Title: Gravity as the square of gauge theory

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Collection: Octonions and the Standard Model

Date: March 01, 2021 - 12:00 PM

URL: http://pirsa.org/21030011

Abstract: Can gravity, in certain regards, be the `product' of two gauge theories, such as those appearing in the Standard Model? I will begin by reviewing the Bernâ€"Carrascoâ€"Johansson colourâ€"kinematics duality conjecture, which implies that one can write the scattering amplitudes of Einstein-Hilbert gravity (coupled to a Kalb-Ramond 2-form and dilaton scalar) as the double copy of Yangâ€"Mills amplitudes. Although the colourâ€"kinematics duality, and therefore the double copy, was quickly established at the tree level, it remains a longstanding open problem at the loop level, despite highly non-trivial explicit examples.

I will then describe how one can take this `gravity = gauge x gauge' amplitude paradigm `off-shellâ€™ as a product of spacetime fields: the Yang-Mills BRST-Lagrangian itself double copies into perturbatively quantised Einstein-Hilbert gravity coupled to a Kalb-Ramond 2-form and dilaton, establishing the validity of the double copy to all orders, tree and loop. I will end by briefly discussing the homotopy algebras underpinning this result and the inclusion of supersymmetry, which reveals fascinating octonionic structures (some very well-known, others completely new) that will be the subject of Mia Hughes's talk in the following week.

Gravity as the square of gauge theory

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Octonions and the Standard Model Workshop
Perimeter Institute, 8 February 2021 to 17 May 2021

Based on joint work 2007.13803 and 2102.11390 with Branislav Jurčo, Hyungrok Kim, Tommaso Macrelli, Christian Saemann, Martin Wolf

Setting the scene for earlier work 1301.4176, 1309.0546, 1312.6523, 1502.05359, Alexandros Anastasiou, Michael Duff, Mia Hughes, Silvia Nagy

Gravity and gauge theory

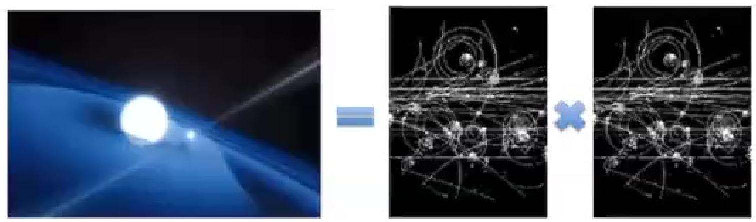
- Gravity as a gauge theory:
  - Gauge theory of Lorentz, (super) Poincaré or de Sitter symmetries [Utiyama '56; Kibble '61; MacDowell-Mansouri '77; Chamseddine-West '77; Stelle-West 79]
  - Holographic principle - AdS/CFT correspondence ['t Hooft '93; Susskind '94; Maldacena '97]
- Here, we appeal to a third and (superficially) independent perspective:

$$\text{Gravity} = \text{Gauge} \times \text{Gauge}$$

- The theme of gravity as the "square" of Yang-Mills has appeared in a variety of guises going back to the KLT relations of string theory [Kawai-Lewellen-Tye '85] Cf. Field theory [Feynman–Morinigo–Wagner; Papini '65]
- Bern-Carrasco-Johansson colour-kinematic (CK) duality and double-copy of (super) Yang-Mills (plus matter) scattering amplitudes [Bern-Carrasco-Johansson '08, '10; Bern-Dennen-Huang-Kiermaier '10]

Gravity = Gauge × Gauge

- Longstanding open question: does the double copy hold to all loop orders?
- Off-shell field theory double copy, as opposed to on-shell amplitudes
- Perturbative quantum Einstein–Hilbert gravity coupled to a Kalb–Ramond 2-form and dilaton[1] *is* the square Yang–Mills theory [BJKMSW '20, '21]



- BV/BRST quantised Yang-Mills $\longrightarrow$ L_∞-algebra that factorises:

$$\mathfrak{colour} \otimes \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$$

$$\mathfrak{L}_{\mathsf{YM}} = \mathfrak{g} \otimes \mathfrak{V} \otimes_\tau \mathfrak{S}$$

Bi-adjoint ϕ^3 theory		YM theory		$\mathcal{N}=0$ supergravity
$\mathfrak{g} \otimes \tilde{\mathfrak{g}} \otimes \mathfrak{S}$	$\longleftarrow$	$\mathfrak{g} \otimes \mathfrak{V} \otimes_\tau \mathfrak{S}$	$\longrightarrow$	$\tilde{\mathfrak{V}} \otimes_{\tilde{\tau}} \otimes \mathfrak{V} \otimes_\tau \mathfrak{S}$

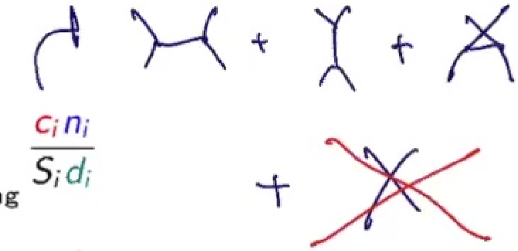
[1] Which I'll call '$\mathcal{N}=0$ supergravity'

Order of Events

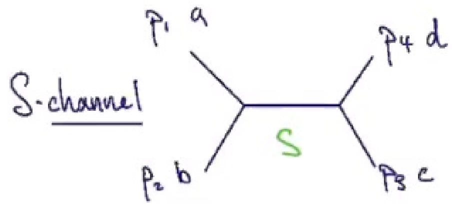
1. Review: BCJ Colour-Kinematic Duality and Double-Copy
2. The BRST Lagrangian Double-Copy: A Heuristic Summary (4 parts)
3. Conclusions - enter the octonions!
4. Homotopy double copy

Amplitudes and cubic diagrams

- Can write n-point L-loop gluon amplitude in terms of only cubic diagrams:



$$A_{\text{YM}}^{n,L} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i}$$



$c_s \sim f^{abx} f_x{}^{cd}$ $\quad s \sim (p_1 + p_2)^2$

- c_i: colour numerator, built from f^{abc}, read off diagram i
- n_i: kinematic numerator, built from p, ε ← not unique
- d_i: propagator, $\prod_{\text{int. lines}} p^2$, read off diagram i

