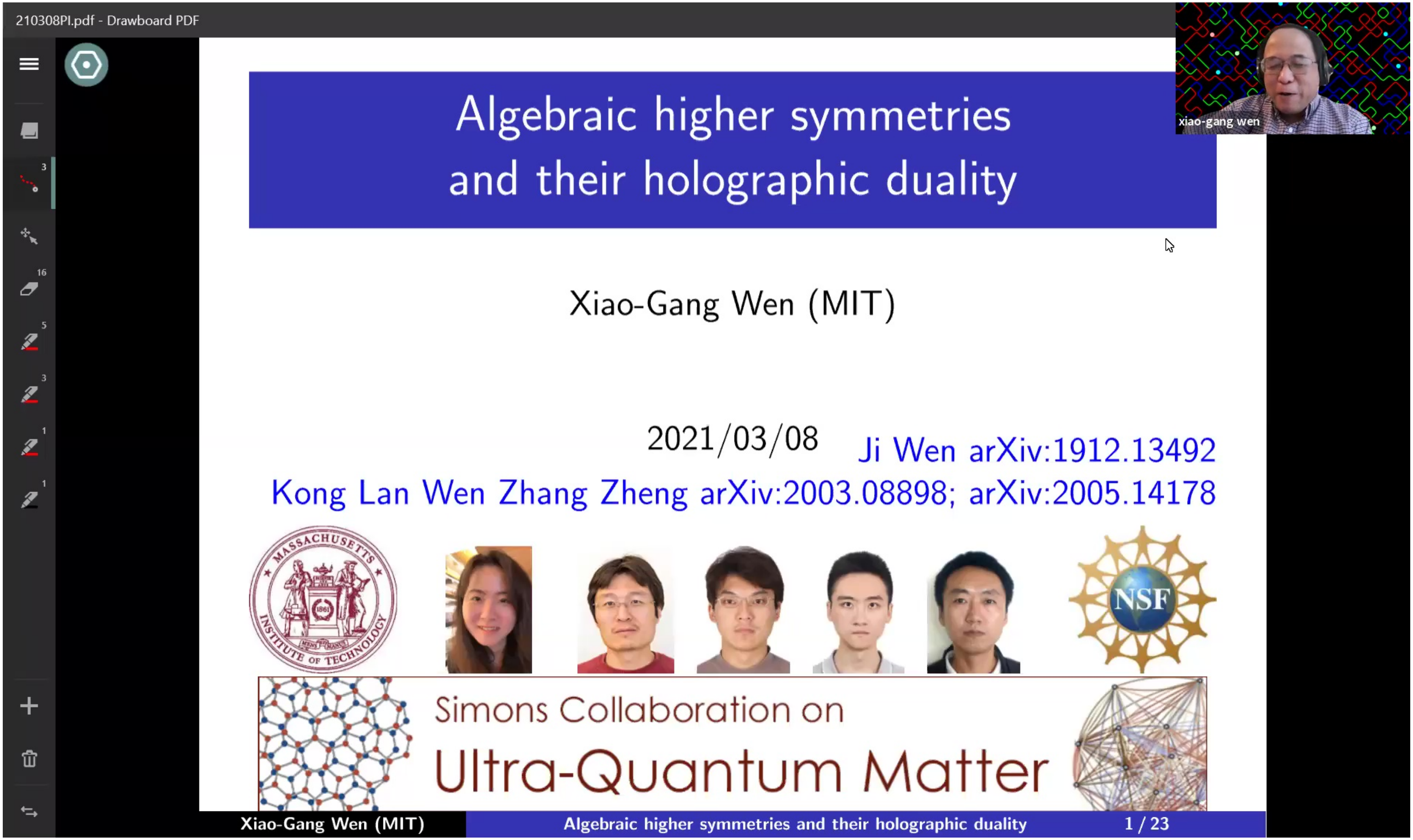
Title: Symmetry as shadow of topological order

Speakers: Xiao-Gang Wen
Date: March 08, 2021 - 12:30 PM

URL: http://pirsa.org/21030001

Abstract: Recently, the notion of symmetry has been extended from 0-symmetry described by group to higher symmetry described by higher group. In this talk, we show that the notion of symmetry can be generalized even further to "algebraic higher symmetry". Then we will describe an even more general point of view of symmetry, which puts the (generalized) symmetry charges and topological excitations at equal footing: symmetry can be viewed gravitational anomaly, or symmetry can be viewed as shadow topological order in one higher dimension.


