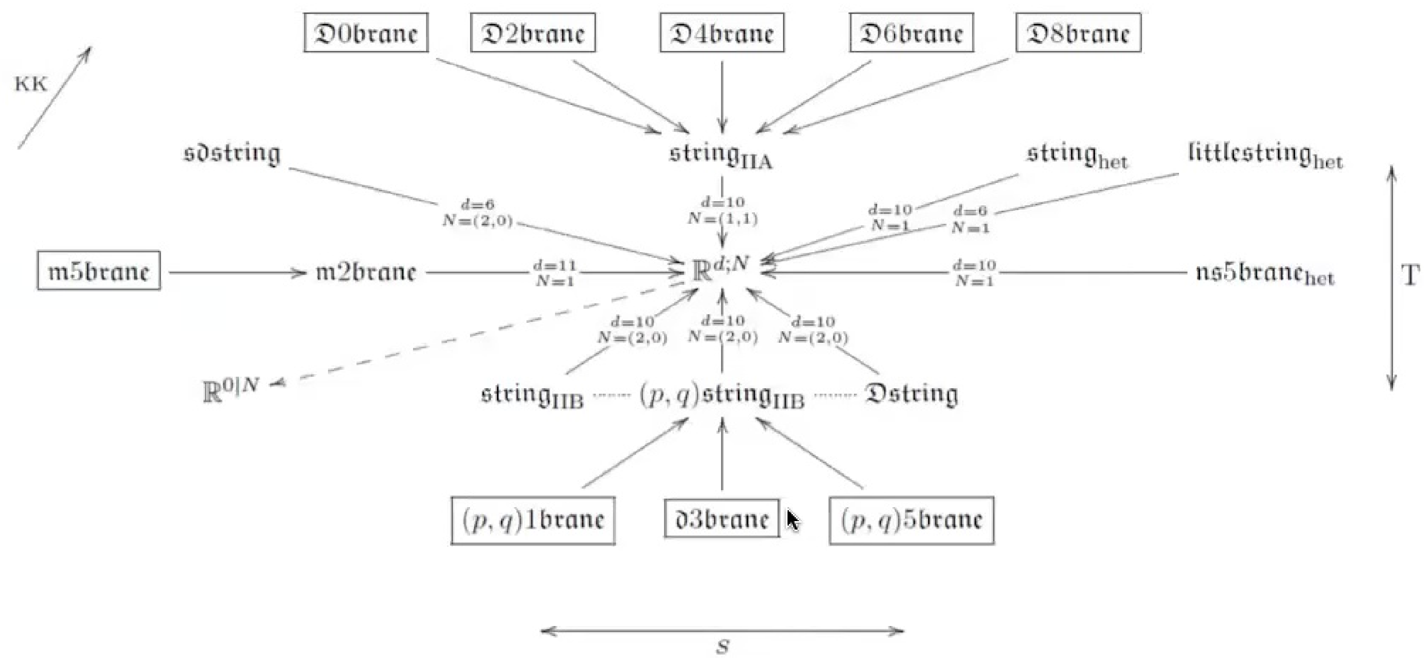
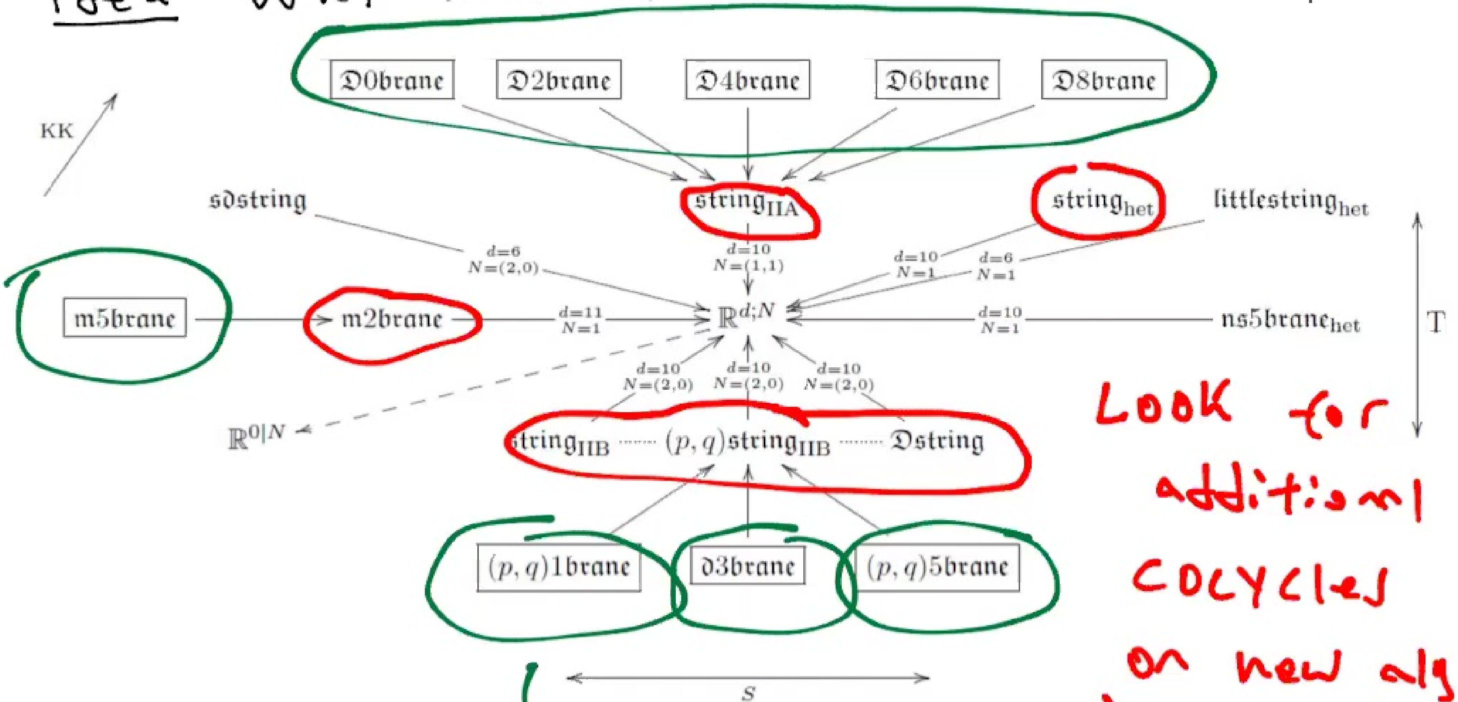


