Title: A gentle introduction to (modular tensor) categories

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Collection: Women at the Intersection of Mathematics and Theoretical Physics

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Abstract: In this talk we will introduce categories, a notion that packages mathematical objects of any kind and provides an abstract language to study them. We will build up our way towards so-called modular tensor categories, which roughly speaking are categories with a tensor product, duals, and quite a bit of extra categorical structure. They arise in (rational) conformal field theory and its study poses many interesting questions on their classification, internal structure and generalizations. I will give an overview of these questions and some current lines of research in this topic.


"WOMEN AT THE INTERSECTION OF
MATHEMATICS + THEORETICAL
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"A gentle introduction to (modular tensor) categories"

(0) Motivation

[Eilenberg-MacLane, '45]: categories

... Why considering them?

(Source: E. Riehl's JHS website)

... Emily Riehl's "Category theory in context", Preface:

"A category is a context for the study of a particular class of mathematical objects (...) but (these classifications) are not the main point. Rather, the action of packaging each variety of objects into a category shifts one's perspective from the particularities of each mathematical sub-discipline to potential commonalities between them"

Which is not such a bad idea in mathematical physics...

E.g. homological mirror symmetry: compare D-branes in string theory via categories,

Coherent sheaves on a CY $\overset{\text{HMS}}{\longleftrightarrow}$ Lagrangian submanifolds

(B-model) (A-model)

① Introducing (modular tensor) categories

Defn: a **category** $\mathcal{C}$ consists of

- a collection of objects $X, Y, Z, \dots$ (Notation: $\mathrm{Ob}(\mathcal{C})$)
- a collection of morphisms $f, g, h \dots$ (Notation: $\mathrm{Mor}(\mathcal{C})$, if explicit $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$)

so that:

- every morphism has specified domain and codomain objects, e.g. $f : X \to Y$,
- every object has a designated identity morphisms, i.e. $\mathrm{id}_X : X \to X \quad \forall X \in \mathrm{Ob}(\mathcal{C})$,
- for any pair of morphisms $f : X \to Y$, $g : Y \to Z$, there exists a composition $g \circ f : X \to Z$, $\forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$,

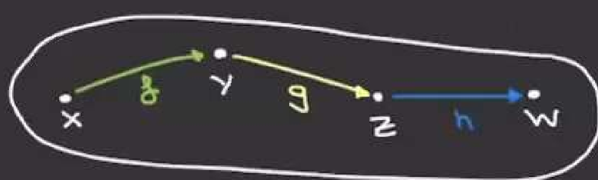
satisfying the following axioms:

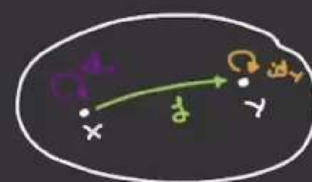
CA1) For any $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ with $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$,

$$h \circ (g \circ f) = h \circ g \circ f = (h \circ g) \circ f ,$$

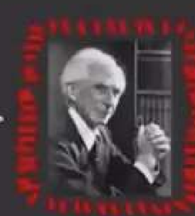
CA2) For any $f : X \to Y$ with $X, Y \in \mathrm{Ob}(\mathcal{C})$,

$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X .$$





Rmk: "collection" here can be class, or set – but careful!



(Source: Wiki)

Some examples:

Algebraic ones

- $\mathcal{C} = \mathrm{Set}$
  objects: sets
  morphisms: functions
- $\mathcal{C} = \mathrm{Vec}_k$
  objects: k-vector spaces
  morphisms: linear transformations
- $\mathcal{C} = \mathrm{Grp}$ (Ab)
  objects: (abelian) groups
  morphisms: group homomorphisms

  ⋮

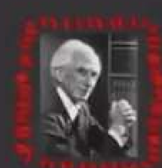
Not so algebraic ones

- $\mathcal{C} = \mathrm{Man}$
  objects: smooth manifolds
  morphisms: smooth maps
- $\mathcal{C} = \mathrm{Top}$
  objects: topological spaces
  morphisms: continuous functions
- $\mathcal{C} = \mathrm{Met}$
  objects: metric spaces
  morphisms: short maps

  ⋮

But also, things like "categories of categories" are possible $\Rightarrow$ higher category theory.

(Careful again!)



One possible way to go. But for today, focus on the kinds of "internal" structures we can have in a category.

