Title: On hidden quantum group symmetries in CFT

Speakers: Eveliina Peltola

Collection: Women at the Intersection of Mathematics and Theoretical Physics

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Abstract: "I discuss applications of a hidden $U_q(\mathfrak{sl}_2)$-symmetry in CFT with central charge $c \leq 1$ (focusing on the generic, semisimple case, with c irrational). This symmetry provides a systematic method for solving Belavin-Polyakov-Zamolodchikov PDE systems, and in particular for explicit calculation of the asymptotics and monodromy properties of the solutions. Using a quantum Schur-Weyl duality, one can understand solution spaces of such PDE systems in a detailed way. The solutions, in turn, are useful both for CFT questions and for rigorous understanding of the connections of 2D CFT with critical models of statistical physics."

On hidden quantum group symmetries in CFT

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**Women at the intersection of
Mathematics and Theoretical Physics,
Online event @ PI**

Based on various joint works with

Steven Flores (LeoLabs), **Kalle Kytölä** (Aalto Uni), **Hao Wu** (Tsinghua Uni)

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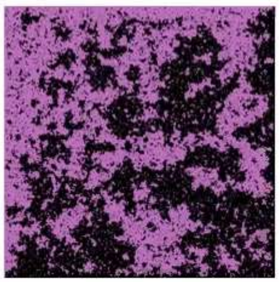
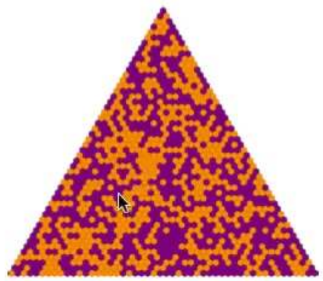
Critical lattice models in 2D statistical physics

- random walks, percolation, Ising model, Gaussian free field, 6-vertex model, Potts model, $O(n)$ model, ...
- phase transitions $\Rightarrow$ **critical phenomena**
  - **critical exponents**: observables have power law behavior
  - **self-similarity**
  - **universality conjecture**: microscopic details irrelevant
- **scaling limits**: **conformal field theories** ???

  [Belavin, Polyakov & Zamolodchikov '84; Cardy '84]


Problem: How to make these things precise?

1

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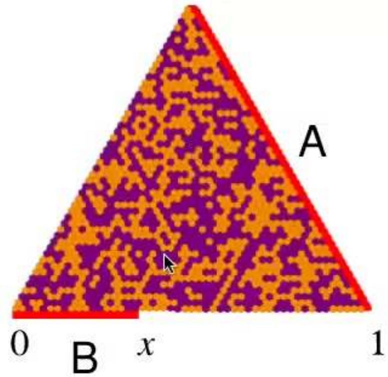
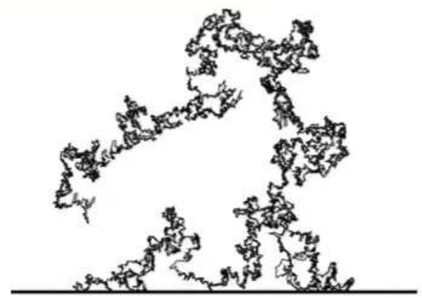
Conformal invariance of critical 2D percolation





Theorem [Cardy '92; Carleson (unpubl.); Smirnov '01]

In the scaling limit, $\mathbb{P}$[yellow crossing from A to B] $= x$

- "**conformal invariance**" of the scaling limit of percolation
  (crit. site perco on triangular lattice; e.g. case of square lattice still open)
- exploration process converges to random curve called **Schramm-Loewner evolution**, SLE_6

"**Geom**" **scaling limit**: loops converge to CLE_6 [Camia, Newman '06]

2

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Crossing probabilities of percolation interfaces

- discrete polygons $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ (on triangular lattice)
- $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \xrightarrow{\delta \to 0} (\Omega; x_1, \ldots, x_{2N})$ (in Carathéodory sense)

