Title: Clifford algebra of the Standard Model

Speakers: Ivan Todorov

Collection: Octonions and the Standard Model

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Abstract: We explore the Z2 graded product C`10 = C`4âŠ—Ë†C`6 (introduced by Furey) as a finite quantum algebra of the Standard Model of particle physics. The gamma matrices generating C`10 are expressed in terms of left multiplication by the imaginary octonion units and the Pauli matrices. The subgroup of Spin(10) that fixes an imaginary unit (and thus allows to write O = CâŠ—C 3 expressing the quark-lepton splitting) is the Pati-Salam group GP S = Spin(4) Ã— Spin(6)/Z2 âŠ‚ Spin(10). If we identify the preserved imaginary unit with the C`6 pseudoscalar Ï‰6 = Î³1...Î³6, Ï‰2 6 = âˆ’1 (cf. the talk of Furey and Hughes), then Pex = 1 2 (1 âˆ’ iÏ‰6) will play the role of the projector on the extended particle subspace including the right-handed (sterile) neutrino. We express the generators of C`4 and C`6 in terms of fermionic oscillators aÎ±, aâˆ— Î±, Î± = 1, 2 and bj , bâˆ— j , j = 1, 2, 3 describing flavour and colour, respectively. The internal space observable algebra (an analog of the algebra of real functions on space-time) is then defined as the Jordan subalgebra of hermitian elements of the complexified Clifford algebra C âŠ— C`10 that commute with the weak hypercharge 1 2 Y = 1 3 P3 j=1 b âˆ— j bj âˆ’ 1 2 P2 Î±=1 a âˆ— Î±aÎ±. We only distinguish particles from antiparticles if they have different eigenvalues of Y . Thus the sterile neutrino and antineutrino (with Y = 0) are allowed to mix into Majorana neutrinos. Restricting C`10 to the particle subspace which consists of leptons with Y < 0 and quarks with Y > 0 allows a natural definition of the Higgs field Î¦, the scalar of Quillenâ€™s superconnection, as an element of C`1 4, the odd part of the first factor in C`10. As an application we express the ratio mH mW of the Higgs and the W-boson masses in terms of the cosine of the theoretical Weinberg angle.
The talk is based on the paper arXiv:2010.15621v3 a copy of which including minor corrections is attached.

Octonions and the Standard Model

The Clifford algebra of the SM

Outline

- Haag - Kastler framework
  Superselection sectors
- Input from the SM (observation)
  2^n complex (chiral) fermions
  ...ercharge Y - central for $\mathcal{G}_{SM}$

2^n complex (chiral) fermions

Weak hypercharge Y - central for $\mathfrak{g}_{SM}$

- Early attempts: GUTs, Octonions (1973-)
- $Cl_{10} = Cl_4 \hat{\otimes} Cl_6$ Fermi oscillators
  $a_\alpha^{(*)} = (a_\alpha, a_\alpha^*), \alpha = 1, 2)$ flavour, $b_j^{(*)}$, $j = 1,2,3$ - colour
- Sterile neutrino $J_{s\nu}$ do not separate (anti)particles
- Simple components of the observable (Jordan) algebra
  Aside: Jordan algebra with symmetry generators
- Higgs field as odd part of a superconnection
- Extended supercurvature satisfying Bianchi identity
- Gauge boson mass spectrum; the Higgs mass
- Concluding remarks. 2010.15621v3

The Clifford algebra of the SM

Search for an internal quantum algebra started with noncommutative geometry in the late 1980s.

Haag's approach:

Field algebra $\mathcal{F}$ - star algebra.

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Superselection rules:

Hermitian elements which commute with all observables.

Algebra of observables $\mathfrak{A} \subset \mathcal{F}$

Superselection sectors - labeled by eigenvalues of the superselection charges. No coherent superposition

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Hermitian elements which commute with all observables.

Algebra of observables $\mathfrak{A} \subset \mathcal{F}$

Superselection sectors – labeled by eigenvalues of the superselection charges. No coherent superposition between different sectors.

