Title: A Magic Pyramid of Supergravity Theories from Yang-Mills Squared

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Collection: Octonions and the Standard Model

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Abstract: "I will begin by reviewing the unified description of pure Super Yang-Mills (SYM) Theory (consisting of just a gauge field and gaugino) in dimensions 3, 4, 6, and 10 over the four normed division algebras R, C, H, and O. Dimensionally reducing these initial theories into dimensions 3, 4, 5, 6, 7, 8, 9, 10 gives a plethora of SYM theories written over the division algebras, with a single master Lagrangian to rule them all. In particular, in D = 3 spacetime dimensions, the SYM theories with N = 1, 2, 4, and 8 supersymmetries enjoy a unified description over R, C, H, and O, respectively. In each spacetime dimension, maximally supersymmetric theories are written over the octonions.

In apparently completely different developments, a popular thread in attempts to understand the quantum theory of gravity is the idea of "gravity as the square of Yang-Mills". The idea in its most basic form is that a symmetric tensor (graviton) can be built from the symmetric tensor product of two vectors (Yang-Mills fields), an idea which can be extended to obtain entire supergravity multiplets from tensor products of SYM multiplets. Having established a division-algebraic description of Super Yang-Mills theories, I will then demonstrate how tensoring these multiplets together results in supergravity theories valued over tensor products of division algebras.

In D = 3, there are 4 SYM theories (N = 1, 2, 4, 8 over R, C, H, O) and so there are 4 x 4 = 16 possible supergravity theories to obtain by "squaring Yang-Mills". The global symmetries of these 16 division-algebraic SYM-squared supergravity theories are precisely those belonging to the 4 x 4 Freudenthal-Rosenfeld-Tits "magic square" of Lie algebras! Furthermore, the scalar fields in these supergravity theories describe non-linear sigma models, whose target space manifolds are division algebraic projective planes! Performing the same tensoring of SYM theories in spacetime dimensions D > 3 results in a whole "magic pyramid" of supergravities, with the magic square at the base in D = 3 and Type II supergravity at the apex in D = 10. This construction gives an explicit octonionic explanation of many of the mysterious appearances of exceptional groups within string/M-theory and supergravity."

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A MAGIC PYRAMID OF SUPERGRAVITIES

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A Unifying View of Simple Lie Groups

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Triality Algebras

The Magic Square

Division Algebras and Super Yang-Mills

A Magic Square from (Yang-Mills)^2

A Magic Pyramid of Supergravity Theories

All simple Lie groups/Lie algebras may be expressed in terms of division algebras

- $\mathfrak{so}(n) = \{X \in \mathbb{R}[n] \mid X^\dagger = -X\}$
- $\mathfrak{u}(n) = \{X \in \mathbb{C}[n] \mid X^\dagger = -X\}$
- $\mathfrak{sp}(n) = \{X \in \mathbb{H}[n] \mid X^\dagger = -X\}$

The **exceptional groups** are constructed from $\mathbb{O}$ (although it's more involved than for the classical groups above)

We will construct $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ using "triality algebras": the Barton-Sudbery approach

Orthogonal Transformations on $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

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Triality Algebras

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A Magic Pyramid of Supergravity Theories

- The normed division algebra $\mathbb{A}$ has a natural Euclidean inner product; for any $v, w \in \mathbb{A}$:

$$\mathrm{Re}(v^* w) = \tfrac{1}{2}(v^* w + v^* w) = v_a w_a, \quad a = 0, 1, \cdots, \dim[\mathbb{A}] - 1$$

- A rep of the orthogonal group $O(\mathbb{A}) \cong O(\dim[\mathbb{A}])$ preserves this, and the action of its Lie algebra $\mathfrak{so}(\mathbb{A})$ can be written as:

$$\delta v = \tfrac{1}{4}\theta^{ab}\left(e_a(e_b^* v) - v(e_a^* e_b)\right), \qquad \theta^{ab} = -\theta^{ba}$$

- In terms of the components of $v = v_a e_a$, this is just multiplication by the antisymmetric matrix θ:

$$\delta v = (\theta^{ab} v_b) e_a$$

$\mathbb{A}$'s Structure Constants as Pauli Matrices

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- In general, the division-algebraic multiplication rule is:

$$e_a\, e_b = t_{abc}\, e_c$$

- If we think of the structure constants as a set of $(\dim[\mathbb{A}]) \times (\dim[\mathbb{A}])$ matrices

$$t_{abc} = (\sigma_a)_{bc} = (\bar{\sigma}_a)_{cb}$$

- Then the multiplication rule may be written as

$$e_a\, e_b = (\sigma_a)_{bc}\, e_c\,, \qquad e_a^*\, e_b = (\bar{\sigma}_a)_{bc}\, e_c$$

- Why would we want to do this?!