210308PI.pdf - Drawboard PDF
xiao-gang wen
Algebraic higher symmetries
and their holographic duality
Xiao-Gang Wen (MIT)
2021/03/08
Ji Wen arXiv:1912.13492
Kong Lan Wen Zhang Zheng arXiv:2003.08898; arXiv:2005.14178
NSF
Simons Collaboration on
Ultra-Quantum Matter
Xiao-Gang Wen (MIT)
Algebraic higher symmetries and their holographic duality
1 / 23

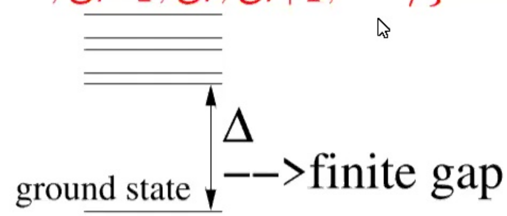
210308PI.pdf - Drawboard PDF

Trivial gapped symmetric state for lattice boson

- **Lattice bosonic quantum system** States on site-i: $|g_i\rangle, g_i \in G$.

- The total Hilbert space $\mathcal{V} = \bigotimes_i \mathcal{V}_i$. Basis $\{|\cdots, g_{i-1}, g_i, g_{i+1}, \cdots\rangle\}$

- A local Hamiltonian $H = -\sum |0\rangle_i\langle 0|_i$,
  where $|0\rangle_i = |G|^{-1/2} \sum_{g_i \in G} |g_i\rangle$



- H has an **on-site symmetry** G:
  $W_h H = H W_h$ (*ie* a global symmetry or a 0-symmetry):
  $$W_h = \prod_i T_h(i), \quad T_h(i)|g_i\rangle = |g_i h\rangle, \quad h \in G$$

- H has a **trivial gapped symmetric** ground state
  $$|\Phi_{\text{grnd}}\rangle = \bigotimes_i |0\rangle_i = |\cdots 0_{i-1} 0_i 0_{i+1} \cdots\rangle$$

- **Excitations** above the ground state $\leftrightarrow$ the representations R_q of G
  $$|\cdots 0_{i-1} q_i 0_{i+1} \cdots\rangle, \quad |q_i\rangle = \sum_{g_i} v_q^\alpha(g_i)|g_i\rangle, \quad v_q^\alpha(g_i h^{-1}) = R_q^{\alpha\beta}(h) v_q^\beta(g_i)$$

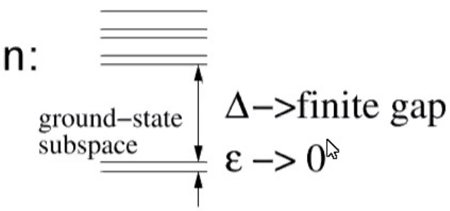
- The **fusion** of representations (excitations) $\rightarrow$ **conservation law**
  $$R_{q_1}(h) \otimes R_{q_2}(h) = \bigoplus_{q_3} N^{q_3}_{q_1 q_2} R_{q_3}(h) \;\rightarrow\; q_1 \otimes q_2 = \bigoplus_{q_3} N^{q_3}_{q_1 q_2} q_3$$

The excitations are described by symmetric fusion category $\mathcal{R}\text{ep}_G$

210308PI.pdf - Drawboard PDF

Spontaneous G-symmetry breaking state

- We may choose another **symmetric** Hamiltonian:
  $H = -\sum_i \delta_{g_i, g_{i+1}}$.



- H has degenerate **spontaneous symmetry breaking** (SSB) ground states labeled by $g \in G$: $|\cdots, g, g, g, \cdots\rangle$
- **Excitations** in the SSB ground states are domain walls labeled by $h = g' g^{-1} \in G$: $|\cdots, g', g', g', g', g, g, g, g, \cdots\rangle$
- The **fusion** of the excitations $\to$ **conservation law**
  $$h_1 \otimes h_2 = h_3, \quad \text{where } h_3 = h_1 h_2.$$
  $\to$ **The excitations are described by fusion category $\mathcal{V}ec_G$.**
- *What is the symmetry of this new conservation law?*
  What is the symmetry transformations of fusion category $\mathcal{V}ec_G$?

\- In comparison, the symmetry transformation of the SFC $\mathcal{R}ep_G$ is
$W_h : |\cdots, g_{i-1}, g_i, g_{i+1}, \cdots\rangle \to |\cdots, g_{i-1}h, g_i h, g_{i+1}h, \cdots\rangle, \ h \in G$
Tannaka duality: SFC $\mathcal{R}ep_G \leftrightarrow$ group G

xiao-gang wen

210308PI.pdf - Drawboard PDF

Algebraic symmetry in G-SSB state

- To see the **symmetry transformation** for $\mathcal{V}\mathrm{ec}_G$, we go to a dual lattice model where degrees of freedom live on link $l \in \mathbb{Z}+\frac{1}{2}$: $|h_l\rangle$.
  The dual relation is $h_l = g_{l-\frac{1}{2}} g^{-1}_{l+\frac{1}{2}}$, $l = i+\frac{1}{2}$: $g_{l-\frac{1}{2}}$ h_l $g_{l+\frac{1}{2}}$, i o———o $i+1$
  Dual model basis: $|\cdots h_{l-1}, h_l, h_{l+1}, \cdots\rangle$
- The dual Hamiltonian is $H = -\sum_i \delta_{g_i, g_{i+1}} \to \widetilde{H} = -\sum_l \delta_{h_l, 1}$.