Amplitudes and cubic diagrams

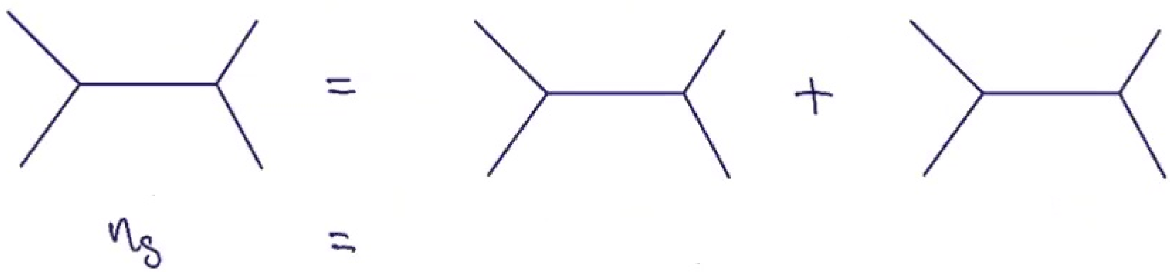
- Can be realised in the Lagrangian through auxiliary fields:

$$\mathcal{L}_{\mathrm{YM}} = \cdots + g^2 [A_\mu, A_\nu][A^\mu, A^\nu] \;\to\; \tfrac{1}{2} B^{\mu\nu\kappa} \Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \frac{1}{\sqrt{2}} \partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu]$$

- Feynman diagrams give 'cubic' amplitudes directly:

$$A^{n,L}_{\mathrm{YM}} = \int_L \sum_{\alpha \in \text{Feynman diag}} \frac{c_\alpha n_\alpha}{S_\alpha d_\alpha} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i}$$

- Example: 4-point s-channel diagram



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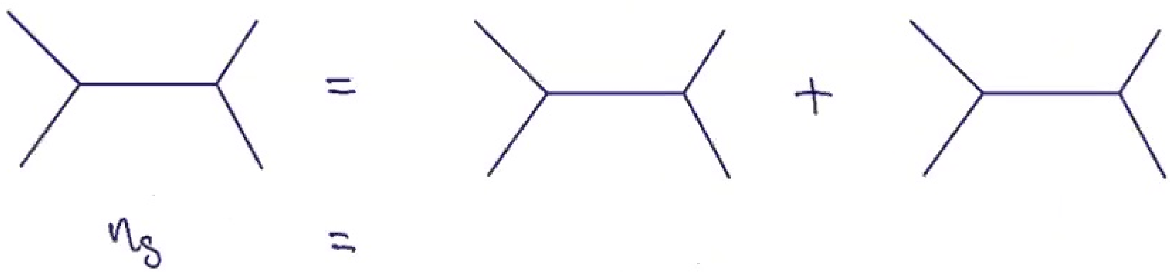

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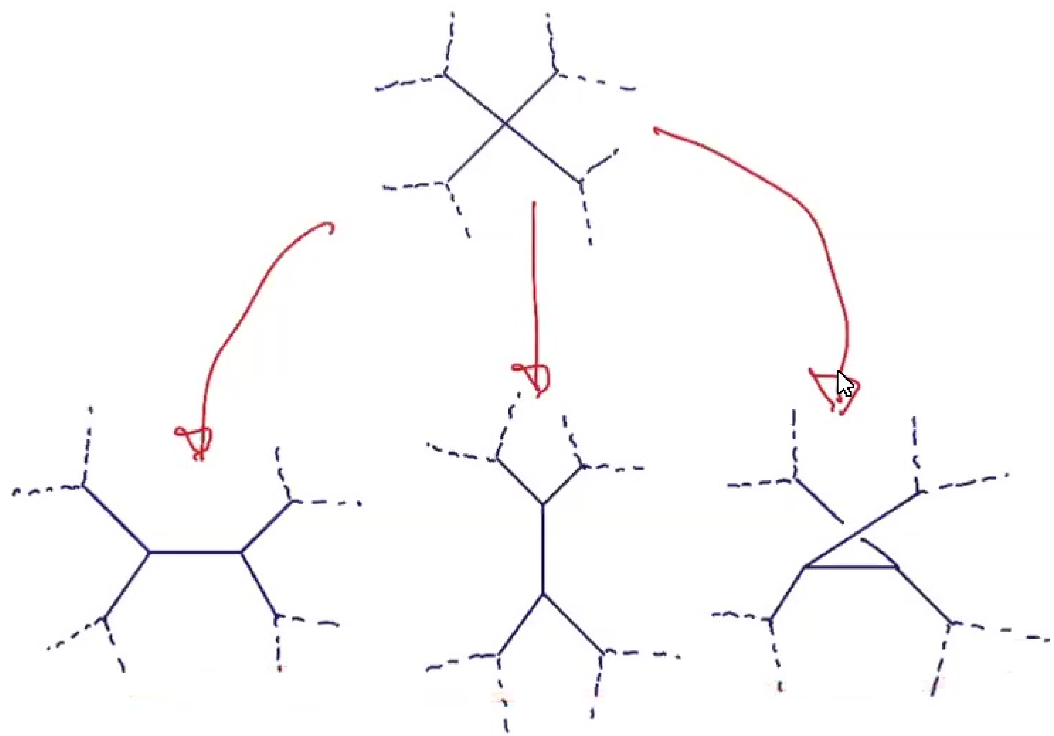