$\mathbb{A}$'s Structure Constants as Pauli Matrices

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- It's easy to see that for any $x \in \mathbb{A}$:

$$e_a(e_b^* x) + e_b(e_a^* x) = 2\,\delta_{ab}\,x$$
$$e_a^*(e_b\,x) + e_b^*(e_a\,x) = 2\,\delta_{ab}\,x$$

- Bearing in mind our multiplication rules,

$$e_a\,e_b = (\sigma_a)_{bc}\,e_c\,, \qquad e_a^*\,e_b = (\bar{\sigma}_a)_{bc}\,e_c$$

- The structure constants must act as Pauli matrices of $\mathfrak{so}(\mathbb{A})$!

$$\sigma^a\bar{\sigma}^b + \sigma^b\bar{\sigma}^a = 2\delta^{ab}\,\mathbb{1}$$
$$\bar{\sigma}^a\sigma^b + \bar{\sigma}^b\sigma^a = 2\delta^{ab}\,\mathbb{1}$$

Spinor Reps of $\mathfrak{so}(\mathbb{A})$

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- We can then write (semi-)spinor reps S_+ and S_- of $\mathfrak{so}(\mathbb{A})$ in terms of $\mathbb{A}$:

$$\delta\psi = \tfrac{1}{4}\theta^{ab} e_a(e_b^* \psi) \quad (\text{cf. } \delta\Psi = \tfrac{1}{4}\theta^{\mu\nu}\sigma_\mu\bar{\sigma}_\nu\Psi)$$
$$\delta\chi = \tfrac{1}{4}\theta^{ab} e_a^*(e_b \chi) \quad (\text{cf. } \delta\mathcal{X} = \tfrac{1}{4}\theta^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu\mathcal{X})$$

- Along with the vector rep V:

$$\delta v = \tfrac{1}{4}\theta^{ab}\left(e_a(e_b^* v) - v(e_a^* e_b)\right),$$

we now have three $\mathfrak{so}(\mathbb{A})$ reps: V, S_+ and S_-, each as a copy of $\mathbb{A}$

- For $\mathbb{A} = \mathbb{O}$ these 3 reps are the $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$

The Trilinear Form

- So in fact division-algebraic multiplication is just Pauli matrix multiplication, intertwining the 3 reps V, S_+, S_-!

$$
\begin{aligned}
v\,\psi &= v^a \psi^c\,(\bar{\sigma}_a)_{bc}\,e_b = (\not{v}\,\psi)_a\,e_a, \\
\chi\,\psi^* &= \chi^b \psi^c\,(\bar{\sigma}_a)_{bc}\,e_a = (\chi^{\mathsf{T}}\bar{\sigma}_a\psi)e_a, \\
v^*\chi &= v^a \chi^b\,(\bar{\sigma}_a)_{bc}\,e_c = (\chi^{\mathsf{T}}\not{v})_a\,e_a,
\end{aligned}
$$

- Multiplying two division algebra elements and taking the inner product with a third, we get a real scalar:

$$\tfrac{1}{2}(\chi^*(v\psi) + (v\psi)^*\chi) = v_a\psi_b\chi_c\,t_{abc} = \chi^{\mathsf{T}}\not{v}\,\psi$$

Non-degenerate Trilinear Forms

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- So each normed division algebra $\mathbb{A}$ gives rise to a trilinear form, called a "normed triality" (Adams 1981):

$$t(v, \psi, \chi) = v_a \psi_b \chi_c t_{abc} = \chi^T \not{v} \psi$$

- Because of the division algebra property, this trilinear form is non-degenerate (and because of the normed property, the triality is "normed")

- Such trilinear forms are extremely rare: there's one for each normed division algebra, and that's it!

Automorphisms of the Normed Triality

- Of course, the normed triality $\chi^{\mathsf{T}} \not{\psi}\, \psi$ is invariant under transformations generated by our $\mathfrak{so}(\mathbb{A})$

- For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ it is also invariant under under (right-)multiplication of the spinors by an arbitrary unit-norm element $u \in \mathbb{A}$:

$$\psi \to \psi u, \qquad \chi \to \chi u, \qquad u^* u = 1.$$

- The Lie algebras of the triality's symmetries are called triality algebras:

$$\begin{aligned}
\mathfrak{tri}(\mathbb{R}) &= \mathfrak{so}(1) &&= \varnothing \\
\mathfrak{tri}(\mathbb{C}) &= \mathfrak{so}(2) \oplus \mathfrak{u}(1) &&= \mathfrak{u}(1) \oplus \mathfrak{u}(1) \\
\mathfrak{tri}(\mathbb{H}) &= \mathfrak{so}(4) \oplus \mathfrak{su}(2) &&= \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \\
\mathfrak{tri}(\mathbb{O}) &= \mathfrak{so}(8) &&= \mathfrak{so}(8)
\end{aligned}$$

More About Triality Algebras

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- Triality Algebras
- The Magic Square
- Division Algebras and Super Yang-Mills
- A Magic Square from (Yang-Mills)^2
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- A useful definition of triality algebras (Barton & Sudbery 2002):

$$\mathfrak{tri}(\mathbb{A}) := \{(A, B, C) \in \mathfrak{so}(\mathbb{A}) \oplus \mathfrak{so}(\mathbb{A}) \oplus \mathfrak{so}(\mathbb{A}) \\ \text{s.t. } A(xy) = B(x)y + x\,C(y) \;\; \forall x, y \in \mathbb{A}\}$$

- We think of elements of $\mathfrak{tri}(\mathbb{A})$ as triples (A, B, C) of $\mathfrak{so}(\mathbb{A})$ elements acting on $3\mathbb{A} \cong S_- \oplus V \oplus S_+$, e.g.

$$A = \tfrac{1}{4}\theta^{ab} e_a^*(e_b \cdot\,)$$
$$B = \tfrac{1}{4}\theta^{ab}((e_a(e_b^* \cdot\,) - (\,\cdot\,)(e_a^*\, e_b))$$
$$C = \tfrac{1}{4}\theta^{ab} e_a(e_b^* \cdot\,)$$

which acts on $(\chi, v, \psi) \in 3\mathbb{A}$

F_4 and Triality

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Triality Algebras

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A Magic Square from (Yang-Mills)^2