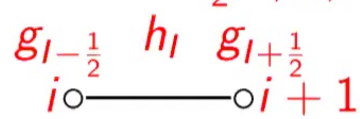
\- (1) The SSB ground state $\to$ The unique trivial ground state $h_l = 1$: $|\Phi_{\text{grnd}}\rangle = |\cdots 111 \cdots\rangle$. (2) An excitation $h_l = 1 \to h_l \neq 1$.

- The symmetry transformations are given by $\widetilde{W}_q = \mathrm{Tr} \prod_l R_q(h_l)$.
  (R_q is an irreducible representation of G) $\widetilde{W}_q$'s do not form a group
  $$\widetilde{W}_{q_1} \widetilde{W}_{q_2} = \sum_{q_3} N^{q_3}_{q_1 q_2} \widetilde{W}_{q_3}$$
  They generate an **algebraic symmetry**, denoted as $\widetilde{G} \cong \mathcal{V}\mathrm{ec}_G$.
  The old G-symmetry can be denoted as $G \cong \mathcal{R}\mathrm{ep}_G$.
- The symmetry requirement, $H\widetilde{W}_q = \widetilde{W}_q H$, prevents terms like $|h_l\rangle \to \#|h_l g\rangle$, but allows terms like $|h_l, h_{l+1}\rangle \to \#|h_l g, g^{-1} h_{l+1}\rangle$.

210308PI.pdf - Drawboard PDF

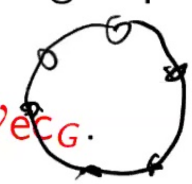
Algebraic symmetry in G-SSB state

- To see the **symmetry transformation** for $\mathcal{V}ec_G$, we go to a dual lattice model where degrees of freedom live on link $l \in \mathbb{Z} + \frac{1}{2}$: $|h_l\rangle$. The dual relation is $h_l = g_{l-\frac{1}{2}} g^{-1}_{l+\frac{1}{2}}$, $l = i + \frac{1}{2}$:



  Dual model basis: $|\cdots h_{l-1}, h_l, h_{l+1}, \cdots\rangle$
- The dual Hamiltonian is $H = -\sum_i \delta_{g_i, g_{i+1}} \to \widetilde{H} = -\sum_l \delta_{h_l, 1}$.
- (1) The SSB ground state $\to$ The unique trivial ground state $h_l = 1$: $|\Phi_{\text{grnd}}\rangle = |\cdots 111 \cdots\rangle$. (2) An excitation $h_l = 1 \to h_l \neq 1$.
- The symmetry transformations are given by $\widetilde{W}_q = \mathrm{Tr} \prod_l R_q(h_l)$. ($R_q$ is an irreducible representation of G) $\widetilde{W}_q$'s do not form a group

$$\widetilde{W}_{q_1} \widetilde{W}_{q_2} = \sum_{q_3} N^{q_3}_{q_1 q_2} \widetilde{W}_{q_3}$$



  They generate an **algebraic symmetry**, denoted as $\widetilde{G} \cong \mathcal{V}ec_G$.
  The old G-symmetry can be denoted as $G \cong \mathcal{R}ep_G$.
- The symmetry requirement, $H\widetilde{W}_q = \widetilde{W}_q H$, prevents terms like $|h_l\rangle \to \#|h_l g\rangle$, but allows terms like $|h_l, h_{l+1}\rangle \to \#|h_l g, g^{-1} h_{l+1}\rangle$.

Symmetry transformations of the symmetry and the algebraic symmetry

- **Algebraic symmetry** $\widetilde{G} \cong \mathcal{V}\mathrm{ec}_G$: $\quad h_l = g_{l-\frac{1}{2}} g_{l+\frac{1}{2}}^{-1}$

- In the dual model: $\widetilde{W}_q = \mathrm{Tr} \prod_l R_q(h_l)$.
- In the original model: $\widetilde{W}_q = \mathrm{Tr} \prod_i R_q(g_i g_{i+1}^{-1}) = \dim R_q$ (trivial)
  $\to$ any local Hamiltonian in the original model has the algebraic symmetry $\widetilde{G} \cong \mathcal{V}\mathrm{ec}_G$.

- **Symmetry** $G \cong \mathcal{R}\mathrm{ep}_G$:

- In the original model: $W_h = \prod_i T_h(i)$, where $T_h(i)|g_i\rangle \to |g_i h\rangle$.
- In the dual model: $W_h = 1$ (no action and trivial)
  $\to$ any local Hamiltonian in the dual model has the symmetry $G \cong \mathcal{R}\mathrm{ep}_G$.