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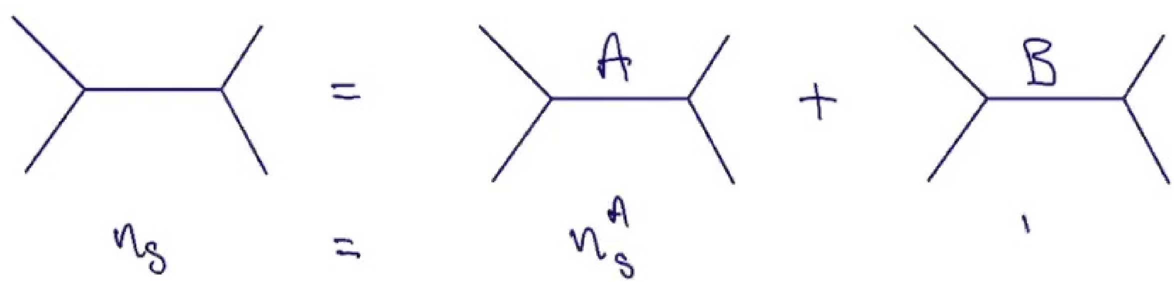


BCJ double-copy prescription

- Given CK dual amplitude of pure Yang-Mills

$$A^{n,L}_{\text{YM}} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i}$$

$$S_{\text{YM}} = \frac{1}{2g^2} \int \text{tr} F \wedge \star F$$

- Double-copy:

$$c_i \longrightarrow n_i$$

Implications

- Conceptually compelling and computationally powerful: $\mathcal{N}=8$ supergravity four-point to 5 loops! (finite) [Bern–Carrasco–Chen–Edison–Johansson–Parra-Martinez–Roiban–Zeng '18]
- Can be explained by supersymmetry and $E_{7(7)}$ U-duality [Bjornsson–Green '10, Bossard–Howe–Stelle '11; Elvang–Freedman–Kiermaier '11; Bossard–Howe–Stelle–Vanhove '11]
- At 7 loops any would-be cancellations are "not consequences of supersymmetry in any conventional sense" [Bjornsson–Green '10]
- $D=4, \mathcal{N}=5$ supergravity finite to 4 loops, contrary to expectations:

"Enhanced" cancellations

[Bern-Davies-Dennen '14]

- Such cancellations not seen for $\mathcal{N}=8$ at 5 loops: implications unclear

Origin, validity, generality, implications and applications

- Solutions: classical (perturbative/non-perturbative) gravity solutions from amplitudes and the double-copy

- → Applications to gravity wave astronomy
  [Monteiro–O'Connell–White '14; Cardoso–Nagy–Nampuri '16;
  Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16;
  Berman–Chacón–Luna–White '18; Kosower–Maybee–O'Connell '18;
  Bern–Cheung–Roiban–Shen–Solon–Zeng '19; Bern–Luna–Roiban–Shen–Zeng '20. . .]

- Geometric/world-sheet pictue: ambitwistor string theories theories and scattering equation formalism

- → Non-trivial gluon and spacetime backgrounds
  [Cachazo–He–Yuan '13 '14; Mason–Skinner '13; Adamo–Casali–Skinner '13;
  Adamo–Casali–Mason–Nekovar '17 '18; Geyer–Monteiro '18; Geyer–Mason '19. . .]

Off-shell BRST-Lagrangian double-copy

- Can the double-copy be realised at the level of the Lagrangian itself?

- Can 'going off-shell' in this way be used to establish the validity of the double-copy to all orders in perturbations theory?

$$\text{tree CK duality} \quad \Rightarrow \quad \text{tree double-copy}$$

$$\text{loop CK duality} \quad \Rightarrow \quad \text{loop double-copy}$$

- Field theory product of BRST gauge theories and Lagrangian double-copy
  [Bern–Dennen–Huang–Kiermaier '10; Anastasiou–LB-Duff–Hughes–Nagy '14; LB '17; Anastasiou–LB–Duff-Nagy–Zoccali '18; LB–Jubb–Makwana–Nagy '20; LB-Nagy '20]

- CK duality manifesting actions and kinematic algebras
  [Bern–Dennen–Huang–Kiermaier '10; Tolotti–Weinzierl '13; Cheung–Shen '16; Luna–Monteiro–Nicholson–Ochirov–O'Connell–Westerberg–White '16] [Monteiro–O'Connell '11, '13; Bjerrum–Bohr–Damgaard–Monteiro–O'Connell '12; Fu–Krasnov '16; Chen–Johansson–Teng–Wang 19]

Lighting overview

Step 1. Cubic tree-level CK duality manifesting Yang-Mills BRST-action:

$$S_{\text{BRST-CK YM}} = c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

Step 2. BRST-action double-copy:

$$S_{\text{BRST-CK}} \times \tilde{S}_{\text{BRST-CK}} = S_{\text{DC}} = C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

Step 3. Double-copy BRST operator:

$$(Q_{\text{YM}}, \tilde{Q}_{\text{YM}}) = Q_{\text{DC}} = Q^{\text{lin}}_{\text{diffeo}} + Q^{\text{lin}}_{\text{2-form}} + \cdots$$

Step 4. Perturbative quantum equivalence:

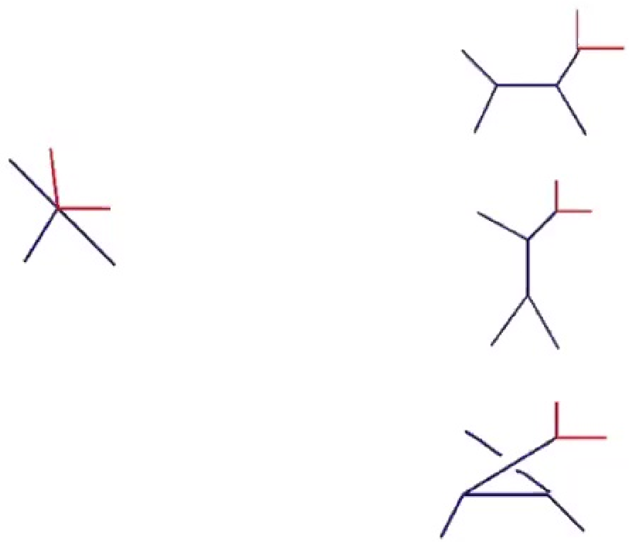
$$\text{tree CK} + \text{BRST Ward identities} \Rightarrow S_{\text{DC}} \cong S_{\text{BRST}\,\mathcal{N}=0} \quad \text{(perturbatively)}$$