A Magic Pyramid of Supergravity Theories

- $\mathfrak{f}_4$ is a Lie algebra built around octonionic triality:

$$\mathfrak{f}_4 \cong \mathfrak{tri}(\mathbb{O}) + \mathbb{O} + \mathbb{O} + \mathbb{O}$$
$$\mathbf{52} \to \mathbf{28} + \mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c$$

- The commutators between $\mathfrak{tri}(\mathbb{O})$ and $3\mathbb{O}$ are given by the natural action of $\mathfrak{tri}(\mathbb{O})$ on $3\mathbb{O}$:

$$[(A, B, C), (\chi, v, \psi)] = (A\chi, Bv, C\psi)$$

- The commutators between different copies of $\mathbb{O}$ are just what one might guess from triality, e.g.

$$[(0, v, 0), (0, 0, \psi)] = (v\psi, 0, 0)$$

The Magic Square

Copying $\mathfrak{f}_4 \cong \mathfrak{tri}(\mathbb{O}) + 3\mathbb{O}$, but swapping $\mathbb{O}$ with a tensor product of two different division algebras $\mathbb{A}_L \otimes \mathbb{A}_R$ results in the Barton-Sudbery version of the (Freudenthal-Rosenfeld-Tits) “magic square”:

$$\mathfrak{M}(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \otimes \mathbb{A}_R)$$

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

The **exceptional groups** *all* come from the **octonions**!

Projective Planes

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It is also possible to define projective *planes* over tensor products of division algebras:

$$(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2$$

The magic square algebras are the isometries of these projective planes!

$$\mathfrak{isom}\left((\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2\right) = \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \otimes \mathbb{A}_R)$$

These planes will show up later in our supergravity theories...

A Nice Lie Algebra Isomorphism: $\mathfrak{sl}(2,\mathbb{A}) \cong \mathfrak{so}(1, \dim\mathbb{A}+1)$

Consider the Hermitian matrix

$$\mathbf{x} = \begin{pmatrix} t+z & x^* \\ x & t-z \end{pmatrix}, \quad \text{where } t, z \in \mathbb{R},\ x \in \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$$

Determinant of $\mathbf{x}$ gives the Minkowski metric for $(\dim\mathbb{A}+2)$-dimensional spacetime:

$$\det\mathbf{x} = t^2 - z^2 - |x|^2$$

$SL(2,\mathbb{A})$ transformations A leave the determinant, and hence the Minkowski norm, invariant:

$$\mathbf{x} \to A\mathbf{x}A^\dagger, \qquad A \in SL(2,\mathbb{A})$$

Kugo, Townsend 1983, Sudbery 1983

A Nice Lie Algebra Isomorphism: $\mathfrak{sl}(2, \mathbb{A}) \cong \mathfrak{so}(1, \dim \mathbb{A} + 1)$

Similarly, a spinor Ψ in $D = \dim \mathbb{A} + 2$ can be written as a 2-component column

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \psi_{1,2} \in \mathbb{A},$$

and transforms as

$$\Psi \to A\Psi$$

Kugo, Townsend 1983, Sudbery 1983

We can use this language to formulate field theories in 3, 4, 6 and 10 dimensions over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

A Nice Lie Algebra Isomorphism: $\mathfrak{sl}(2,\mathbb{A}) \cong \mathfrak{so}(1, \dim\mathbb{A}+1)$

Consider the Hermitian matrix

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Kugo, Townsend 1983, Sudbery 1983

Super Yang-Mills in $D = 3, 4, 6, 10$

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Pure $\mathcal{N} = 1$ super Yang-Mills in $D = 3, 4, 6, 10$ consists only of a gauge field and spinor transforming in the adjoint, so we can write the four theories over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$:

$$\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} - \mathrm{Re}(i\Psi^{\dagger A} \bar{\sigma}^\mu D_\mu \Psi^A)$$

Kugo, Townsend 1983, Evans 1988, Baez, Huerta 2009

with supersymmetry transformations

$$\delta A^A_\mu = \mathrm{Re}(i\Psi^{A\dagger}\bar{\sigma}_\mu \epsilon), \qquad \delta\Psi^A = \tfrac{1}{4} F^A_{\mu\nu}\sigma^\mu(\bar{\sigma}^\nu \epsilon).$$

For $\mathbb{C}, \mathbb{H}$ (and $\mathbb{R}$) this has an additional R-symmetry:

$$\Psi \to \Psi u, \qquad \epsilon \to \epsilon u, \qquad u^* u = 1.$$

Triality Algebras and Super Yang-Mills

- The on-shell degrees of freedom with momentum $p_\mu = (E, 0, \cdots, 0, E)$ are reps of the little group $\mathfrak{so}(\mathbb{A})$ inside $\mathfrak{sl}(2, \mathbb{A})$:

$$A \to a \in \mathbb{A}^{\mathbb{I}}, \qquad \Psi \to \psi \in \mathbb{A}, \qquad \epsilon \to \chi \in \mathbb{A}$$

  with supersymmetry transformations

$$\delta a = -i\chi\psi^*, \qquad \delta\psi = iE\, a^*\chi$$

- Including the R-symmetry that means **the on-shell symmetry of SYM is** $\mathfrak{so}(\mathbb{A}) \oplus \mathfrak{R} \cong \mathfrak{tri}(\mathbb{A})$!