Take the example of (finite dimensional) k-vector spaces, Vec_k

Behaviour to mimic	Categorical notion
Tensor product of vector spaces over k ($\otimes_k$)	Monoidal category
Dual vector space ($^*V, V^*$)	Rigid category
$\forall X, Y \in Ob(\mathcal{C})$, $Hom_{\mathcal{C}}(X,Y)$ is a k-vector space	k-linear category
Of finite dimension	Locally finite category
There is a notion of kernel and cokernel (and short exact sequences)	Abelian category
	+ k cannot be decomposed in smaller vector spaces, i.e. k "simple object" + finitely many simples $/\sim$ + $\forall X \in Ob(\mathcal{C})$, $X = \bigoplus \{$simple obj's in $\mathcal{C}\}$

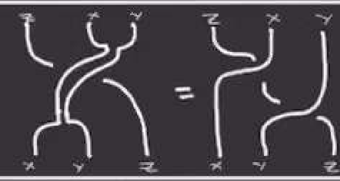
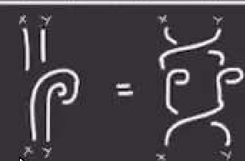
Roughly speaking!

$\Rightarrow$ FUSION CATEGORY

Close enough, but we want a particular case of fusion categories:

modular tensor categories

For this, we need to include extra structure:

	Pictionary	What about it?
$\exists\, c_{X,Y}: X\otimes Y \longrightarrow Y\otimes X$ $\forall X,Y \in Ob(\mathcal{C})$ "braiding"		Needs to satisfy: 
$\exists\, \theta_X: X \longrightarrow X$ $\forall X \in Ob(\mathcal{C})$ "twist"		Needs to satisfy: 
$\exists$ "S-matrix" ("$c_{X,Y}\circ c_{Y,X}$") well-behaved		A test of how "nice" the braiding is!

And that's it! :)

Example: Fibonacci category, notation: Fib

- Only 2 objects, $\mathbb{1}$ and X.
- Multiplication table:

	$\mathbb{1}$	X
$\mathbb{1}$	$\mathbb{1}$	X
X	X	$\mathbb{1} \oplus X$

- Why Fibonacci? The categorical data depends on $\phi = \frac{1+\sqrt{5}}{2}$.

A more physics-motivated example of a MTC: Ising category

- Inspired by the critical limit of the Ising model in 2 dimensions. ($\rightsquigarrow$ 2d conformal field theory)
- 3 objects (primary fields): $\mathbb{1}, \sigma, \varepsilon$.
- Multiplication table:

	$\mathbb{1}$	σ	ε
$\mathbb{1}$	$\mathbb{1}$	σ	ε
σ	σ	$\mathbb{1} \oplus \varepsilon$	σ
ε	ε	σ	$\mathbb{1}$

Speaking of examples ...

② MTCs and conformal field theory (in 2d)

The coolest example ever of a modular tensor category:

2-dimensional conformal field theory

Symmetries?

Chiral algebras / algebras of chiral symmetry (see Ward identities!)

Mathematically formalized as

Vertex operator algebras [Borcherds '86, Frenkel-Meurman-Lepowsky '88]

Fields Medal '98 (Source: wiki)

Modular tensor categories

> Thm [Huang]: the category of representations of a (nice enough) vertex operator algebra is a modular tensor category.

Giving an alternative road to study 2d CFTs using categorical methods...

- which is where yours faithfully kicks into this story! :)

③ What to do with MTCs?

- Classification problems:
  - [Bruillard, Galindo, Plavnik, Rowell, and MANY more]: up to 6 objects almost complete.

(Hamburg, 2013)

- Evidence of MANY more [Gannon-Höhn-Yamauchi, ...]

The online database of Vertex Operator Algebras and Modular Categories (Version 0.5)

Table of content

Some links

↳ Study of invariants associated [..., Delaney, Wang and many more]

- Building bridges between CFT and other physical theories. ([ERC]: "Landau-Ginzburg/CFT correspondence", AMA?, ...)

- Look for interesting internal structures, like e.g. algebras (⇔ bdry conditions).

[S. Hannah's PhD project] Here soon!

WINART2 team, looking for them in some examples of fusion categories

(Leeds, 2019)

- And beyond! 3d TFTs

THANK YOU VERY MUCH
FOR YOUR ATTENTION!

Questions? :)

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