Corollary: Loop amplitude (integrands) computed from Feynman diagrams of $S_{\text{BRST-CK}}$ double-copy correctly:

Step 1. Colour-Kinematic Duality Redux

- There is YM action such that the Feynman diagrams yield amplitudes manifesting CK duality for tree-level amplitudes:

$$S^{\infty}_{\mathrm{CK\ YM}} = \int \mathcal{L}^{(2)}_{\mathrm{YM}} + \mathcal{L}^{(3)}_{\mathrm{YM}} + \frac{\Box}{\Box}\mathcal{L}^{(4)}_{\mathrm{YM}} + \sum_{n=5}^{\infty} \mathcal{L}^{(n)}_{\mathrm{YM}}$$



+ 13 diagrams

[Bern–Dennen–Huang–Kiermaier 1004.0693; Tolotti–Weinzierl 1306.2975]

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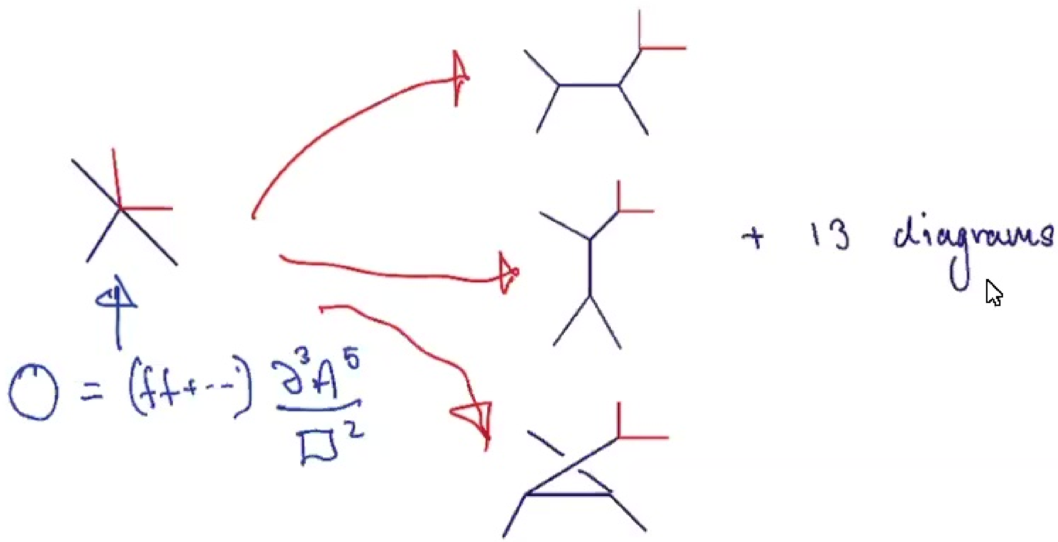

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Step 1. Colour-Kinematic Duality Redux

- This can be "strictified" to have only cubic interactions through infinite tower of auxiliaries [2007.13803]

$$S_{\text{CK YM}} = c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

  - i, j, k: DeWitt indices over position, all fields including the auxiliaries:

$$A^{ai} = (A^a{}_\mu(x), B^a{}_{\mu\nu\rho}(x), \ldots)$$

  - c_{ab}, f_{abc}: Lie gauge algebra Cartan-Killing form and structure constants
  - C_{ij}, F_{ijk}: Bi- and tri-linear differential operators

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- Example: 5-points

$$\mathcal{L}^{(5)}_{\text{YM}} \longrightarrow C^{\mu\nu} \Box \bar{C}_{\mu\nu} + C^{\mu\nu\kappa} \Box \bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda} \Box \bar{C}_{\mu\nu\kappa\lambda} + \\ + gC^{\mu\nu}[A_\mu, A_\nu] + g\partial_\mu C^{\mu\nu\kappa}[A_\nu, A_\kappa] - \tfrac{g}{2}\partial_\mu C^{\mu\nu\kappa\lambda}[\partial_{[\nu} A_{\kappa]}, A_\lambda] \\ + g\bar{C}^{\mu\nu}\left(\tfrac{1}{2}[\partial^\kappa \bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa \bar{C}_{\kappa\lambda\nu\mu}, A^\lambda]\right)$$

[Bern–Dennen–Huang–Kiermaier '10]

Step 1. Colour-Kinematic Duality Redux

Tree-level gluon CK duality

- Cubic action manifesting tree-level CK duality for physical gluon states

$$S_{\text{CK YM}} = c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

$$S_{\text{CK YM}} \longrightarrow \sum_{i \in \text{cubic}} \frac{c_i n_i}{d_i}$$

Generalise: Tree-level BRST CK duality

- Cubic action manifesting tree-level CK duality for physical gluons and unphysical longitudinal gluons and ghosts:

$$S_{\text{BRST-CK YM}} = c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

- Now i, j, k runs also over the BRST ghosts $c, \bar{c}$, the Nakanishi-Lautrup auxiliary b and auxiliary ghosts

Step 1. Colour-Kinematic Duality Redux

BRST Quantization

- BRST operator: $\delta A = D\theta \to Q_{\mathrm{YM}} A = Dc$

$$Q_{\mathrm{YM}} S_{\mathrm{YM}} = 0, \quad {Q_{\mathrm{YM}}}^2 = 0$$

$$Q_{\mathrm{YM}} c = cc \quad Q_{\mathrm{YM}} \bar{c} = b \quad Q_{\mathrm{YM}} b = 0$$

- Gauge fixing

$$\mathcal{L}_{\mathrm{BRST,\,YM}} = \mathcal{L}_{\mathrm{YM}} + Q_{\mathrm{YM}} \Psi, \qquad \Psi = \bar{c}\,(G[A] + \xi b)$$

$$\mathcal{L}_{\mathrm{BRST,\,YM}} = \mathcal{L}_{\mathrm{YM}} + b\,(G[A] + \xi b) - \bar{c} Q_{\mathrm{YM}} G[A]$$

- Extended BRST Fock space (note, abuse of notation):