A Nice Lie Algebra Isomorphism: $\mathfrak{sl}(2, \mathbb{A}) \cong \mathfrak{so}(1, \dim \mathbb{A} + 1)$

Consider the Hermitian matrix

$$\mathbf{x} = \begin{pmatrix} t+z & x^* \\ x & t-z \end{pmatrix}, \quad \text{where } t, z \in \mathbb{R},\, x \in \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$$

Determinant of $\mathbf{x}$ gives the Minkowski metric for $(\dim \mathbb{A} + 2)$-dimensional spacetime:

$$\det \mathbf{x} = t^2 - z^2 - |x|^2$$

$SL(2, \mathbb{A})$ transformations A leave the determinant, and hence the Minkowski norm, invariant:

$$\mathbf{x} \to A\mathbf{x}A^\dagger, \qquad A \in SL(2, \mathbb{A})$$

Kugo, Townsend 1983, Sudbery 1983

Super Yang-Mills Over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

Dimensional reduction gives whole stacks of SYM theories:

$$
\begin{array}{r|llll}
D=10 & \mathcal{N}=1\ (\mathbb{O}) & & & \\
D=9 & \mathcal{N}=1\ (\mathbb{O}) & & & \\
D=8 & \mathcal{N}=1\ (\mathbb{O}) & & & \\
D=7 & \mathcal{N}=1\ (\mathbb{O}) & & & \\
D=6 & \mathcal{N}=2\ (\mathbb{O}) & \mathcal{N}=1\ (\mathbb{H}) & & \\
D=5 & \mathcal{N}=2\ (\mathbb{O}) & \mathcal{N}=1\ (\mathbb{H}) & & \\
D=4 & \mathcal{N}=4\ (\mathbb{O}) & \mathcal{N}=2\ (\mathbb{H}) & \mathcal{N}=1\ (\mathbb{C}) & \\
D=3 & \mathcal{N}=8\ (\mathbb{O}) & \mathcal{N}=4\ (\mathbb{H}) & \mathcal{N}=2\ (\mathbb{C}) & \mathcal{N}=1\ (\mathbb{R})
\end{array}
$$

Dickson-doubling the division algebra doubles the number of supersymmetries; maximal SUSY $\Leftrightarrow$ octonions

Anastasiou, Borsten, Duff, MJH, Nagy 2013

Super Yang-Mills Over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

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Example: $D = 3$ SYM with $\mathcal{N} = 1, 2, 4, 8$ over $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{2}D_\mu\phi^{A*}D^\mu\phi^A - i\bar{\Psi}^A\sigma^\mu D_\mu\Psi^A \\ &-\frac{1}{16}g^2 f_{BC}{}^A f_{DE}{}^A(\phi^{B*}\phi^D + \phi^{D*}\phi^B)(\phi^{C*}\phi^E + \phi^{E*}\phi^C) \\ &-gf_{BC}{}^A\mathrm{Re}\left(i\bar{\Psi}^A\phi^B\Psi^C\right),\end{aligned}$$

where $\Psi \in \mathbb{A}^2$ and $\phi \in \mathrm{Im}(\mathbb{A})$

Borsten, Duff, MJH, Nagy 2013

Yang-Mills Symmetries

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A Magic Square from (Yang-Mills)^2

A Magic Pyramid of Supergravity Theories

In the dimensionally reduced theories the on-shell states have a spacetime symmetry of $\mathfrak{so}(D-2)$ plus internal symmetry $\mathfrak{int}_D(\mathbb{A})$, which is the commutant of $\mathfrak{so}(D-2)$ inside $\mathfrak{tri}(\mathbb{A})$

$$\mathfrak{so}(D-2) \oplus \mathfrak{int}_D(\mathbb{A}) \subseteq \mathfrak{tri}(\mathbb{A})$$

In $D=3$ the $\mathfrak{so}(D-2)$ becomes trivial and so the internal symmetry is the whole of $\mathfrak{tri}(\mathbb{A})$ again! In the $D=3$ linearised theory the field strengths of the real vector $F^A_{\mu\nu} \in \mathbb{R}$ and imaginary scalars $\partial_\mu \phi^A \in \mathrm{Im}(\mathbb{A})$ may be reunited as a full $\mathbb{A}$ element again:

$$\tfrac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho\,A} + \partial_\mu \phi^A \in \mathbb{A}$$

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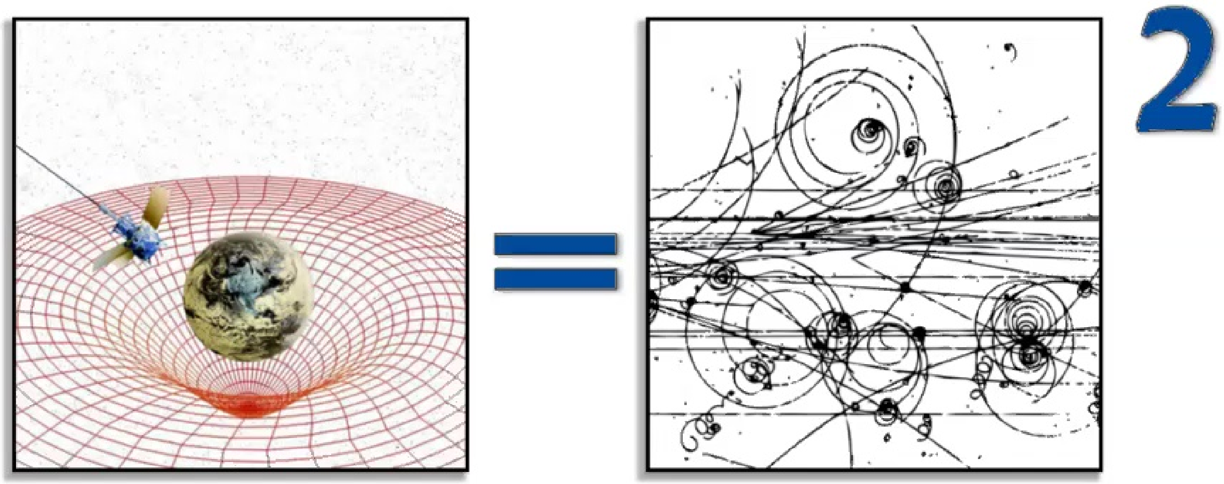
Gravity = (Yang-Mills)^2



