Title: Change the coefficients of conditional entropies in extensivity

Speakers: Asuka Takatsu

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Abstract: The Boltzmann--Gibbs entropy is a functional on the space of probability measures. One characterization of the Boltzmann--Gibbs entropy is given by the Shannon--Khinchin axioms, which consist of continuity, maximality, expandability and extensivity. The extensivity is expressed in terms of the linear combinations of conditional probabilities. Replacing the coefficients in the linear combinations with a power function provides a characterization of the Tsallis entropy. I talk about the impossibility to replace the coefficients with a non-power function.

Change the coefficients of conditional entropies in extensivity

Asuka TAKATSU

$\mathcal{P}_n := \{ p = (p_j)_{j=1}^n \in \mathbb{R}^n \mid p_j \geq 0 \ \forall j, \ \sum_{j=1}^n p_j = 1 \}$

Thm. (Khinchin '57) / (Suyari '04)

Let $S : \bigsqcup_{n \in \mathbb{N}} \mathcal{P}_n \to \mathbb{R}$ satisfy

1) S is conti on $\mathcal{P}_n$ $\forall n \in \mathbb{N}$.

2) $S(p) \leq S((\frac{1}{n}, \cdots, \frac{1}{n}))$ $\forall p \in \mathcal{P}_n$

3) $S(p) = S((p_1, \cdots, p_n, 0))$ $\forall p = (p_j)_{j=1}^n \in \mathcal{P}_n$

4) For $P = (p^i_j)_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathcal{P}_{nm}$ with $p_j = \sum_{i=1}^m p^i_j > 0$

$$S(P) = S((p_1, \cdots, p_n)) + \sum_{j=1}^n p_j^q S\left(\left(\frac{p^1_j}{p_j}, \cdots, \frac{p^m_j}{p_j}\right)\right) \qquad q > 0$$

$\Rightarrow \exists \lambda > 0$ s.t. $S_1(p) = -\lambda \sum_{j=1}^n p_j \log p_j$ $\forall p = (p_j)_{j=1}^n \in \mathcal{P}_n$ (Boltzmann entropy)

$\uparrow q \to 1$

$q \neq 1$ $S_q(p) = -\frac{\lambda}{1-q} \sum_{j=1}^n (p_j^q - p_j)$ … (Tsallis entropy)

Ex. roll 2 dices X_m & X_n (value $1 \sim m$, $1 \sim n$)

p^i_j = prob. of $X_m = i$ & $X_n = j$

p_j = prob. of $X_n = j$

$\frac{p^i_j}{p_j}$ = prob of $X_m = i$ given $X_n = j$

(= prob. of $X_m = i$)

$\uparrow$ if X_n, X_m : indep

Thm.

$s \in C([0,1]) \cap C^2((0,1])$ with $s'' < 0$ on $(0,1]$ & $s(0) = 0$.

$s_1(r) = -r\log r$

$s_q(r) = -\dfrac{r^q - r}{1-q} \quad (q \neq 1)$

Define $S : \bigsqcup_{n\in\mathbb{N}} \mathcal{P}_n \to \mathbb{R}$ by $S(p) = \sum_{j=1}^{n} s(p_j)$ $\quad {}^\forall p = (p_j)_{j=1}^{n} \in \mathcal{P}_n$

Then S satisfies (1) ~ (3) i.e.

1) S is conti on $\mathcal{P}_n$ $\ {}^\forall n \in \mathbb{N}$.

2) $S(p) \leq S((\frac{1}{n}, \cdots, \frac{1}{n}))$ $\ {}^\forall p \in \mathcal{P}_n$

3) $S(p) = S((p_1, \cdots, p_n, 0))$ $\ {}^\forall p = (p_j)_{j=1}^{n} \in \mathcal{P}_n$

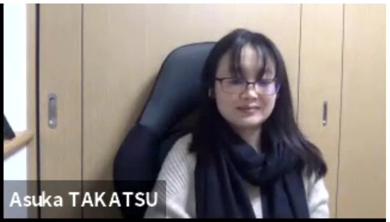
If ${}^\exists f : [0,1) \to \mathbb{R}$ s.t.

4) For $P = (p_j^i)_{1\leq i\leq m,\, 1\leq j\leq n} \in \mathcal{P}_{nm}$ with $p_j = \sum_{i=1}^{m} p_j^i > 0$

$$S(P) = S((p_1, \cdots, p_n)) + \sum_{j=1}^{n} f(p_j)\, S\left(\left(\frac{p_j^1}{p_j}, \cdots, \frac{p_j^m}{p_j}\right)\right)$$

then $f(r) = r^q$ $\ ({}^\exists q > 0)$ hence S : Boltzmann $(s = s_1)$

Tsallis $(s = s_q \quad q \neq 1)$



$$s(\rho) = \int_{\Omega} s(\rho(\omega))\, d\mathbb{P}(\omega)$$

$$= \int_{\mathbb{R}^n} s(\rho(x))\, dx$$

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$\mathrm{Ric} \geq K \iff \exists$ convexity of Boltzmann entropy $\quad q = 1$

$\iff$ " of entropy $\in DC_{\infty} \qquad DC_{\frac{1}{1-q}}$

displacement convexity

Sturm, Lott – Villani