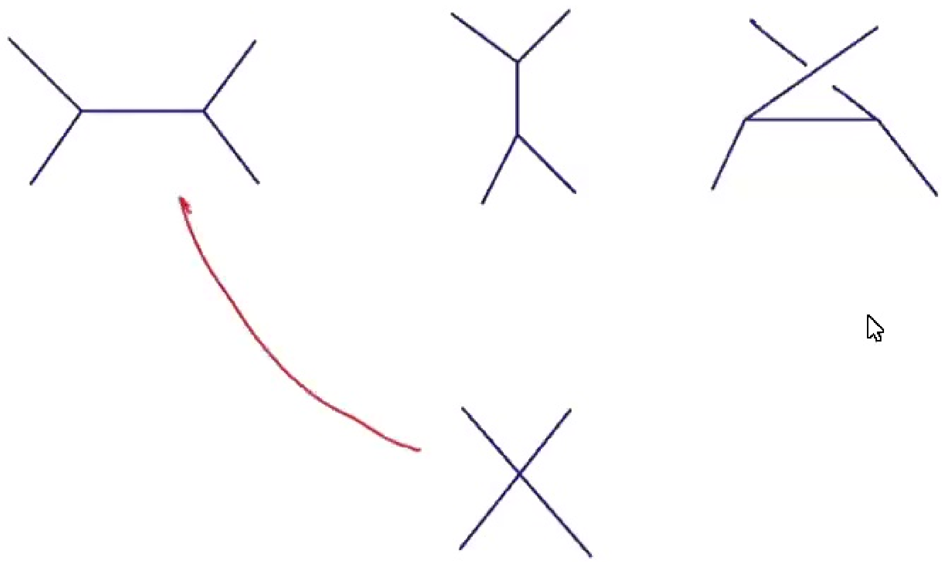
$$A_{\mathrm{phys}}, \quad A_{\mathrm{f}}, \quad A_{\mathrm{b}} \equiv b, \quad c, \quad \bar{c},$$

- Allow ourselves to put $A_{\mathrm{f}}, b, c, \bar{c}$ on external points of tree-amplitudes and show that they can made CK dual

Step 1. Colour-Kinematic Duality Redux

Tree-level CK duality for longitudinal gluons

- Relax transversality $p_n \cdot \varepsilon_n \neq 0 \Rightarrow$ tree CK duality fails
- By analogy can compensate with new *non-zero* vertices [2007.13803]:



- We can add them to the action without changing the physics [2007.13803]

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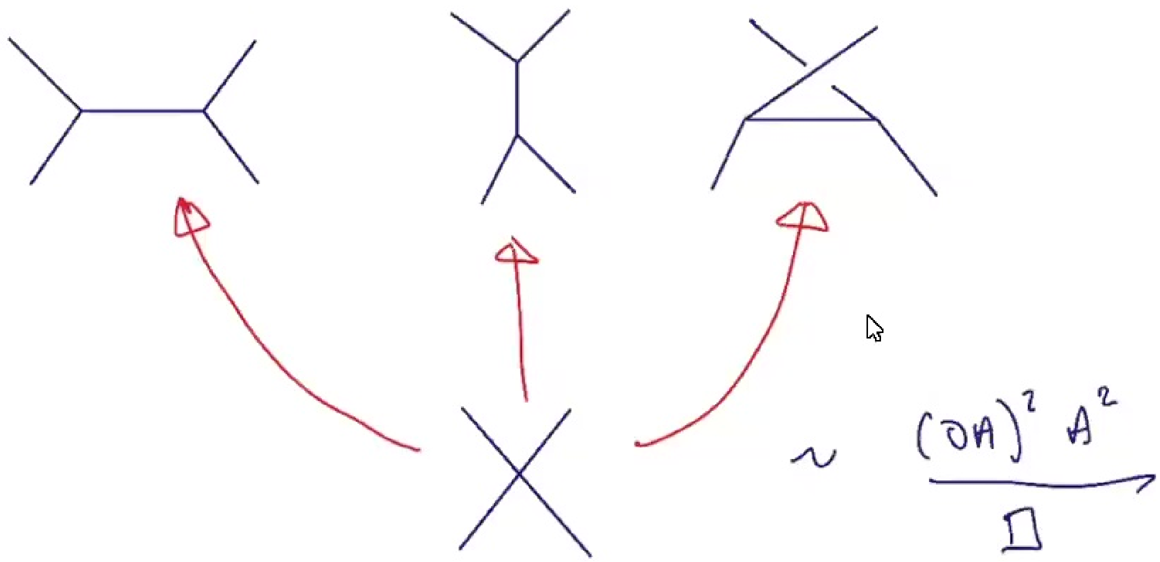

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Step 1. Colour-Kinematic Duality Redux

Tree-level CK duality for longitudinal gluons

- The CK failures are always of the form

$$\partial^\mu A_\mu Y[A]$$

- Can add them through the gauge-fixing functional

$$\begin{aligned} G[A] = \partial^\mu A_\mu \quad &\mapsto \quad G'[A] = \partial^\mu A_\mu - 2\xi Y \\ b \quad &\mapsto \quad b' = b + Y \end{aligned}$$

Step 1. Colour-Kinematic Duality Redux

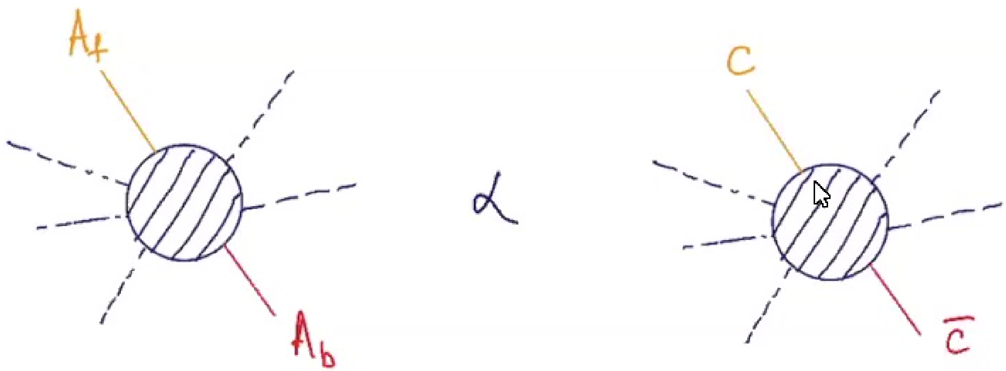
Tree-level CK duality for ghost

- Use on-mass-shell BRST Ward identities

$$Q^{\rm lin}_{\rm YM} A_{\sf phys} = 0, \quad Q^{\rm lin}_{\rm YM} A_{\sf f} = c, \quad Q^{\rm lin}_{\rm YM} b = \bar{c}$$