- To see both the symmetries in a non-trivial forms, we need to introduce **patch symmetry transformation**

Ji Wen arXiv:1912.13492

210308PI.pdf - Drawboard PDF

xiao-gang wen

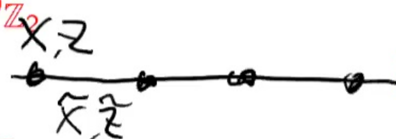
Patch symmetry transformations (for $G = \mathbb{Z}_2$)

The model $(X_i, Z_i Z_{i+1}) \leftrightarrow (\widetilde{X}_I \widetilde{X}_{I+1}, \widetilde{Z}_I)$ **The dual model**

$H = -\sum_i [J Z_i Z_{i+1} + B X_i]$ $\quad$ $H = -\sum_I [J \widetilde{Z}_I + B \widetilde{X}_I \widetilde{X}_{I+1}]$

- The $\mathbb{Z}_2$-symmetry $\mathcal{R}ep_{\mathbb{Z}_2}$ $\quad$ The $\mathbb{Z}_2$-symmetry $\mathcal{R}ep_{\mathbb{Z}_2}$

  $W(i,j) = \prod_{k=i}^{j} X_k$ $\quad$ $W(I,J) = \widetilde{X}_{I(\text{empty-bulk})} \widetilde{X}_J$



- The $\widetilde{\mathbb{Z}}_2$-symmetry $\mathcal{V}ec_{\mathbb{Z}_2}$ $\quad$ The $\widetilde{\mathbb{Z}}_2$-symmetry $\mathcal{V}ec_{\mathbb{Z}_2}$

  $\widetilde{W}(i,j) = Z_{i(\text{empty-bulk})} Z_j$ $\quad$ $\widetilde{W}(I,J) = \mathrm{Tr} \prod_I R_q(h_I) = \prod_{K=I}^{J} \widetilde{Z}_K$

- $\widetilde{W}(i,j) \to$ a pair of $\mathbb{Z}_2$-charge $\quad$ $W(I,J) \to$ a pair of $\widetilde{\mathbb{Z}}_2$-charge

- Local symmetric operators $\quad$ Local symmetric operators

  $O_k W(i,j) = W(i,j) O_k$ $\quad$ $O_K W(I,J) = W(I,J) O_K$

  $O_k \widetilde{W}(i,j) = \widetilde{W}(i,j) O_k$ $\quad$ $O_K \widetilde{W}(I,J) = \widetilde{W}(I,J) O_K$

  where k is far away from i, j $\quad$ where K is far away from I, J

The symmetric Hamiltonian is a sum of local symmetric operators

The model and the dual model have both the $\mathbb{Z}_2$-symmetry and the $\widetilde{\mathbb{Z}}_2$-symmetry. They have the $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ symmetry.

210308PI.pdf - Drawboard PDF

Patch symmetry transformations (for $G = \mathbb{Z}_2$)

The model $(X_i, Z_i Z_{i+1}) \leftrightarrow (\widetilde{X}_I \widetilde{X}_{I+1}, \widetilde{Z}_I)$ **The dual model**

$H = -\sum_i [J Z_i Z_{i+1} + B X_i]$ $\quad$ $H = -\sum_I [J \widetilde{Z}_I + B \widetilde{X}_I \widetilde{X}_{I+1}]$

- The $\mathbb{Z}_2$-symmetry $\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$ $\quad$ The $\mathbb{Z}_2$-symmetry $\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$

  $W(i,j) = \prod_{k=i}^{j} X_k$ $\quad$ $W(I,J) = \widetilde{X}_{I(\text{empty-bulk})} \widetilde{X}_J$

- The $\widetilde{\mathbb{Z}}_2$-symmetry $\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$ $\quad$ The $\widetilde{\mathbb{Z}}_2$-symmetry $\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$

  $\widetilde{W}(i,j) = Z_{i(\text{empty-bulk})} Z_j$ $\quad$ $\widetilde{W}(I,J) = \mathrm{Tr} \prod_I R_q(h_I) = \prod_{K=I}^{J} \widetilde{Z}_K$

- $\widetilde{W}(i,j) \to$ a pair of $\mathbb{Z}_2$-charge $\quad$ $W(I,J) \to$ a pair of $\widetilde{\mathbb{Z}}_2$-charge

- Local symmetric operators $\quad$ Local symmetric operators

  $O_k W(i,j) = W(i,j) O_k$ $\quad$ $O_K W(I,J) = W(I,J) O_K$

  $O_k \widetilde{W}(i,j) = \widetilde{W}(i,j) O_k$ $\quad$ $O_K \widetilde{W}(I,J) = \widetilde{W}(I,J) O_K$

  where k is far away from i, j $\quad$ where K is far away from I, J

The symmetric Hamiltonian is a sum of local symmetric operators

The model and the dual model have both the $\mathbb{Z}_2$-symmetry and the $\widetilde{\mathbb{Z}}_2$-symmetry. They have the $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ symmetry.