Recent progress in scattering amplitudes shows that in a certain sense gravity is the “square of” Yang-Mills

SYM^2 in $D = 3$

Tensoring SYM multiplets in $D = 3$:

$\mathbb{A}_L/\mathbb{A}_R$	$A_\mu(R) \in \text{Re}\mathbb{A}_R$	$\phi(R) \in \text{Im}\mathbb{A}_R$	$\Psi(R) \in \mathbb{A}_R$
$A_\mu(L) \in \text{Re}\mathbb{A}_L$	$g_{\mu\nu} + \varphi \in \text{Re}\mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\varphi \in \text{Re}\mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\Psi_\mu + \chi \in \text{Re}\mathbb{A}_L \otimes \mathbb{A}_R$
$\phi(L) \in \text{Im}\mathbb{A}_L$	$\varphi \in \text{Im}\mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\varphi \in \text{Im}\mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\chi \in \text{Im}\mathbb{A}_L \otimes \mathbb{A}_R$
$\Psi(L) \in \mathbb{A}_L$	$\Psi_\mu + \chi \in \mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\chi \in \mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\varphi \in \mathbb{A}_L \otimes \mathbb{A}_R$

Borsten, Duff, MJH, Nagy 2013

Grouping spacetime fields of the same type,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}, \quad \varphi, \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}$$

we find the field content of a supergravity theory with $\mathcal{N} = \dim[\mathbb{A}_L] + \dim[\mathbb{A}_R] = \mathcal{N}_L + \mathcal{N}_R$ supersymmetries

SYM^2 in $D = 3$

Tensoring SYM multiplets in $D = 3$:

$\mathbb{A}_L/\mathbb{A}_R$	$A_\mu(R) \in \text{Re}\mathbb{A}_R$	$\phi(R) \in \text{Im}\mathbb{A}_R$	$\Psi(R) \in \mathbb{A}_R$
$A_\mu(L) \in \text{Re}\mathbb{A}_L$	$g_{\mu\nu} + \varphi \in \text{Re}\mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\varphi \in \text{Re}\mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\Psi_\mu + \chi \in \text{Re}\mathbb{A}_L \otimes \mathbb{A}_R$
$\phi(L) \in \text{Im}\mathbb{A}_L$	$\varphi \in \text{Im}\mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\varphi \in \text{Im}\mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\chi \in \text{Im}\mathbb{A}_L \otimes \mathbb{A}_R$
$\Psi(L) \in \mathbb{A}_L$	$\Psi_\mu + \chi \in \mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\chi \in \mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\varphi \in \mathbb{A}_L \otimes \mathbb{A}_R$

Borsten, Duff, MJH, Nagy 2013

Grouping spacetime fields of the same type,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}, \quad \varphi, \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}$$

we find the field content of a supergravity theory with $\mathcal{N} = \dim[\mathbb{A}_L] + \dim[\mathbb{A}_R] = \mathcal{N}_L + \mathcal{N}_R$ supersymmetries

A Magic Square from Yang-Mills Squared

Supergravity theories have non-compact global symmetry "U-duality" groups G and scalars living in G/H, where H is the maximal compact subgroup of G