- Analogous to global SUSY Ward identities

$$0 = \langle 0 | [Q^{\rm lin}_{\rm YM}, O_1 \cdots O_n] | 0 \rangle$$



- Transfers CK duality onto ghosts through

$$\mathcal{L}_{\rm YM, ghost} = \bar{c} Q_{\rm YM}(\partial^\mu A_\mu - 2\xi Y)$$

Step 1. Colour-Kinematic Duality Redux

Summary: tree-level CK manifesting BRST action

- Introduce new auxiliary gluons and ghosts:

$$S_{\text{BRST-CK YM}} = c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

- i, j, k: DeWitt indices now run over all BRST fields including $b, c, \bar{c}$ and the tower of ghost auxiliaries

- Feynman diagrams yield CK dual tree amplitudes for physical gluons and unphysical longitudinal modes and ghosts

Step 2. BRST-Lagrangian double-copy

- Given cubic form and "colour-kinematic" factorisation:

$$S = c_{ab} C_{ij} \Phi^{ai} \Box \Phi^{aj} + f_{abc} F_{ijk} \Phi^{ai} \Phi^{bj} \Phi^{ck}$$

$$\tilde{S} = \tilde{c}_{\tilde{a}\tilde{b}} \tilde{C}_{\tilde{\imath}\tilde{\jmath}} \Phi^{\tilde{a}\tilde{\imath}} \Box \tilde{\Phi}^{\tilde{a}\tilde{\jmath}} + \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \tilde{\Phi}^{\tilde{a}\tilde{\imath}} \tilde{\Phi}^{\tilde{b}\tilde{\jmath}} \tilde{\Phi}^{\tilde{c}\tilde{k}}$$

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- Replace left (right) sector with second copy of right (left) sector:

Double-copy	$c_{ab} \to \tilde{C}_{\tilde{\imath}\tilde{\jmath}}$	$f_{abc} \to \tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}}$	$\Phi^{ai} \to \Phi^{i\tilde{\imath}}$
Zeroth-copy	$C_{ij} \to \tilde{c}_{\tilde{a}\tilde{b}}$	$F_{ijk} \to \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}$	$\Phi^{ai} \to \Phi^{a\tilde{a}}$

- Double/zeroth copy Lagrangians:

$$S \times \tilde{S} \to \begin{cases} S_{\mathrm{DC}} = C_{ij} \tilde{C}_{\tilde{\imath}\tilde{\jmath}} \Phi^{i\tilde{\imath}} \Box \Phi^{j\tilde{\jmath}} + F_{ijk} \tilde{F}_{\tilde{\imath}\tilde{\jmath}\tilde{k}} \Phi^{i\tilde{\imath}} \Phi^{j\tilde{\jmath}} \Phi^{k\tilde{k}} \\ S_{\mathrm{ZC}} = c_{ab} \tilde{c}_{\tilde{a}\tilde{b}} \Phi^{a\tilde{a}} \Box \Phi^{a\tilde{b}} + f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}} \end{cases}$$

Step 2. BRST-Lagrangian double-copy

- $S_{\text{BRST-CK YM}} \times \tilde{S}_{\text{BRST-CK YM}} \to \mathcal{N} = 0$ supergravity

$$A^{ai} \quad \to \quad A^{i\tilde{\imath}} \quad = \quad h_{\mu\nu} \oplus B_{\mu\nu} \oplus \varphi \oplus \text{ghosts} \oplus \text{auxiliaries}$$

$$S_{\text{CK YM}} \quad \to \quad S_{\text{DC}}^{\mathcal{N}=0} \quad = \quad C_{ij} C_{\tilde{\imath}\tilde{\jmath}} A^{i\tilde{\imath}} \Box A^{j\tilde{\jmath}} + F_{ijk} F_{\tilde{\imath}\tilde{\jmath}\tilde{k}} A^{i\tilde{\imath}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

- $G \times \tilde{G}$ bi-adjoint scalar theory,

$$S_{\text{DC}}^{\text{bi-adj}} = c_{ab} \tilde{c}_{\tilde{a}\tilde{b}} \Phi^{a\tilde{a}} \Box \Phi^{a\tilde{b}} + f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}}$$

- Cf. scattering equation formalism [Hodges '11; Cachazo–He–Yuan '13 '14]

Step 2. BRST-Lagrangian double-copy

- Conclusion: $\mathcal{N} = 0$ supergravity is the double-copy of Yang-Mills?
- But wait, you should be suspicious!
- No mention of CK duality - isn't this overly general?
- Semi-classical equivalence needs tree-level CK duality of $S_{\text{CK YM}}$:

$$f_{abc} F_{i'j'k'} A^{ai'} A^{bj'} A^{ck'} \quad \to \quad F_{ijk} F_{i'j'k'} A^{ii'} A^{jj'} A^{kk'}$$

$$\sum \frac{cn}{d} \quad \to \quad \sum \frac{nn}{d}$$

- Implies by construction the physical (h, B, φ) tree-level amplitudes are those of $\mathcal{N} = 0$ supergravity
- Cf. [Bern-Dennen-Huang-Kiermaier 1004.0693] for gravitons up to 6 points
- What about quantum consistency? $\Rightarrow$ Double-copy BRST operator