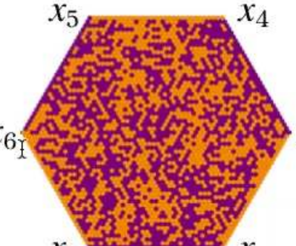
Theorem [P. & Wu]

For the critical Bernoulli percolation model on Ω^δ with alternating boundary conditions, for all connectivities $\alpha \in \mathrm{LP}_N$, we have

$$\lim_{\delta \to 0} \mathbb{P}[\text{ connectivity of interfaces} = \alpha] \quad = \quad \mathcal{Z}_\alpha^{(\kappa=6)}(\Omega; x_1, \ldots, x_{2N})$$

where $\{\mathcal{Z}_\alpha^{(\kappa=6)} : \alpha \in \mathrm{LP}_N\}$ are so-called "pure partition functions".

- $\alpha \in \mathrm{LP}_N$ label planar pairings of $x_1, \ldots, x_{2N}$
- $\#\mathrm{LP}_N = C_N = \frac{1}{N+1}\binom{2N}{N}$ Catalan number
- **Idea:** $\mathcal{Z}_\alpha$ are **CFT correlation functions** (central charge $c = 0$, weight $h_{1,2} = 0$)
- $N = 1$: $\mathcal{Z} = 1$, and $N = 2$: Cardy's formula (prev.slide)



3

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How to get grasp on this?

1. BPZ partial differential equations

   [Belavin, Polyakov & Zamolodchikov '84]

2. Coulomb gas formulas for solutions

   [Feigin & Fuchs (unpubl.) and Dotsenko & Fateev '84, ...]

   Problem: find "physical" solutions...?

3. Tool: Hidden $\mathcal{U}_q(\mathfrak{sl}_2)$-symmetry

   ['80s–'90s: Felder, Fröhlich, Keller; Moore, Reshetikin; Bouwknegt, McCarthy, Pilch; Gomez, Sierra; Felder, Wieczerkowski; Varchenko; Ramirez, Ruegg, Ruiz-Altaba; Pasquier, Saleur; ...]

4

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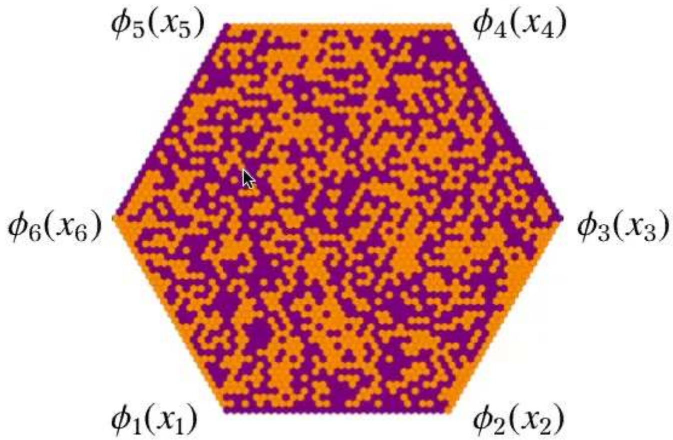
CFT correlation functions



Correlation functions $\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle = F(z_1, \ldots, z_n)$ in CFT

- encode **physical information** ("observables")
- are **conformally covariant**: e.g. for (chiral) primaries ϕ_j, there are *conformal weights* $\Delta_j \in \mathbb{R}$ s.t. for conf. maps f,

$$F(f(z_1), \ldots, f(z_n)) = \prod_{j=1}^{n} \left| f'(z_j) \right|^{-\Delta_j} \times F(z_1, \ldots, z_n)$$

5

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BENOIT & SAINT-AUBIN PARTIAL DIFFERENTIAL EQUATIONS

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- many interesting observables satisfy **linear homog. PDEs**
- Benoit, Saint-Aubin '88: explicit formulas, e.g.