210308PI.pdf - Drawboard PDF

xiao-gang wen

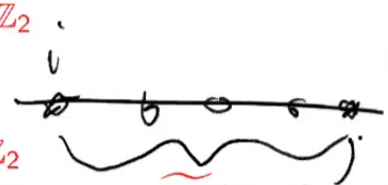
Patch symmetry transformations (for $G = \mathbb{Z}_2$)

The model $(X_i, Z_i Z_{i+1}) \leftrightarrow (\widetilde{X}_I \widetilde{X}_{I+1}, \widetilde{Z}_I)$ **The dual model**

$H = -\sum_i [J Z_i Z_{i+1} + B X_i]$ $H = -\sum_I [J \widetilde{Z}_I + B \widetilde{X}_I \widetilde{X}_{I+1}]$

- The $\mathbb{Z}_2$-symmetry $\mathcal{R}\text{ep}_{\mathbb{Z}_2}$ The $\mathbb{Z}_2$-symmetry $\mathcal{R}\text{ep}_{\mathbb{Z}_2}$

  $W(i,j) = \prod_{k=i}^{j} X_k$ $W(I,J) = \widetilde{X}_{I(\text{empty-bulk})} \widetilde{X}_J$



- The $\widetilde{\mathbb{Z}}_2$-symmetry $\mathcal{V}\text{ec}_{\mathbb{Z}_2}$ The $\widetilde{\mathbb{Z}}_2$-symmetry $\mathcal{V}\text{ec}_{\mathbb{Z}_2}$

  $\widetilde{W}(i,j) = Z_{i(\text{empty-bulk})} Z_j$ $\widetilde{W}(I,J) = \mathrm{Tr} \prod_I R_q(h_I) = \prod_{K=I}^{J} \widetilde{Z}_K$

- $\widetilde{W}(i,j) \to$ a pair of $\mathbb{Z}_2$-charge $W(I,J) \to$ a pair of $\widetilde{\mathbb{Z}}_2$-charge

- Local symmetric operators Local symmetric operators

  $O_k W(i,j) = W(i,j) O_k$ $O_K W(I,J) = W(I,J) O_K$

  $O_k \widetilde{W}(i,j) = \widetilde{W}(i,j) O_k$ $O_K \widetilde{W}(I,J) = \widetilde{W}(I,J) O_K$

  where k is far away from i, j where K is far away from I, J

The symmetric Hamiltonian is a sum of local symmetric operators

The model and the dual model have both the $\mathbb{Z}_2$-symmetry and the $\widetilde{\mathbb{Z}}_2$-symmetry. They have the $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ symmetry.

210308PI.pdf - Drawboard PDF

Patch symmetry transformations (for $G = \mathbb{Z}_2$)

The model $(X_i, Z_iZ_{i+1}) \leftrightarrow (\tilde{X}_I\tilde{X}_{I+1}, \tilde{Z}_I)$ **The dual model**

$H = -\sum_i [JZ_iZ_{i+1} + BX_i]$ $\quad$ $H = -\sum_I [J\tilde{Z}_I + B\tilde{X}_I\tilde{X}_{I+1}]$

- The $\mathbb{Z}_2$-symmetry $\mathcal{R}\text{ep}_{\mathbb{Z}_2}$ $\quad$ The $\mathbb{Z}_2$-symmetry $\mathcal{R}\text{ep}_{\mathbb{Z}_2}$

  $W(i,j) = \prod_{k=i}^{j} X_k$ $\quad$ $W(I,J) = \tilde{X}_{I(\text{empty-bulk})}\tilde{X}_J$

- The $\tilde{\mathbb{Z}}_2$-symmetry $\mathcal{V}\text{ec}_{\mathbb{Z}_2}$ $\quad$ The $\tilde{\mathbb{Z}}_2$-symmetry $\mathcal{V}\text{ec}_{\mathbb{Z}_2}$

  $\widetilde{W}(i,j) = Z_{i(\text{empty-bulk})}Z_j$ $\quad$ $\widetilde{W}(I,J) = \text{Tr}\prod_I R_q(h_I) = \prod_{K=I}^{J} \tilde{Z}_K$

- $\widetilde{W}(i,j) \to$ a pair of $\mathbb{Z}_2$-charge $\quad$ $W(I,J) \to$ a pair of $\tilde{\mathbb{Z}}_2$-charge

- Local symmetric operators $\quad$ Local symmetric operators

  $O_k W(i,j) = W(i,j) O_k$ $\quad$ $O_K W(I,J) = W(I,J) O_K$

  $O_k \widetilde{W}(i,j) = \widetilde{W}(i,j) O_k$ $\quad$ $O_K \widetilde{W}(I,J) = \widetilde{W}(I,J) O_K$

  where k is far away from i, j $\quad$ where K is far away from I, J

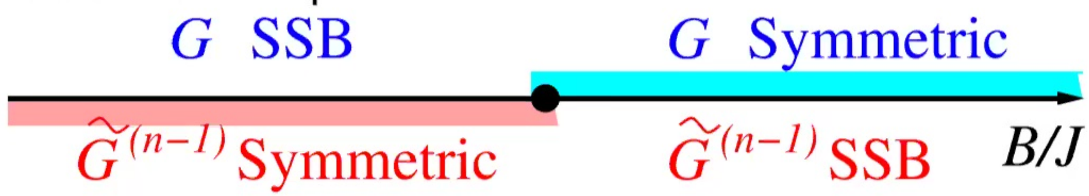
The symmetric Hamiltonian is a sum of local symmetric operators

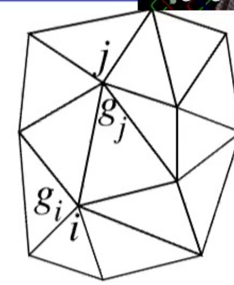
The model and the dual model have both the $\mathbb{Z}_2$-symmetry and the $\tilde{\mathbb{Z}}_2$-symmetry. They have the $\mathbb{Z}_2 \vee \tilde{\mathbb{Z}}_2$ symmetry.