Step 3. BRST operator double-copy

- How do we we know that there exists some BRST Q such that:

$$QS_{\text{DC}} = 0, \qquad Q^2 = 0$$

- Double-copy yields a putative BRST operator Q_{DC}

$$Q\Phi^{ai} = q^a{}_b Q^i{}_j \Phi^{bj} + q^a{}_{bc} Q^i{}_{jk} \Phi^{bj}\Phi^{ck} + \cdots$$

$$\tilde{Q}\tilde{\Phi}^{\tilde{a}i} = \tilde{q}^{\tilde{a}}{}_{\tilde{b}} Q^{\tilde{\imath}}{}_{\tilde{\jmath}} \tilde{\Phi}^{\tilde{b}\tilde{\jmath}} + \tilde{q}^{\tilde{a}}{}_{\tilde{b}\tilde{c}} \tilde{Q}^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} \tilde{\Phi}^{\tilde{b}\tilde{\jmath}}\tilde{\Phi}^{\tilde{c}\tilde{k}} + \cdots$$

$$Q_{\text{DC}} = Q_{\text{L}} + Q_{\text{R}}$$

$$Q^i{}_j \delta^{\tilde{\imath}}{}_{\tilde{\jmath}} \Phi^{j\tilde{\jmath}} + Q^i{}_{jk} F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} \Phi^{j\tilde{\jmath}}\Phi^{k\tilde{k}} \qquad \delta^i{}_j Q^{\tilde{\imath}}{}_{\tilde{\jmath}} \Phi^{j\tilde{\jmath}} + F^i{}_{jk} Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} \Phi^{j\tilde{\jmath}}\Phi^{k\tilde{k}}$$

Step 3. BRST operator double-copy

- How do we we know that there exists some BRST Q such that:

$$QS_{\mathrm{DC}} = 0, \qquad Q^2 = 0$$

$$f^{abc} \to \tilde{F}^{\tilde{i}\tilde{j}\tilde{k}}$$

- Double-copy yields a putative BRST operator Q_{DC}

$$q^{a}{}_{bc} \to \tilde{F}^{\tilde{i}}{}_{\tilde{j}\tilde{k}}$$

$$Q\Phi^{ai} = q^{a}{}_{b} Q^{i}{}_{j} \Phi^{bj} + q^{a}{}_{bc} Q^{i}{}_{jk} \Phi^{bj}\Phi^{ck} + \cdots$$

$$\tilde{Q}\tilde{\Phi}^{\tilde{a}\tilde{i}} = \tilde{q}^{\tilde{a}}{}_{\tilde{b}} Q^{\tilde{i}}{}_{\tilde{j}} \tilde{\Phi}^{\tilde{b}\tilde{j}} + \tilde{q}^{\tilde{a}}{}_{\tilde{b}\tilde{c}} \tilde{Q}^{\tilde{i}}{}_{\tilde{j}\tilde{k}} \tilde{\Phi}^{\tilde{b}\tilde{j}}\tilde{\Phi}^{\tilde{c}\tilde{k}} + \cdots$$

$$Q_{\mathrm{DC}} = Q_{\mathrm{L}} + Q_{\mathrm{R}}$$

$$Q^{i}{}_{j}\delta^{\tilde{i}}{}_{\tilde{j}}\Phi^{j\tilde{j}} + Q^{i}{}_{jk}F^{\tilde{i}}{}_{\tilde{j}\tilde{k}}\Phi^{j\tilde{j}}\Phi^{k\tilde{k}} \qquad \delta^{i}{}_{j}Q^{\tilde{i}}{}_{\tilde{j}}\Phi^{j\tilde{j}} + F^{i}{}_{jk}Q^{\tilde{i}}{}_{\tilde{j}\tilde{k}}\Phi^{j\tilde{j}}\Phi^{k\tilde{k}}$$

Step 3. BRST operator double-copy

- For Yang-Mills

$$Q_{\mathrm{DC}} A^{i\tilde{\imath}} = Q^{i}{}_{j} \delta^{\tilde{\imath}}{}_{\tilde{\jmath}} A^{j\tilde{\jmath}} + Q^{i}{}_{jk} F^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}} + \delta^{i}{}_{j} Q^{\tilde{\imath}}{}_{\tilde{\jmath}} A^{j\tilde{\jmath}} + F^{i}{}_{jk} Q^{\tilde{\imath}}{}_{\tilde{\jmath}\tilde{k}} A^{j\tilde{\jmath}} A^{k\tilde{k}}$$

- Linear diffeomorphisms and 2-form gauge (and gauge-for-gauge) symmetry:

$$Q^{\mathrm{lin}}_{\mathrm{DC}} = Q^{\mathrm{lin}}_{\mathrm{diffeo}} + Q^{\mathrm{lin}}_{\text{2-form}}$$

- Require $Q_{\mathrm{DC}} S_{\mathrm{DC}} = 0$, $Q^2_{\mathrm{DC}} = 0$

- Holds if F^{ijk} satisfies the same identities as f^{abc} as *operators equations*,

$$c_{ab} = c_{(ab)} \qquad f_{abc} = f_{[abc]} \qquad f_{[ab|d} c^{dd'} f_{d'|c]e} = 0$$

$$C_{ij} = C_{(ij)} \qquad F_{ijk} = F_{[ijk]} \qquad F_{[ij|l} C^{ll'} F_{l'|k]m} = 0$$

- Tree-level CK for extended BRST Fock space: $Q_{\mathrm{DC}} S_{\mathrm{DC}} \in \mathrm{Ideal}(\Box A^{i\tilde{\imath}})$

Step 4. Perturbative quantum equivalence

Claim: The BRST Lagrangian $\mathcal{L}_{\text{DC}}$ and the canonical BRST Lagrangian $\mathcal{L}_{\mathcal{N}=0}$ are related by field redefinitions preserving quantum equivalence

1. **Kinematic equivalence** BRST field complexes and kinetic Lagrangians straightforwardly related:

$$\mathcal{L}_{\text{DC, kin}} \leftrightarrow \mathcal{L}_{\mathcal{N}=0,\text{kin}}$$

