Title: From gauge fields to direct connections on gauge groupoids

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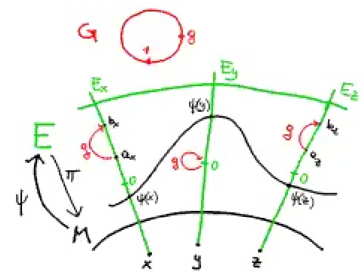
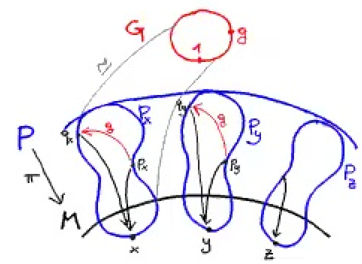
Abstract: "Geometrically, a gauge theory consists of a spinor bundle describing the matter fields, associated to some principal bundle whose gauge group rules the internal symmetries of the system. The gauge fields are the local expressions of a principal connection inducing a covariant derivative which settles the dynamics of the matter fields.
Principal connections can be seen as parallel displacements on the fibres of the principal bundle along curves on the base manifold. In this talk I shortly present a generalisation of gauge fields given by direct connections on gauge groupoids, based on a work in progress with S. Azzali, Y. Boutaib, A. Garmendia and S. Paycha."

Geometric model underlying field and gauge theories



- space-time manifold M
- vector/spinor bundle $E \to M$ with fibres V
- (matter) field $\psi : M \to E$ section of E
  $\Rightarrow$ configuration space $\mathcal{E} = \Gamma(M, E)$
- dynamics via action $S[\psi]$ (isolated particles)
  $\Rightarrow$ use derivatives $D_X\psi$ i.e. linear connection on E



- symmetries by Lie group G acting on fibres of E
- principal G-bundle $P \to M$ s.t. $E = P \times_G V$
  (associated bundle)
- gauge bosons (force carriers) $A \in \Omega^1_{\text{loc}}(M, \mathfrak{g})$
  local form of principal connection on P
- gauge group $\widehat{G} = \mathrm{Aut}_M(P) = \Gamma(M, P^{ad}) = \mathrm{Aut}_M(E)$
  $\Rightarrow$ acts on ψ and A
- dynamics via $\widehat{G}$-inv. action $S[\psi, A]$ (interacting particles)
  $\Rightarrow$ new covariant derivative $D^A_X\psi$

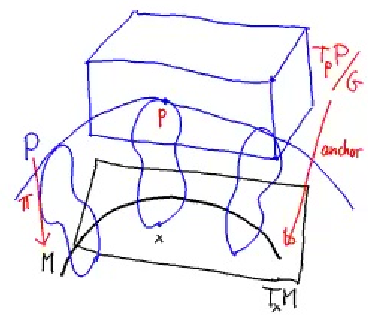
Idea: replace $\left\{\begin{array}{l} \text{connection on } P \\ + \text{ gauge group} \\ (+ \text{ Lie algebra } \mathfrak{g}) \end{array}\right.$ by $\left\{\begin{array}{l} \text{direct connection} \\ \text{on gauge groupoid of } P \\ (+ \text{ Lie algebroid of } P) \end{array}\right.$.

Infinitesimal structure: Lie algebroids

- **Lie algebroid**: vector bundle $\boxed{A \to M}$ with a Lie bracket $[\ ,\ ]_A$ on sections $\Gamma(A)$ and an **anchor map** $\boxed{a : A \to TM}$ inducing a derivation on sections w.r.t. vector fields on M:

$$[X, fY]_A = f[X, Y]_A + a(X)(f)\,Y$$

- Atiyah Lie algebroid of a principal G-bundle $\pi : P \to M$:
$\boxed{A(P) = TP/G \to M}$ with fibres $A_x(P) \cong T_{p_x}P$,
anchor $A(P) \to TM$ induced by $d\pi : TP \to TM$
via quotient map $j : TP \to TP/G$
and Lie bracket of G-invariant vector felds on P.



- Trivial G-bundle: $\quad P = M \times G \to M \quad \Rightarrow \quad A(M \times G) = TM \oplus (M \times \mathfrak{g}) \overset{id+0}{\to} TM$
where $\mathfrak{g} = \mathrm{Lie}(G)$.