Input from the Standard Model:

- Fundamental fermions 2^n

4 leptons ν_L e_L, ν_R e_R (8 with the antileptons)

4 quarks in 3 colours $12 + 4 = 2^4$

16 fermions of each generation

$\Rightarrow$ Spinor representations of Spin $(2n)$
(or Spin (p, q))

- Complex - chiral representations

$\Rightarrow$ Spin (p, q) with $p - q = \pm 2 \mod (8)$

Spin (n) $n = 2, 6, 10$

(or Spin(p, q))

- Complex chiral representations

$\Rightarrow$ Spin(p, q) with $p - q = \pm 2 \mod (8)$

Spin(n) $n = 2, 6, 10$

$n=2$ complex numbers $\mathbb{C}$

Dirac equation = Cauchy - Riemann eqs

$n=6$ Spin(6) acts on $\mathbb{C}\otimes\mathbb{H}^2$

$n=10$ Spin(10) acts on $\mathbb{C}\otimes\mathbb{O}^2$

Early attempts to understand where does the SM come from?

(A) D T: - Salam

Dirac equation = Cauchy - Riemann eqs

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Early attempts to understand where does the SM come from?

First GUTs: $\left\{ \begin{array}{l} \dfrac{\mathrm{Spin}(6) \times \mathrm{Spin}(4)}{\mathbb{Z}_2} \quad \text{Pati-Salam } 1973-1974 \\ \mathrm{Spin}(10) \quad \text{Georgi } 1974\,? \quad \text{Minkowski } 1975 \end{array} \right.$

$(SU(5) \to)\ \Lambda \mathbb{C}^5$ $(2^5 = 32)$ Georgi-Glashow 1974

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$(SU(5) \to)\ \Lambda \mathbb{C}^5\ (2^5 = 32)$ Georgi-Glashow 1974

One can obtain an extension G of the gauge group $G_{SM} = S(U(2)\times U(3))$ of the Standard Model as the stabilizer of the lepton-quark splitting of the octonions in J_2^8

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3 \to G_{SM} \subset G \subset Spin(10)$$

J_2^8: Michel Dubois-Violette, I.T. 1806.09450

2. Michel Dubois-Violette, I.T. 1806.09450

Spin(9): Kirill Krasnov 1912.11282v2

Spin(10) $\subset E_6$ Latham Boyle 2006.16265

Here, in 2010.15621 v3,
we postulate superselection of the
weak hypercharge Y, generator of center of $\mathfrak{g}_{SM}$:
It appears naturally in the picture
of Fermi oscillator – CAR
Equivalent to complex structure D-V 1992

$Cl_{10} = Cl_4 \hat{\otimes} Cl_6$, Cl_4 a_α, a^*_α $\alpha = 1,2$ flavour; Cl_6 b_j, b^*_j $j=1,2,3$ colour

Cohl Furey 1806.00612

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Weak isospin $I = I^L$

$$I_+ = a_1^* a_2 \,, \quad I_- = a_2^* a_1 , \quad 2 I_3 = a_1^* a_1 - a_2^* a_2$$

$$B - L = \frac{1}{3} \sum_{j=1}^{3} [b_j^*, b_j]$$

Hypercharge = difference of number operators:

$$\frac{1}{2} Y = \frac{1}{3} \sum_{j=1}^{3} b_j^* b_j - \frac{1}{2} \sum_{\alpha=1}^{2} a_\alpha^* a_\alpha$$

$$U(1)_Y : \quad b_j \to e^{-i\frac{\varphi}{3}} b_j \,, \quad b_j^* \to e^{i\frac{\varphi}{3}} b_j^*$$

$$a_\alpha \to e^{i\frac{\ }{2}} a_\alpha, \quad a^*_\alpha \to e^{-i\frac{\ }{2}} a^*_\alpha$$

Observables and generators of the gauge Lie algebra are $U(1)_Y$ invariant

$$\Longrightarrow \mathfrak{g}_{SM} \subset \mathfrak{g} = \mathfrak{u}(2) \oplus \mathfrak{u}(3) \subset \mathfrak{so}(10)$$

SM gauge symmetry extended by $U(1)_{B-L}$.

$[a^*_\alpha, a_\beta]$, $[b^*_j, b_k]$ are $U(1)_Y$ invariant.