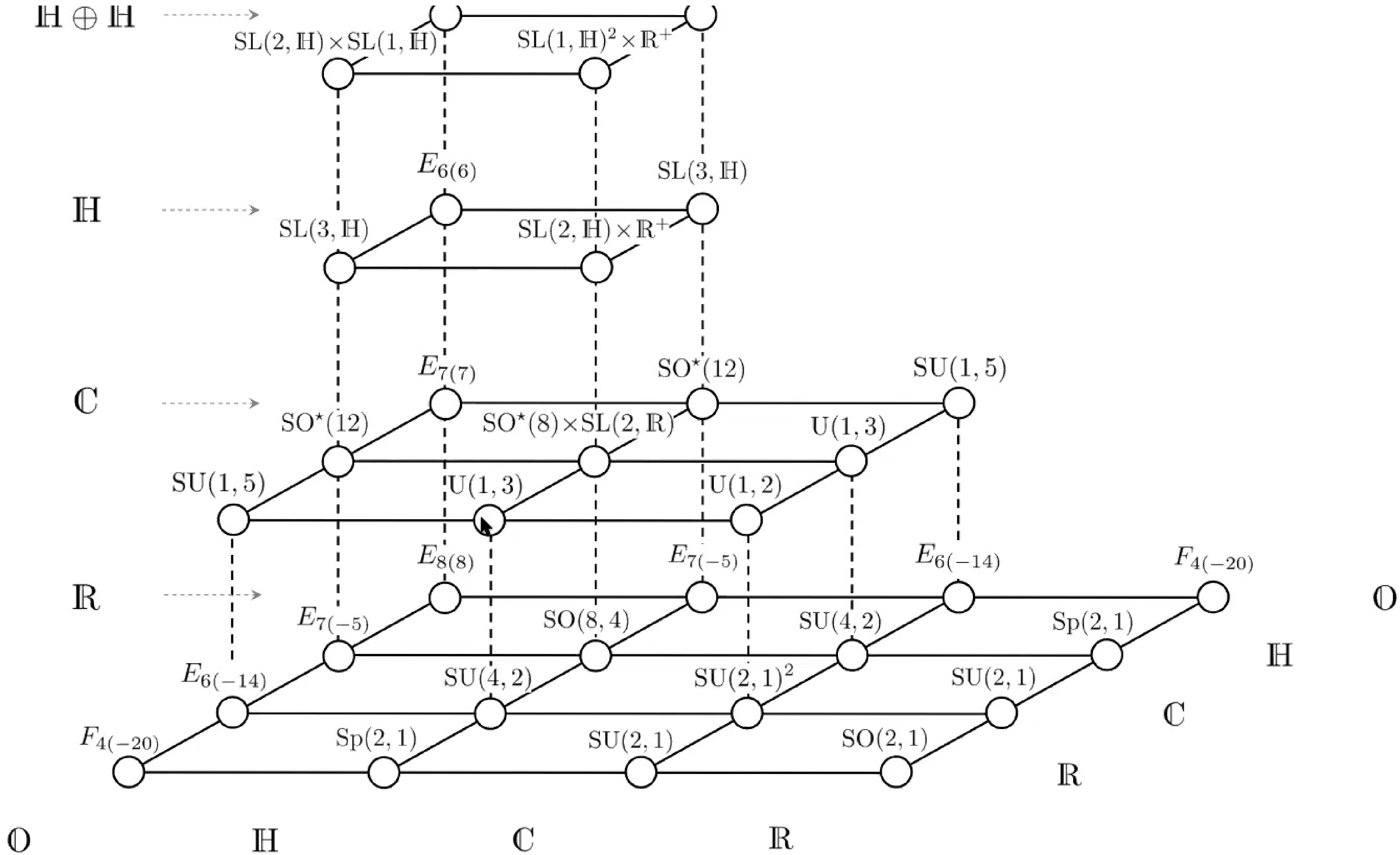
$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SU}(2,1)$	$\mathrm{USp}(4,2)$	$F_{4(-20)}$
$\mathbb{C}$	$\mathrm{SU}(2,1)$	$\mathrm{SU}(2,1)\times\mathrm{SU}(2,1)$	$\mathrm{SU}(4,2)$	$E_{6(-14)}$
$\mathbb{H}$	$\mathrm{USp}(4,2)$	$\mathrm{SU}(4,2)$	$\mathrm{SO}(8,4)$	$E_{7(-5)}$
$\mathbb{O}$	$F_{4(-20)}$	$E_{6(-14)}$	$E_{7(-5)}$	$E_{8(8)}$

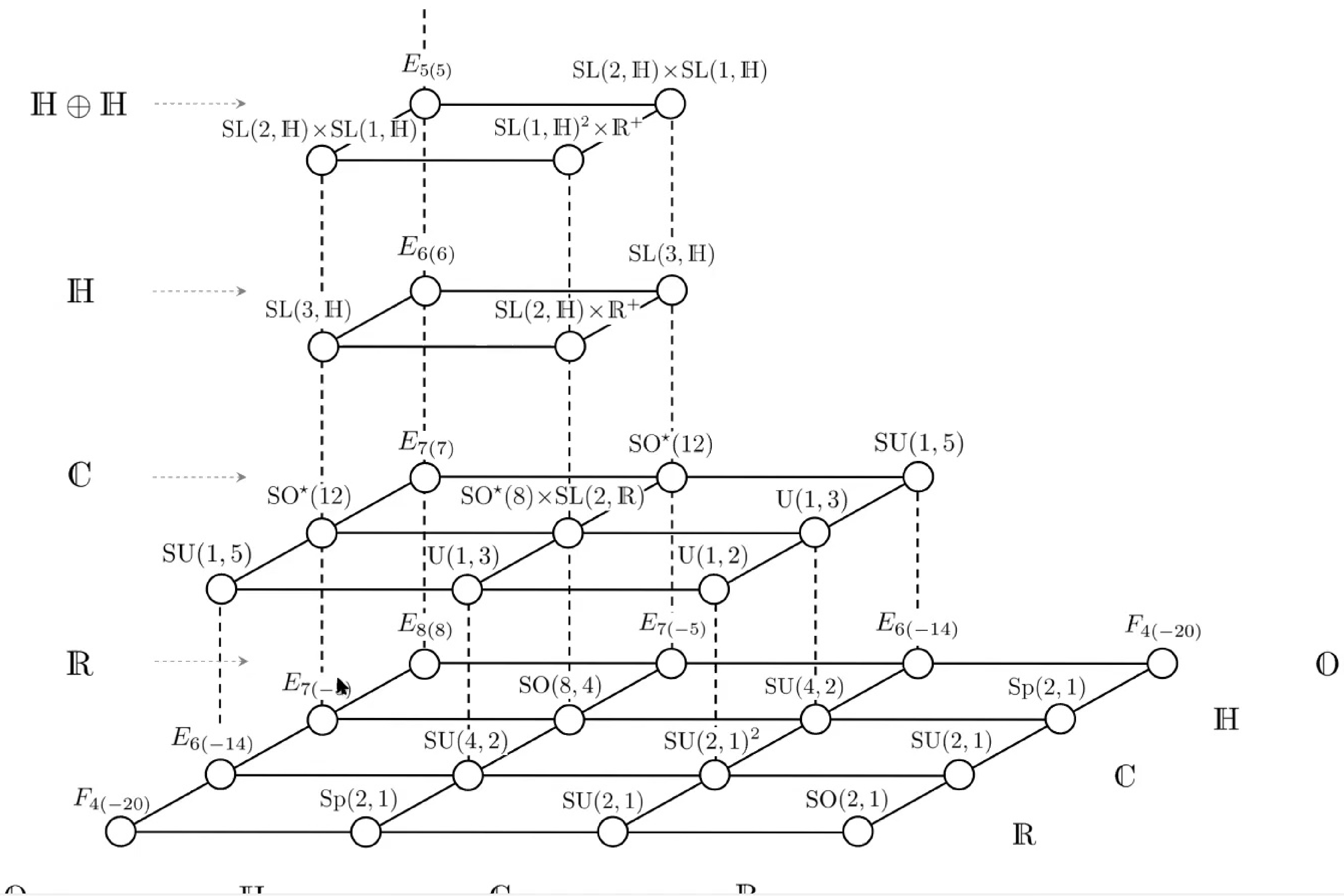
The U-dualities G fill out the magic square!