2. **Semi-classical** Tree-level CK duality $\Rightarrow$ physical tree-level amplitudes match and there is an invertible field redefinition $\mathcal{L}_{\text{DC, phys}} \leftrightarrow \mathcal{L}_{\mathcal{N}=0,\text{phys}}$

   - Formally: if two field theories have the same tree amplitudes, then the minimal models of their L_∞-algebras coincide and are thus quasi-isomorphic, cf. [Jurčo-Raspollini-Saemann-Wolf 1809.09899]

   - Local (in fact non-derivative) field redefinition:

$$\mathcal{L}_{\mathcal{N}=0,\text{phys}} \rightarrow \mathcal{L}'_{\mathcal{N}=0,\text{phys}} = \mathcal{L}_{\text{DC, phys}}$$

Step 4. Perturbative quantum equivalencee

3. **Quantum equivalence** Now add gauge-fixing and ghost sectors to

$$\mathcal{L}'_{\mathcal{N}=0,\text{phys}}$$

3.i **Gauge-fixing sector** Choose gauge-fixing fermion $\Psi_{\mathcal{N}=0}$ such that

$$Q'_{\mathcal{N}=0}\Psi_{\mathcal{N}=0} = \mathcal{L}'_{\mathcal{N}=0,\text{gf}} + \mathcal{L}'_{\mathcal{N}=0,\text{gh}}$$

3.ii **Ghost sector** Since $Q_{\text{DC}} \stackrel{?}{=} Q'_{\mathcal{N}=0}$ it is not clear that $\mathcal{L}'_{\mathcal{N}=0,\text{gh}} = \mathcal{L}_{\text{DC},\text{gh}}$

- But $Q^{\text{Lin}}_{\text{DC}} = Q'^{\text{Lin}}_{\mathcal{N}=0}$ so we have matching on-mass-shell BRST Ward identities $\rightarrow$ tree-level ghost amplitudes match
- Final subtlety: auxiliary-ghost amplitudes $\rightarrow$ the set of auxiliary-ghosts and their tree amplitudes are both determined by the non-auxiliary ghost amplitudes
- Conclude: Both Lagrangians are local, cubic and have matching tree amplitudes for *all* fields $\rightarrow$ perturbatively equivalent (tree and loop)

Conclusions

- BRST-Lagrangian picture of the double-copy
- Tree-level BRST-CK duality $\rightarrow$ perturbative quantum equivalence
- Quantum gravity *is* the square of Yang-Mills
  (well, perturbatively and coupled to a 2-form and dilaton)

Corollary: $S_{\text{BRST-CK YM}} \rightarrow$ 'almost BCJ numerators' that correctly double-copy:

$$A^{n,L}_{\text{YM}} = \int_L \sum_{i \in \text{cubic diag}} \frac{c_i n_i}{S_i d_i} \longrightarrow \int_L \sum_{i \in \text{cubic diag}} \frac{n_i n_i}{S_i d_i} = A^{n,L}_{\mathcal{N}=0}$$

- 'Almost': construction doesn't imply n_i satisfy perfect loop CK duality, but close enough for double-copy
  Cf. generalised CK duality [Bern–Carrasco–Chen–Johansson–Roiban '18]
- Only tree-level CK duality required to construct loop almost BCJ n_i - complicated, but purely algebraic
- Is there a precise weaken notion of on-mass-shell loop CK duality?

Conclusions

- BV quantisable field theory $\rightarrow L_\infty$ homotopy algebra:

Vector space	Graded vector space
$\mathfrak{g} = V_0$	$\mathfrak{L} = \bigoplus_n V_n$
Bracket	Higher brackets
$\mu_2 = [-,-]$	$\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-], \ldots$

Higher μ_i satisfy homotopy Jacobi relations [Zwiebach '93; Hinich–Schechtman '93]

- Homotopy algebra realisation of colour-kinematics

$$\mathfrak{L}_{\mathsf{YM}} = \underbrace{\underbrace{\mathfrak{colour}}_{L_\infty} \otimes \underbrace{\mathfrak{kinematics} \otimes_\tau \underbrace{\mathfrak{scalar}}_{A_\infty}}_{C_\infty}}_{L_\infty}$$

- τ kinematic twist, cf. kinematic algebras of [Monteiro–O'Connell '11, '13]

- Homotopy double-copy: $\mathfrak{L}_{\mathsf{DC}} = \widetilde{\mathfrak{kinematics}} \otimes_\tau \mathfrak{kinematics} \otimes_\tau \mathfrak{scalar}$

Future Work

- When sending $\mathfrak{colour} \to \widetilde{\mathfrak{kinematics}}$ it was not necessary that $\widetilde{\mathfrak{kinematics}} \cong \mathfrak{kinematics}$
- $\widetilde{\mathfrak{kinematics}}$ could have come from any other $\tilde{\mathfrak{L}}$ satisfying tree-level CK duality:
  - Super Yang-Mills $\to$ supergravity
  - Non-linear sigma model $\to$ special Galileon theory [BJKMSW '21]
  - Massive Yang-Mills $\to$ massive gravity(?)
  - Higher derivative Yang-Mills $\to$ conformal gravity(?)
- Growing zoology $\to$ Unity through homotopy and double-copy
- Counter-terms and renormalisation
- Ultimately, open/closed string field theory

A Kind of Magic

- Introduce $\mathcal{N} = 1, 2, 4, 8$ supersymmetry through $\mathbb{A} \cong \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \Rightarrow$

  Magic Square $\mathbb{A}_L \otimes \mathbb{A}_R$ of supergravity theories

  [LB–Duff–Hughes–Nagy '13]

- Note, *not* the famous magic supergravities of Günaydin–Sierra–Townsend (different theories and different real form of magic square)

- Introduce $D = 3, 4, 6, 10$ spacetime through $\mathbb{A} \cong \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \Rightarrow$

  Magic Pyramid $\mathbb{A}_L \otimes \mathbb{A}_R$ of supergravity theories

  [Anastasiou–LB–Duff–Hughes–Nagy '13; Anastasiou–LB–Duff–Hughes–Nagy '15]

- Tune into Mia Hughes' talk next week!