210308PI.pdf - Drawboard PDF

Emergent symmetry at G-SSB critical point

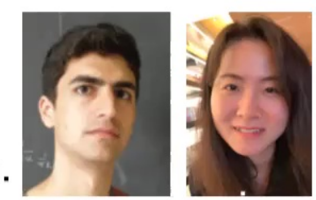
- $n+1$D G-symmetric $H = -\sum_{\langle ij\rangle} J\delta_{g_i,g_j} - B\sum_i |0_i\rangle\langle 0_i|$ have a SSB critical point.





- Definition: **spontaneous breaking** of algebraic higher symmetry 1) **Degenerate ground states** on some spaces M^n. 2) Some symmetry trans. are **not identity** in the ground state subspace

- The Hamiltonians have both the symmetries.



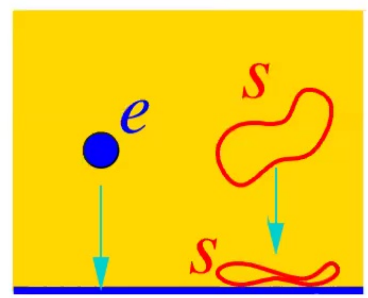
- For gapped state, symmetry, dual symmetry, or a certain mixture of them, must be spontaneously broken.
- The critical point has both the symm $G \cong \mathcal{R}\mathrm{ep}_G$ and the dual algebraic symm $\widetilde{G} \cong \mathcal{V}\mathrm{ec}_G$. Levin arXiv:1903.09028; Ji Wen arXiv:1912.13492

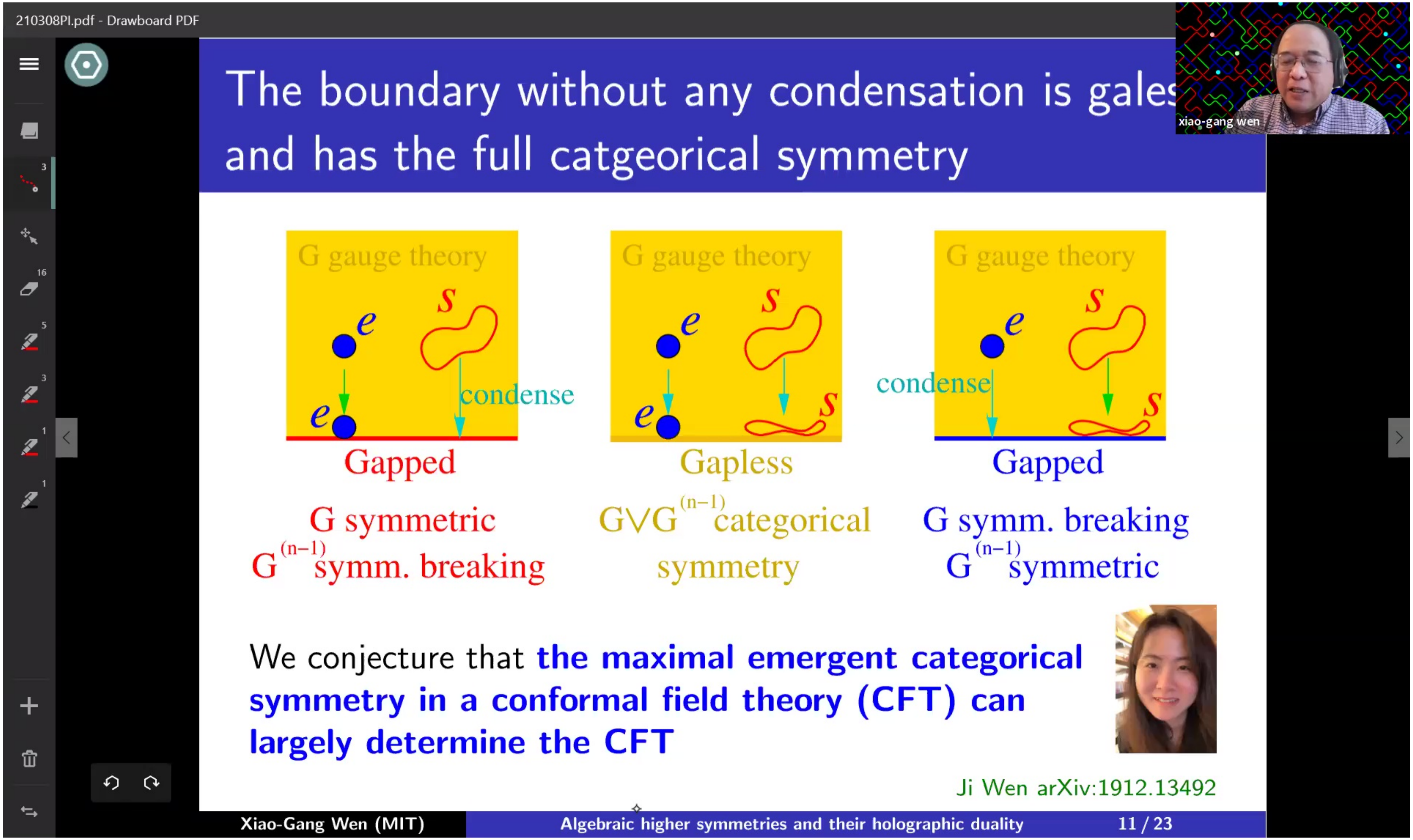
  The critical point has an unbroken **categorical symmetry** $\mathcal{R}\mathrm{ep}_G \vee \mathcal{V}\mathrm{ec}_G = G \vee \widetilde{G}$. ($\vee \to$ not an independent combination)