$$\mathfrak{g} = \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \otimes \mathbb{A}_R)$$

Furthermore, the scalar manifolds are (non-compact versions of) the projective planes:

$$G/H \cong (\mathbb{A}_L \otimes \mathbb{A}_R)\tilde{\mathbb{P}}^2$$


$\mathbb{H} \oplus \mathbb{H}$
$\mathrm{SL}(2,\mathbb{H}) \times \mathrm{SL}(1,\mathbb{H})$
$\mathrm{SL}(1,\mathbb{H})^2 \times \mathbb{R}^+$
$\mathbb{H}$
$E_{6(6)}$
$\mathrm{SL}(3,\mathbb{H})$
$\mathrm{SL}(3,\mathbb{H})$
$\mathrm{SL}(2,\mathbb{H}) \times \mathbb{R}^+$
$\mathbb{C}$
$E_{7(7)}$
$\mathrm{SO}^\star(12)$
$\mathrm{SU}(1,5)$
$\mathrm{SO}^\star(12)$
$\mathrm{SO}^\star(8) \times \mathrm{SL}(2,\mathbb{R})$
$\mathrm{U}(1,3)$
$\mathrm{SU}(1,5)$
$\mathrm{U}(1,3)$
$\mathrm{U}(1,2)$
$\mathbb{R}$
$E_{8(8)}$
$E_{7(-5)}$
$E_{6(-14)}$
$F_{4(-20)}$
$\mathbb{O}$
$E_{7(-5)}$
$\mathrm{SO}(8,4)$
$\mathrm{SU}(4,2)$
$\mathrm{Sp}(2,1)$
$\mathbb{H}$
$E_{6(-14)}$
$\mathrm{SU}(4,2)$
$\mathrm{SU}(2,1)^2$
$\mathrm{SU}(2,1)$
$\mathbb{C}$
$F_{4(-20)}$
$\mathrm{Sp}(2,1)$
$\mathrm{SU}(2,1)$
$\mathrm{SO}(2,1)$
$\mathbb{R}$
$\mathbb{O}$
$\mathbb{H}$
$\mathbb{C}$
$\mathbb{R}$



$E_{5(5)}$
$\mathrm{SL}(2,\mathbb{H})\times\mathrm{SL}(1,\mathbb{H})$
$\mathbb{H}\oplus\mathbb{H}$
$\mathrm{SL}(2,\mathbb{H})\times\mathrm{SL}(1,\mathbb{H})$
$\mathrm{SL}(1,\mathbb{H})^2\times\mathbb{R}^+$
$E_{6(6)}$
$\mathrm{SL}(3,\mathbb{H})$
$\mathbb{H}$
$\mathrm{SL}(3,\mathbb{H})$
$\mathrm{SL}(2,\mathbb{H})\times\mathbb{R}^+$
$E_{7(7)}$
$\mathrm{SO}^\star(12)$
$\mathrm{SU}(1,5)$
$\mathbb{C}$
$\mathrm{SO}^\star(12)$
$\mathrm{SO}^\star(8)\times\mathrm{SL}(2,\mathbb{R})$
$\mathrm{U}(1,3)$
$\mathrm{SU}(1,5)$
$\mathrm{U}(1,3)$
$\mathrm{U}(1,2)$
$E_{8(8)}$
$E_{7(-5)}$
$E_{6(-14)}$
$F_{4(-20)}$
$\mathbb{R}$
$\mathbb{O}$
$\mathrm{SO}(8,4)$
$\mathrm{SU}(4,2)$
$\mathrm{Sp}(2,1)$
$\mathbb{H}$
$E_{6(-14)}$
$\mathrm{SU}(4,2)$
$\mathrm{SU}(2,1)^2$
$\mathrm{SU}(2,1)$
$\mathbb{C}$
$F_{4(-20)}$
$\mathrm{Sp}(2,1)$
$\mathrm{SU}(2,1)$
$\mathrm{SO}(2,1)$
$\mathbb{R}$