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$$a_\alpha \to e^{i\frac{e}{2}} a_\alpha, \quad a_\alpha^* \to e^{-i\frac{e}{2}} a_\alpha^*$$

Observables and generators of the gauge Lie algebra are $U(1)_Y$ invariant

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Unexpected invariants:

$$\Omega = a_1 a_2 b_1 b_2 b_3 , \quad \Omega^* = b_3^* b_2^* b_1^* a_2^* a_1^* , \quad \Omega^2 = 0 = \Omega^{*2}$$

$$S_\theta = e^{i\theta}\Omega + e^{-i\theta}\Omega^* \in J$$ – Jordan algebra of observables

$$S_\theta^2 = \Omega\Omega^* + \Omega^*\Omega = |\nu_R\rangle\langle\nu_R| + |\bar{\nu}_L\rangle\langle\bar{\nu}_L|$$

sterile (anti)neutrino – allowed to mix into a Majorana neutrino

... both $|\nu_R\rangle$ and $|\bar{\nu}_L\rangle$ have $Y=0$.

$\Omega = a_1 a_2 b_1 b_2 b_3 ,$ …

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sterile (anti)neutrino – allowed to mix into a Majorana neutrino since both $|\nu_R\rangle$ and $|\bar{\nu}_L\rangle$ have $Y=0$.

$J_{s\nu} = J_2^2$ $\quad \dim J_{s\nu} = 4 \quad \mathfrak{g}_{SM} J_{s\nu} = 0.$

Particles and antiparticles are only distinguished if $Y \neq 0$.

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Projector on the 15 dimensional particle subspace:

$P = \ell + q$ $\quad \ell \, (= \ell^2)$ projects on 3 leptons:

$\nu_L, e_L \; (Y = -1); \; e_R \; (Y = -2)$

$q = q_1 + q_2 + q_3$ $\quad q_j$ - coloured quark in 4 flavours

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Elementary projectors:

$$\pi_\alpha = a_\alpha a_\alpha^*, \quad \pi_\alpha' = a_\alpha^* a_\alpha, \quad \alpha = 1, 2$$

$$p_j = b_j b_j^*, \quad p_j' = b_j^* b_j, \quad j = 1, 2, 3.$$

$$\pi_\alpha^2 = \pi_\alpha, \quad {\pi_\alpha'}^2 = \pi_\alpha', \quad \pi_\alpha \pi_\alpha' = 0, \quad \pi_\alpha + \pi_\alpha' = 1$$

$$p_j^2 = p_j, \quad {p_j'}^2 = p_j', \quad p_j + p_j' = 1, \quad p_j p_j' = 0.$$

In terms of these

$q_j = p_j p_k' p_l'$ where $(jkl) \in P$ ...

$$\pi_\alpha = a_\alpha a_\alpha^*, \quad \pi_\alpha' = a_\alpha^* a_\alpha, \quad \alpha = 1, 2$$

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In terms of these

$$q_j = p_j p_k' p_\ell' \quad \text{where } (j, k, \ell) \in \mathrm{Perm}(1,2,3)$$

$$\ell = p_1 p_2 p_3 (1 - \pi_1 \pi_2) = p_1 p_2 p_3 - |\nu_R\rangle\langle \nu_R|$$

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$\ell = p_1 p_2 p_3 (1-\pi_1 \pi_2) = p_1 p_2 p_3 - |\nu_R\rangle\langle\nu_R|$

Primitive projectors (leptons):

$|\nu_L\rangle\langle\nu_L| = \pi_1'\pi_2 p_1 p_2 p_3 = \pi_1'\pi_2 \ell$ Products of 5 elementary

$|e_L\rangle\langle e_L| = \pi_1\pi_2'\ell$, $|e_R\rangle\langle e_R| = \pi_1\pi_2\ell$

Jordan algebra J of superselected observables $J = J_p \oplus J_{\bar{p}} \oplus J_{s\nu}$

Particles + antiparticle + sterile neutrino

D-V.T. 2003.06591 have taken $P_0 = \ell_0 + q$, $\ell_0 = p_1 p_2 p_3 = \ell + (\nu_R)$

$P P_0 = P$..