210308PI.pdf - Drawboard PDF

Symmetry = shadow of topological order

- $n+1$D G 0-symmetry implies $G \vee \widetilde{G}^{(n-1)}$ categorical symmetry
- $G \vee \widetilde{G}^{(n-1)}$ categorical symmetry $=$ $n+2$D topological order (G gauge theory) $\to$ $n+1$**D** G 0**-symmetry = a shadow of** $n+2$**D topological order (**G **gauge theory)**
- $n+1$D $\widetilde{G}^{(n-1)}$ algebraic $(n-1)$-symmetry implies $G \vee \widetilde{G}^{(n-1)}$ categorical symmetry
- $G \vee \widetilde{G}^{(n-1)}$ categorical symmetry $=$ $n+2$D topological order (G gauge theory) $\to$ $n+1$**D** $\widetilde{G}^{(n-1)}$ **algebraic** $(n-1)$**-symmetry = a shadow of** $n+2$**D topological order (**G **gauge theory)**
- **Example**: $2+1$D $2\mathcal{V}ec_{\mathbb{Z}_2} \cong \mathbb{Z}_2^{(1)}$ 1-symmetry $=$ shadow of $3+1$D Z_2-gauge theory
  Realized by the charge condensed boundary of $3+1$D Z_2-gauge theory. The boundary excitations are described by $\{\mathbf{1}_s, s\}$, $s \otimes s = \mathbf{1}_s \to \mathbb{Z}_2^{(1)}$ 1-symmetry.





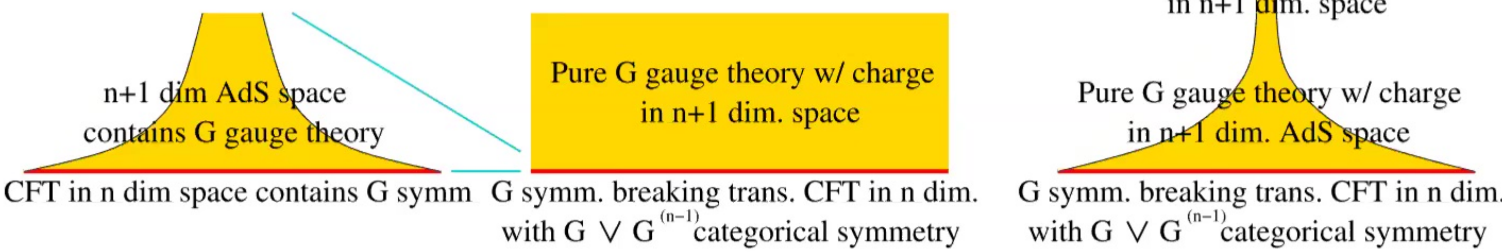
210308PI.pdf - Drawboard PDF
The boundary without any condensation is gales
and has the full catgeorical symmetry
G gauge theory
e
s
condense
Gapped
Gapless
G symmetric
G(n−1) symm. breaking
G∨G(n−1) categorical
symmetry
G symm. breaking
G(n−1) symmetric
We conjecture that the maximal emergent categorical symmetry in a conformal field theory (CFT) can largely determine the CFT
Ji Wen arXiv:1912.13492
Xiao-Gang Wen (MIT)
Algebraic higher symmetries and their holographic duality
11 / 23
xiao-gang wen

210308PI.pdf - Drawboard PDF

Apply categorical symmetry to AdS/CFT duality

- **Witten**: "for gauge theory, suppose the AdS theory has a gauge group G, [...] Then in the scenario of [13], the group G is a global symmetry group of the conformal field theory on the boundary."