Figure by Urs Schreiber.

The brane bouquet

Fiorenza - Sati - Schreiber

Idea: Why stop at just one extension?



Look for additional cocycles on new alg, & extend!

not in brane scan: now includes branes w/ fields on their worldvol.

Figure by Urs Schreiber.

The equivariant brane bouquet

Idea • Branes on spacetimes w/ singularities are interesting [Acharya – Witten].

• singularities are often of orbifold type: $\mathbb{R}^{D|N} / G$ G finite grp.

• So make the L∞-alg data equivariant in this finite grp

⇝ black branes.

Nice Thinking $\Gamma^{1\mu}$, 11D Γ-matrices, ℂ'ly often leads to nice G.

An L_∞-action of supersymmetry

Elliott – Safronov – Williams

- "L_∞-action" of supersymmetry on the BV version of SYM.

<u>BV SYM $D=10$, $N=1$</u>

$$\left\{\begin{array}{ll} A & \text{gauge boson} \\ \psi & \text{gaugino} \\ c & \text{ghost} \end{array}\right. \qquad \left.\begin{array}{l} A^* \\ \psi^* \\ c^* \end{array}\right\} \text{antifields} \sim \mathcal{E}$$

graded vector space.

An L_∞ action of $\mathfrak{T}$ on $\mathcal{E}$:

$$\rho_n : \mathfrak{T}^{\otimes n} \otimes \mathcal{E} \longrightarrow \mathcal{E}$$

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antifields

graded vector space.

An L_∞ action of $\widetilde{\mathfrak{L}}$ on $\mathcal{E}$:

$\rho_n : \widetilde{\mathfrak{L}}^{\otimes n} \otimes \mathcal{E} \longrightarrow \mathcal{E}$, obeying eqns.

ψ gaugino $\qquad \psi \qquad c$

c ghost $\qquad c^*$ $\qquad$ graded vector space.

An L_∞ action of $\tilde{L}$ on $\mathcal{E}$:

$$\rho_n : \tilde{L}^{\otimes n} \otimes \mathcal{E} \longrightarrow \mathcal{E},$$ obeying eqns.

- ρ_n obey generalized Jacobi-like eqns.
- EWS prove this making heavy use of $[\psi, \psi] \cdot \psi = 0$.
- <u>Virtue</u> This action holds off shell, unlike the usual which requires EOM.

Factorization algs are C & G

approach to observables in QFT.

It's a machine $\mathcal{O}$ that to any open

$$U \subseteq M \rightsquigarrow \mathcal{O}(U)$$

$U, V \subseteq M$ disjoint

$$\mathcal{O}(U) \otimes \mathcal{O}(V) \longrightarrow \mathcal{O}(U \amalg V)$$