THE MAGIC PYRAMID

Mia Hughes

Triality Algebras

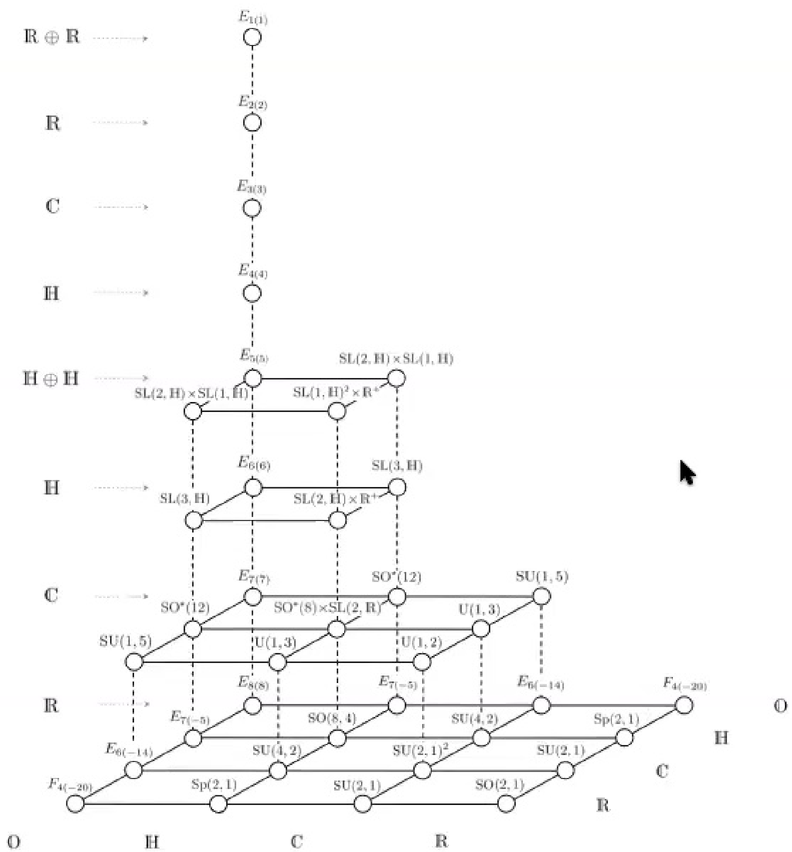
The Magic Square

Division Algebras and Super Yang-Mills

A Magic Square from (Yang-Mills)^2

A Magic Pyramid of Supergravity Theories

A Magic Pyramid of Supergravities



Anastasiou, Borsten, Duff, MJH, Nagy 2013

A Magic Pyramid of Supergravities

⊗ THE MAGIC PYRAMID

Mia Hughes

- The Lie algebras of the groups in the pyramid are subalgebras of $\mathfrak{M}(\mathbb{A}_L, \mathbb{A}_R)$, which can be found as follows
- Inside each of $\mathfrak{tri}(\mathbb{A}_L)$ and $\mathfrak{tri}(\mathbb{A}_R)$ there is an $\mathfrak{so}(D-2)$ subalgebra that represents the on-shell spacetime symmetry
- These two copies of $\mathfrak{so}(D-2)$ must be identified; we take the diagonal subalgebra:

$$\mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) \supset \mathfrak{so}(D-2) \oplus \mathfrak{so}(D-2) \supset \mathfrak{so}(D-2)$$

- In dimension D the pyramid algebra $\mathfrak{Pyramid}_D(\mathbb{A}_L, \mathbb{A}_R)$ is then the commutant of this $\mathfrak{so}(D-2)$ subalgebra within $\mathfrak{M}(\mathbb{A}_L, \mathbb{A}_R)$

A Magic Pyramid of Supergravities

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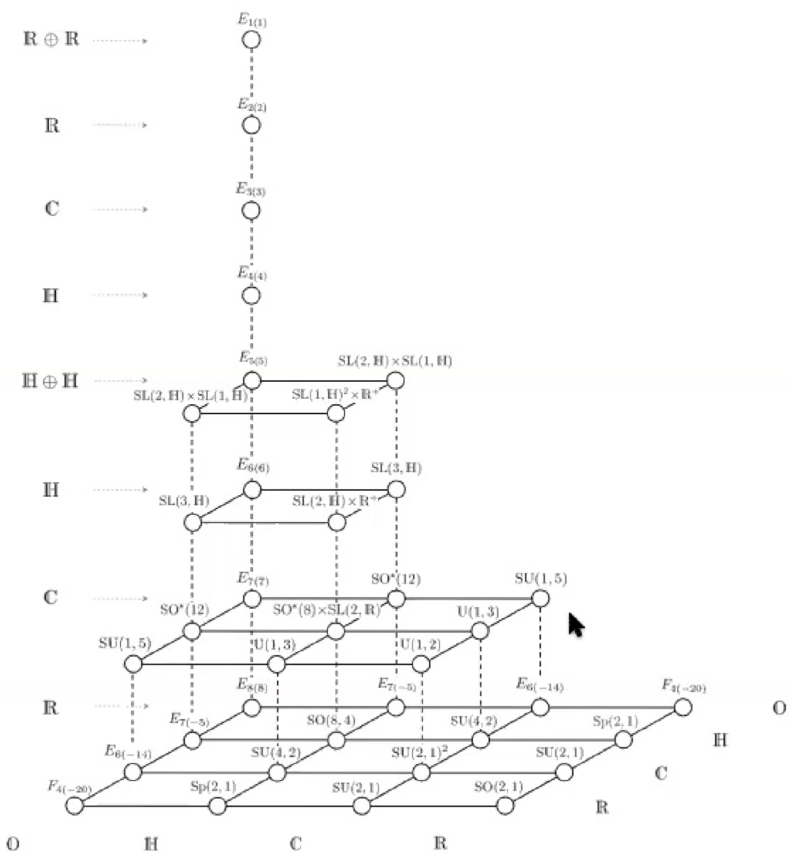

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Summary

- Pure Super Yang-Mills Theories in $D = 3, 4, 6, 10$ have a unified description in terms of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

- Dimensionally reducing these gives a whole bunch of SYM theories in $D = 3, 4, 5, 6, 7, 8, 9, 10$

- Tensoring these division-algebraic SYM theories together gives a whole pyramid of supergravity theories, with the magic square at the base in $D = 3$ and Type II supergravity at the apex in $D = 10$

- This provides a novel octonionic "explanation" of exceptional groups in supergravity

The End

THE MAGIC PYRAMID

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Thanks for listening!