$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial z^2} + \sum_{i=1}^{n} \left(\frac{2}{z_i - z} \frac{\partial}{\partial z_i} - \frac{2h_{1,2}}{(z_i - z)^2} \right) \right\} \langle \phi_{1,2}(z) \phi_1(z_1) \cdots \phi_n(z_n) \rangle = 0$$

where $h_{1,2} = \frac{6-\kappa}{2\kappa}$, central charge $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1$, with $\kappa > 0$

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Benoit & Saint-Aubin partial differential equations

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Problem: **There are many solutions! Find physical ones?**

Idea: **Impose boundary conditions, or specify monodromy...**

6

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Crossing probabilities of percolation interfaces

- $(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \overset{\delta\to 0}{\longrightarrow} (\Omega; x_1, \ldots, x_{2N})$ (in Carathéodory sense)

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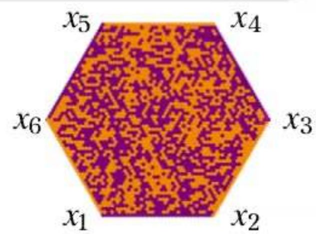

Idea: $\mathcal{Z}_\alpha^{(\kappa)}$ are **CFT correlation functions**

(central charge $c = 0$, weight $h_{1,2} = 0$)

Mathematically: solns to 2nd order PDEs with prescribed asymptotic properties:

$$|x_{j+1} - x_j|^{\frac{6-\kappa}{\kappa}} \mathcal{Z}_\alpha(x_1, \ldots, x_{2n})$$
$$\underset{x_j, x_{j+1}\to\xi}{\longrightarrow} \begin{cases} \mathcal{Z}_{\alpha\setminus\{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2n}) & \text{if } \{j, j+1\} \in \alpha \\ 0 & \text{if } \{j, j+1\} \notin \alpha \end{cases}$$

7

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Benoit & Saint-Aubin partial differential equations

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8

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Correlation functions of (certain) primary fields

- **Benoit & Saint-Aubin PDE system**: with $\varsigma = (s_1, \ldots, s_n)$,

$$\mathcal{D}^{(z_j)}_{s_j+1} F_\varsigma(z;\bar{z}) = 0, \qquad \mathcal{D}^{(\bar{z}_j)}_{s_j+1} F_\varsigma(z;\bar{z}) = 0, \qquad \text{for all } j = 1, \ldots, n$$

where $s_j + 1$ = degree of differential operator $\mathcal{D}^{(z_j)}_{s_j+1}$

- **covariance**: for all conformal maps f,

$$F_\varsigma(f(z), f(\bar{z})) = \left(\prod_{i=1}^{n} \left(\partial f(z_i)\, \bar{\partial} f(\bar{z}_i) \right)^{-\Delta_{s_i}} \right) F_\varsigma(z, \bar{z})$$

where Δ_s = conf. weight of type $h_{1,s+1}$ or $h_{s+1,1}$ in the Kac table

- F_ς is defined for $\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j \}$

Theorem. [Flores, P. (in prep.)]

Suppose $\kappa \in (0,8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1$ irrational).

There exists a unique single-valued, conformally covariant solution $F_\varsigma(z, \bar{z})$ to the Benoit & Saint-Aubin PDEs.

(Uniqueness holds up to normalization in a natural solution space.)

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General method:

Hidden quantum group symmetry

Feigin & Fuchs (unpubl.); Dotsenko & Fateev '84; Felder, Fröhlich, Keller '89; Felder, Wieczerkowski '91; Kytölä, P. '14

Main ref: see [arXiv:1408.1384] / JEMS 22(1): 55–118, 2020.

Kytölä, P., *Conformally covariant boundary correlation functions with a quantum group.*

10

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How to find solutions to BPZ/BSA PDEs

$$\mathcal{D}^{(z_i)} F = 0 \qquad \text{for all } i = 1, \ldots, n$$

???