- Frame bundle of a vector bundle $E \to M$ of rank r: $\quad F(E) = \bigcup_{x \in M} \mathrm{Iso}(\mathbb{R}^r, E_x) \to M$

  $\Rightarrow \quad A(F(E)) = \mathrm{Der}(E) \to TM$ bundle of derivative endomorphisms
  s.t. $\Gamma(\mathrm{Der}(E)) =$ derivations of $\Gamma(E)$.

Principal connections, gauge fields and covariant derivative

- **Principal connection on P:** five equivalent presentations

1) G-equivariant **horizontal subbundle** $\boxed{HP \subset TP \to P}$ s.t. $TP = HP \bigoplus VP$, where the **vertical bundle** VP (spaces tangent to the fibres) is canonical.

2) G-equivariant **connection 1-form** $\boxed{\omega \in \Omega^1(P, \mathfrak{g})}$ s.t. $\omega_p(\hat{A}_p) = A$ if $\hat{A}$ is the (vertical) fundamental vector field on P det. by $A \in \mathfrak{g}$, and $\omega_p(B_p) = 0$ if $B_p \in H_p P$ is horizontal.

3) **infinitesimal connection** $\boxed{\delta : TM \to A(P) \text{ s.t. } j^{-1}(\delta(X)) \in HP}$ is the horizontal lift of $X \in TM$.

4) **parallel transport** $\boxed{\tau_\gamma(y, x) : P_x \xrightarrow{\cong} P_y}$ along a curve γ of M from x to y given by the horizontal lift of γ.

5) (local) **gauge fields** $\boxed{\{A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\}}$ = pull back of ω along local sections of P.

- **Covariant derivative on sections of E:** bundle map $\boxed{D^A : TM \to \mathrm{Der}(E)}$ equivalent to $C^\infty(M)$-derivation $D^A{}_X : \Gamma(E) \to \Gamma(E)$ for $X \in \Gamma(TM)$ given only locally by

$$\boxed{D^A{}_X(\psi)_{|U_\alpha} = \sum_{\mu, i, j} (X^\mu \partial_\mu \psi^i + X^\mu A^i_{\mu j} \psi^j) e_i}$$

if $X = X^\mu \partial_\mu$ in coordinates x^μ on $U_\alpha \subset M$
$\psi = \psi^i e_i$ on a local basis (e^i) of E_{U_α}
and $A^i_j = A^i_{\mu j} dx^\mu$ are the components of the gauge field A in terms of generators of $\mathfrak{g}$.

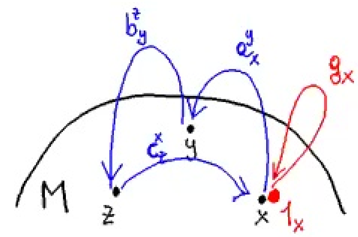
Lie groupoids

- **Lie groupoid** $\boxed{\mathcal{G} \rightrightarrows M}$: bi-fibred manifold $\mathcal{G} = \bigcup_{(y,x)\in M\times M} \mathcal{G}_x^y$

  with elements $a_{yx} \in \mathcal{G}_x^y$ called **arrows**

  projections $s, t : \mathcal{G} \to M \begin{cases} \textbf{source} & s(a_{yx}) = x \\ \textbf{target} & t(a_{yx}) = y \end{cases}$

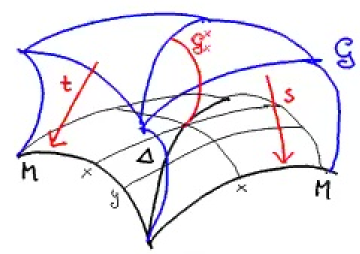
  such that



- arrows can be **composed**: $b_{zy}\, a_{yx} \in \mathcal{G}_x^z$ if $s(b_{zy}) = t(a_{yx})$ (composition is associative),
- there are **units** $u(x) = 1_x \in \mathcal{G}_x^x$ and $M \equiv u(M) \subset \mathcal{G}$,
- each arrow $a_{yx} \in \mathcal{G}_x^y$ has an **inverse** $a_{yx}^{-1} \in \mathcal{G}_y^x$.

The induced map $(t, s) : \mathcal{G} \to M \times M$ is called the **anchor**.