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Decomposition of the particle Jordan algebra into simple (irreducible) components J_r^d

r = rank, d = degree (labeled by Y)

$$J_p = J_2^2 \oplus J_6^2 \text{ (left chiral)}; \quad \oplus \mathbb{R}e_R \oplus J_3^2 \oplus J_3^2 \text{ right chiral}$$

$$Y = -1 \quad \tfrac{1}{3} \qquad\qquad -2 \quad \tfrac{4}{3} \quad \tfrac{2}{3}$$

Sterile neutrinos form a simple component $J_{s\nu} = J_2^2$, $\underline{\mathfrak{g}_{SM} J_{s\nu} = 0}$

Aside: What give us (special) Jordan algebras?

$$X \circ Y = Y \circ X \qquad X^2 = X \circ X, \; X^{n+1} = X^n \circ X$$

Power associativity; partially ordered $X^2 > 0$ for $X \neq 0$. For finite dimensional $\operatorname{tr} X^2 > 0$ (if $X \neq 0$).

Observables in J play a dual role: they generate symmetry:

"α - product"

$$\frac{dX}{dt} = \frac{1}{i\hbar}[X, H] =: X \alpha H = - H \alpha X$$

Two products symmetric and skewsymmetric hermitian matrices

$$\dim J_r^2 = r^2$$

J_r^2 are all special = Jordan subalgebras of associative algebras with Jordan product

$$X \circ Y = \frac{1}{2}[X,Y]_+ = \frac{1}{2}(XY + YX)$$

10

$Cl_{10} = R[32]$ – there are 32 primitive idempotents in Cl_{10} corresponding to first generation (anti) fermions.

Particle subspace (including a sterile neutrino $\mathbb{C}\mathbb{O}^2$ { Bryant 2011.05568
Krasnov 2021

The Higgs field

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(10)

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The Higgs field

Finite quantum algebra was

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first envisaged in the framework of noncommutative geometry (Connes; Kerner, Madore, Dubois-Violette, 1988–) with the idea to give a geometric meaning of the Higgs boson.

The Higgs field finds a natural place in the odd part of a $\mathbb{Z}_2$-graded Clifford algebra.

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$$Cl_{10} = Cl_{10}^0 \oplus Cl_{10}^1, \quad \mathfrak{g} \subset spin(10) \subset Cl_{10}^0$$

$$Higgs \in Cl_{10}^1, \quad \mathfrak{g} \subset Cl_{10}^0$$

Physical Fermi fields are tensor products of Dirac fields with internal space vectors. (cf. Chamseddine, Iliopoulos, Suijlekom 2009.03367v2 who use 384×384 matrices)

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Physical Fermi fields are tensor products of Dirac fields with internal space vectors. (Cf. Chamseddine, Iliopoulos, Suijlekom 2009.03367v2 who use 384x384 matrices)

Dirac fields include antiparticles — fermion doubling problem.

Particle field algebra: $\mathcal{F}_p = P\mathcal{F}P$

$P = \ell + q$, $P f_j^{(*)} P = 0 \Rightarrow$ No symmetry breaking for QCD!

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$\ell\, a_2^* \ell = b\, a_2^* a_1 - \ldots$

$$[F_\pm, \bar{F}_\pm]_+ = I_\pm\,, \quad [F_\pm, \bar{F}_\mp]_+ = Z \mp I_3\,, \quad 2Z = \ell Y$$

Lie superalgebra proposed in 1979 by Ne'eman and Fairly (independently)

<u>Superconnection</u> rediscovered (and named) by Quillen 1985 in pure math, used by Coquereaux (1990) ... most recently by

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by Ne'eman and Fairley
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Jean Thierry-Mieg; 2003.12234, 2005.04754

Original observation

vanishing supertrace $tr\, Q|_{Y_{Left}} = tr\, Q|_{Y_{Right}}$

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Original observation

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Superconnection

$$\mathbb{D} = \chi D + \Phi, \quad D = d + A = dx^\mu(\partial_\mu + A_\mu)$$

$$iA_\mu = W_\mu + B_\mu + G_\mu \quad (\in Cl^0_{10}), \quad \Phi \in Cl^1_4$$

$$\pi_\alpha = a_\alpha a_\alpha^*$$

$$\chi = [a_1^*, a_1][a_2^*, a_2] = (\pi_1' - \pi_1)(\pi_2' - \pi_2) \qquad \pi_\alpha' = a_\alpha^* a_\alpha$$

projection of chirality on Cl_4

$$P_L = \pi_1' \pi_2 + \pi_1 \pi_2', \quad P_R = \pi_1 \pi_2 + \pi_1' \pi_2'$$

$\Longleftrightarrow$ associativity: $\mathbb{D}\mathbb{D}^2=\mathbb{D}^2\mathbb{D}$