Write solution in integral form & use Stokes theorem:

$$F(z_1, \ldots, z_n) = \int_\Gamma f(z_1, \ldots, z_n; w_1, \ldots, w_\ell)\, \mathrm{d}w_1 \cdots \mathrm{d}w_\ell$$

- find f s.t. for all j

  $\mathcal{D}^{(z_i)} f$ = "total derivative"

  $\Rightarrow$ then $\mathcal{D}^{(z_i)} f\, \mathrm{d}\mathbf{w} = \mathrm{d}\eta$ **is exact ℓ-form**

- find **closed ℓ-surface** Γ: $\quad \partial\Gamma = \emptyset$

  $$\Rightarrow \text{ then } \mathcal{D}^{(z_i)} F = \int_\Gamma \mathcal{D}^{(z_i)} f\, \mathrm{d}\mathbf{w} = \int_\Gamma \mathrm{d}\eta$$
  $$= \int_{\partial\Gamma} \eta = \int_\emptyset \eta$$
  $$= 0$$

11

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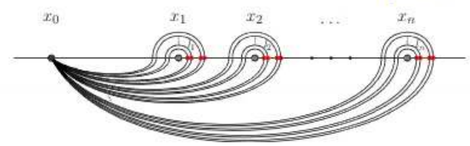
Hidden quantum group symmetry $\quad q = e^{4\pi i/\kappa} \in \mathbb{C} \setminus e^{\pi i \mathbb{Q}}$

hidden quantum group: $\mathcal{U}_q(\mathfrak{sl}_2)$ (= "quantum" $\mathfrak{sl}_2(\mathbb{C})$)

vectors $v \in \mathrm{M}_{s_n} \otimes \cdots \otimes \mathrm{M}_{s_1} \quad \longleftrightarrow \quad$ functions $\mathcal{F}[v](x_1, \ldots, x_n)$

basis vectors $e_{\ell_n} \otimes \cdots \otimes e_{\ell_1} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad$ integral functions $\displaystyle\int_{\Gamma^{(x_0)}_{\ell_1,\ldots,\ell_n}} f\, dw$

$$\mathcal{F}[v](\boldsymbol{x}) = \int_{\Gamma(v)} f(\boldsymbol{x}; \boldsymbol{w}) d\boldsymbol{w}$$



Theorem [Kytölä, P. '14]

- highest weight vectors ($E.v = 0$) $\overset{\mathcal{F}}{\longleftrightarrow}$ solutions of PDEs
- weight ($K.v = q^{\lambda} v$) $\overset{\mathcal{F}}{\longleftrightarrow}$ conformal transformation properties
- projections onto subrepresentations $\overset{\mathcal{F}}{\longleftrightarrow}$ asymptotics

$\mathcal{U}_q(\mathfrak{sl}_2)$ gen. by E, F, K, K^{-1}; and M_s its simple module (dim. $s+1$)

12

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1

IRREDUCIBLE REPRESENTATIONS OF $\mathcal{U}_q(\mathfrak{sl}_2)$

- $\mathrm{M}_s = \mathrm{span}\{e_0, \ldots, e_s\}$
- e_0 is **highest weight vector**: $E.e_0 = 0$ and $K.e_0 = q^s e_0$
- e_0 generates M_s: $e_k = F^k.e_0$
- associate contours to e_k:

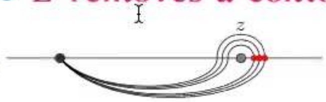
e_k =


- ***F* adds a contour**:

$F.$


=


- ***E* removes a contour**:

$E.$


$= \; [k]_q[s+1-k]_q \times$


13

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Asymptotics $\left|x_{j+1} - x_j\right| \to 0$

$$\frac{\mathcal{F}[v](x_1, \ldots, x_n)}{|x_{j+1} - x_j|^{\Delta_{s, s_j, s_{j+1}}}}$$

$$\downarrow \quad x_j\,,\, x_{j+1} \;\to\; \xi$$

$$\mathcal{F}[\hat{v}](x_1, \ldots, x_{j-1}, \xi, x_{j+2} \ldots, x_n)$$