Each $\mathcal{G}_x^x$ is a Lie group, called the **vertex group** or **isotropy**.



- Infinitesimal structure of Lie groupoid = Lie algebroid: $\boxed{\mathcal{L}\mathcal{G} = \bigcup_{x\in M} T_{1_x}\mathcal{G}_x \to TM}$

Gauge groupoids

- Gauge groupoid of principal G-bdl $P \to M$: $\boxed{\mathcal{G}(P) = P \times_G P \rightrightarrows M}$
  contains equivalence classes $[p,q]$ under $(p,q) \sim (pg, qg)$ for $g \in G$.
  $\Rightarrow$ 1) $\boxed{\mathcal{L}(\mathcal{G}(P)) = A(P) \to TM}$
  2) $\mathcal{G}(P)$ **acts on** $E = P \times_G V$: $\boxed{\mathcal{G}(P) \times_M E = \{([p,q],[r,v]),\ \pi(q) = \pi(r)\} \to E}$
  with action $\rho_{[p,q]}([r,v]) = [p, gv]$ where $g \in G$ s.t. $r = qg$,
  3) $\boxed{\hat{G} = \mathrm{Aut}_M(P) \subset \mathcal{G}(P)}$ given by $\boxed{\Phi \mapsto [\Phi(p), p]}$ for any $p \in P$.

- Pair groupoid: $\boxed{\mathrm{Pair}(M) = M \times M \rightrightarrows M}$ for $P = M \times \{1\} \to M$
  $\Rightarrow$ Lie algebroid $= \mathcal{L}(\mathrm{Pair}(M)) = TM \xrightarrow{id} TM$.

- Trivial Lie groupoid with fibre G: $\boxed{M \times G \times M \rightrightarrows M}$ for $P = M \times G \to M$
  $\Rightarrow$ Lie algebroid $= TM \oplus (M \times \mathfrak{g}) \xrightarrow{id+0} TM$.

- Frame groupoid of $E \to M$: $\boxed{\mathrm{Iso}(E) = \bigcup_{x,y} \mathrm{Iso}(E_y, E_x)}$ for $P = F(E)$
  $\Rightarrow$ Lie algebroid $= \mathrm{Der}(E) \to TM$

  If the structure gp of E reduces to $G \subset GL_r(\mathbb{R})$ and $P \subset F(E)$ then $\boxed{\mathcal{G}(P) \hookrightarrow \mathrm{Iso}(E)}$

Direct connections on Lie groupoids

- **Local map** $\boxed{\psi : \mathcal{G} \twoheadrightarrow \mathcal{G}'}$ between two groupoids: map $\psi : \mathcal{U} \subset \mathcal{G} \to \mathcal{G}'$ defined on an open neighborhood $\mathcal{U}$ of the units $u(M) \subset \mathcal{G}$, which commutes with s, t and u.

  Local morphism: local map which also preserves composition (hence inversion).

- [Teleman 2004 in the linear case, Kock 2007 similar, ABFP general]

  Direct connection on $\mathcal{G} \rightrightarrows M$: local right inverse of the anchor which preserves units, i.e. $\boxed{\Gamma : \mathrm{Pair}(M) \twoheadrightarrow \mathcal{G}}$ defined on an open n. $\mathcal{U}_\Delta$ of the diagonal $\Delta \subset \mathrm{Pair}(M)$ s.t.

  $$\boxed{\Gamma(y,x) \in \mathcal{G}_x^y} \text{ for all } (y,x) \in \mathcal{U}_\Delta \quad \text{and} \quad \boxed{\Gamma(x,x) = 1_x \in \mathcal{G}_x^x} \text{ for all } x \in M.$$

- A Lie groupoid with a direct connection is a gauge groupoid.

- If $\mathcal{G} \times_M E \to E$ is a linear action, then a direct connection Γ on $\mathcal{G}$ induces a transport on fibres $E_x \to E_y$ which is not necessarily a parallel displacement!

Relationship to usual connections

Assume M is a manifold with affine connection ∇^M and local geodesics.