$\mathbb{D}^2$ - canonical curvature

The bosonic Lagrangian is the square of the curvature. The square of the canonical curvature gives a Higgs potential with minimum at $\Phi=0 \Rightarrow$ no symmetry breaking. Happily the Bianchi identity is still

square of the curvature. [illegible]
of the canonical curvature
gives a Higgs potential with
minimum at $\Phi = 0 \Rightarrow$ no
symmetry breaking. Happily
the Bianchi identity is still
satisfied if we add to $\mathbb{D}^2$ a
constant matrix operator

$$\hat{m}^2 = m^2(\ell + g^2 q).$$

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Higgs mass

$$V(\Phi) = \frac{1+6\rho^4}{2}(\phi\bar{\phi})^2 - (1+6\rho^4)\bar{\phi}\phi m^2 + (1+6\rho^4)m^4$$

$$\Rightarrow \quad m_H^2 = (1+6\rho^4)m^2 \quad \Big\} \text{coefficient to } -\bar{\phi}\phi$$

$$m_W^2 = \frac{1+6\rho^2}{4} m^2$$

$$\Longrightarrow \quad m_H^2 = 4\,\frac{1+6\rho^4}{1+6\rho^2}\, m_W^2 \underset{\text{for } \rho^2 = \frac{1}{2}}{=} \frac{5}{2} m_W^2 = 4\cos^2\theta_W\, m_W^2$$

$$m_H = 2\cos\theta_W\, m_W = \sqrt{\frac{5}{2}}\, m_W$$

Satisfied within 1% accuracy.

18

Higgs mass

$$V(\Phi) = \frac{1+6\rho^4}{2}(\bar{\phi}\phi)^2 - (1+6\rho^4)\bar{\phi}\phi\, m^2 + (1+6\rho^4)m^4$$

$$\Rightarrow \quad m_H^2 = (1+6\rho^4)m^2$$ coefficient to $-\bar{\phi}\phi$

$$m_W^2 = \frac{1+6\rho^2}{4} m^2$$

$$\Rightarrow \quad m_H^2 = 4\,\frac{1+6\rho^4}{1+6\rho^2}\, m_W^2 = \frac{5}{2} m_W^2 = 4\cos^2\theta_W\, m_W^2$$

for $\rho^2 = \frac{1}{2}$

$$m_H = 2\cos\theta_W\, m_W = \sqrt{\frac{5}{2}}\, m_W$$

Satisfied within 1% accuracy.

$$m_H = 2\cos\theta_W \, m_W = \sqrt{\tfrac{5}{2}}\, m_W .$$

Satisfied within 1% accuracy.

Concluding remarks

- $Cl_{10} = Cl_4 \hat{\otimes} Cl_6 \quad Spin(10) \subset Cl^0_{10}, \; Higgs \in Cl^1_4$
- Fermions $\in \mathbb{C}\mathbb{O}^2$ but ν_R may mix with ν_L.
- GUT relations at unification recovered.
- New relation $m_H^2 = 4\cos^2\theta_W \, m_W^2 = \frac{5}{2} m_W^2$.
- Open problem: three generations - CKM mixing matrix?

The role of triality in Yukawa
... reduce the number

Concluding [remarks]

- $Cl_{10} = Cl_4 \hat{\otimes} Cl_6$ $\quad Spin(10) \subset Cl^0_{10}$, Higgs $\in Cl^1_4$
- Fermions $\in \mathbb{C}\mathbb{O}^2$ but ν_R may mix with D_L.
- GUT relations at unification recovered.
- New relation $m_H^2 = 4\cos^2\theta_W \, m_W^2 = \frac{5}{2} m_W^2$.
- Open problem: three generations - CKM mixing matrix? The role of triality in Yukawa interactions. → Reduce the number of parameters?! Understand the mass spectrum of fundamental fermions?

→ Extension of G_{SM} - Will we need more Higgses?