Maldacena arXiv:hep-th/9711200; Witten arXiv:hep-th/9802150



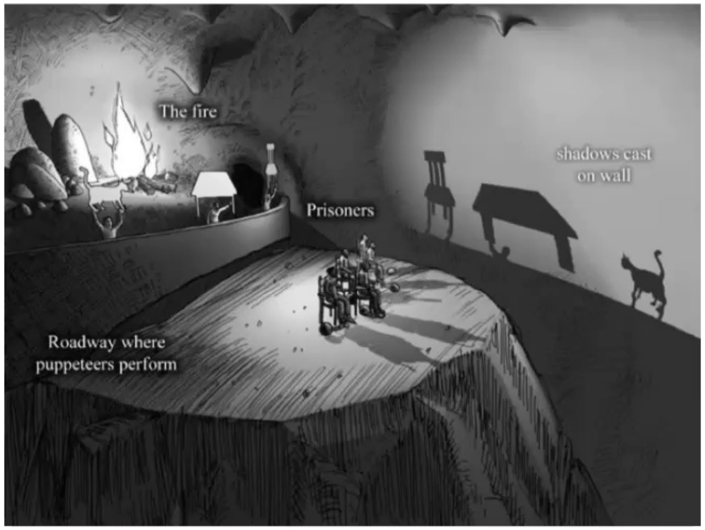
- G-symm.-breaking-transition CFT has a categorical symmetry described by the G-gauge theroy in one higher dimension, *which uniquely determines the bulk theory*.

  Pure G-gauge theory (w/ charge fluc. & gravity) in $(n+1)$-dim. AdS space = CFT at the G-symm.-breaking-transition in n-dim. space, not other CFT's with G-symmetry. Ji Wen arXiv:1912.13492

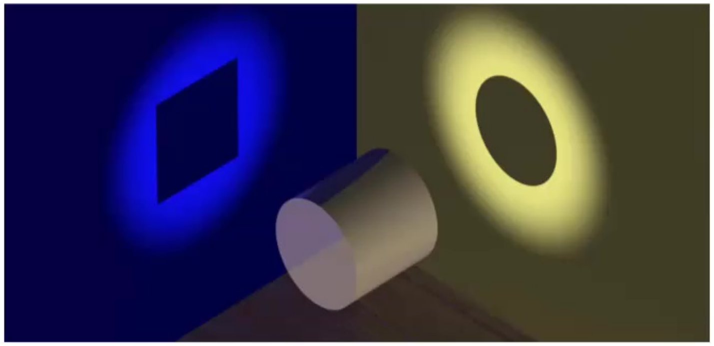
210308PI.pdf - Drawboard PDF

The essence of a symmetry

A symmetry is the shadow of a topological order in one higher dimension (*ie* categorical symmetry)



Categorical symmetry
$\to$ symmetry and dual symmetry

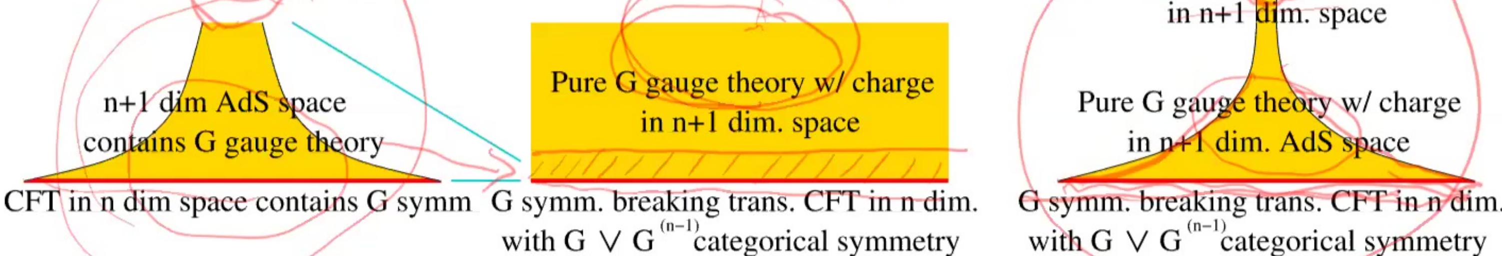


The same topological order (in one higher dimensions) can have different shadows $\to$ **dual-equivalent** symmetries.

Apply categorical symmetry to AdS/CFT duality

- **Witten**: "for gauge theory, suppose the AdS theory has a gauge group G, [...] Then in the scenario of [13], the group G is a global symmetry group of the conformal field theory on the boundary."

Maldacena arXiv:hep-th/9711200; Witten arXiv:hep-th/9802150



- G-symm.-breaking-transition CFT has a categorical symmetry described by the G-gauge theroy in one higher dimension, *which uniquely determines the bulk theory*.
  Pure G-gauge theory (w/ charge fluc. & gravity) in $(n+1)$-dim. AdS space = CFT at the G-symm.-breaking-transition in n-dim. space, not other CFT's with G-symmetry. Ji Wen arXiv:1912.13492