- Parallel displacement τ on $P \to M$ along small geodesics (equivalent to a principal connection ω on P hence to gauge fields A) defines a direct connection Γ^τ on $\mathcal{G}(P) \rightrightarrows M$ by

$$\Gamma^\tau(y,x) = \big[\tau(y,x)(p), p\big] \quad \text{for any choice of } p \in P_x$$

Same for $E \to M$ and $\mathrm{Iso}(E)$ [Teleman 2004].

- Viceversa, a direct connection Γ on $\mathcal{G}(P) \rightrightarrows M$ induces an infinitesimal connection on the Lie algebroid $A(P) \to TM$ by

$$\nabla^\Gamma(\dot\gamma(0)) = D\Gamma_{|M}(\dot\gamma(0)) = \frac{d}{dt}_{|t=0}\Gamma(\gamma(t),x),$$

hence a principal connection ω^Γ on P.

- Apply maps $\omega \mapsto \tau \mapsto \Gamma^\tau \mapsto \omega^{\Gamma^\tau}$, then $\boxed{\omega^{\Gamma^\tau} = \omega}$ on P.
- Viceversa, if apply maps $\Gamma \mapsto \omega^\Gamma \mapsto \tau^\Gamma \mapsto \Gamma^{\tau^\Gamma}$, then $\boxed{\Gamma^{\tau^\Gamma} \neq \Gamma}$ on $\mathcal{G}(P)$ in general. There are direct connections on $\mathcal{G}(P)$ which are not parallel displacements!

Examples

- $M = \mathbb{R}$ with flat connection $\nabla^M_{\partial_x}(h(x)\,\partial_x) = h'(x)\,\partial_x$.
- $E = M \times \mathbb{R} \to M$ with global section $e_1(x) = (x, 1) \in E_x$ and linear connection $\nabla^E_{\partial_x} : \Gamma(E) \to \Gamma(E)$ given by $f \in C^\infty(M)$ s.t. $\nabla^E_{\partial_x} e_1 = f\, e_1$.
- The induced **parallel transport** along a geodesic from x to y is the isomorphism $\tau(y,x) : E_x \to E_y$ defined by $\tau(y,x)\,\xi_0\, e_1(x) = \xi(y)\, e_1(y)$ solution of the ODE

  $$\nabla^E_{\partial_x}\big(\xi(x)\, e_1(x)\big) = \big(\xi'(x) + \xi(x) f(x)\big) e_1(x) = 0$$

  with initial value $\xi(x)\, e_1(x) = \xi_0\, e_1(x)$. Set $F(x) = \int -f(x)dx$.
  Then the direct connection on $\mathrm{Iso}(E)$ is

  $$\boxed{\tau(y,x) : E_x \to E_y, \quad e_1(x) \mapsto \tau(y,x)\, e_1(x) = e^{F(y)-F(x)}\, e_1(y)}$$

  The associated direct connection is flat. For instance:
  $\nabla^E_{\partial_x} e_1(x) = -2x\, e_1(x) \quad \Rightarrow \quad \tau(y,x) e_1(x) = e^{y-x+y^2-x^2} e_1(y)$,
  $\nabla^E_{\partial_x} e_1(x) = -3x^2\, e_1(x) \quad \Rightarrow \quad \tau(y,x) e_1(x) = e^{y-x+y^3-x^3} e_1(y)$.

- Instead, the following direct connections **are not parallel transports**:
  $\Gamma(y,x) = e^{y-x+(y-x)^2}$ non natural $\big(\Gamma(x,y)\Gamma(y,x) = e^{2(y-x)^2} \neq 1_x\big)$,
  $\Gamma(y,x) = e^{y-x+(y-x)^3}$ natural but non-flat.

Conclusion: there is a surjective functor

Gauge groupoids with direct connections	$\longrightarrow$	Principal bundles with connections

which admits an inverse, but it is not an equivalence of categories.

Further results:

- Jet prolongation of direct connections to jet groupoids $J^n\mathcal{G} \rightrightarrows M$ (existence, examples).
- Applications to geometric regularity structures for solving stochastic PDEs (cf. M. Hairer and coll.).

Next:

- Look for more examples of direct connections which are not parallel displacements.
- Adapt to α-Hölder sections of bundles i.e. define distributional direct connections and compare to usual propagators.
- Study the whole geometry of **groupoids with direct connections** and compare with **usual gauge theory**.

Thank you for the attention!