Title: On generalized hyperpolygons

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Collection: Women at the Intersection of Mathematics and Theoretical Physics

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Abstract: In this talk we will introduce generalized hyperpolygons, which arise as Nakajima-type representations of a comet-shaped quiver, following recent work joint with Steven Rayan. After showing how to identify these representations with pairs of polygons, we shall associate to the data an explicit meromorphic Higgs bundle on a genus-g Riemann surface, where g is the number of loops in the comet. We shall see that, under certain assumptions on flag types, the moduli space of generalized hyperpolygons admits the structure of a completely integrable Hamiltonian system.

ON GENERALIZED HYPERPOLYGONS

AND THE APPEARENCE OF MEROMORPHIC
HIGGS BUNDLES ON CURVES, BRANES
AND INTEGRABLE SYSTEMS.

JOINT WORK WITH STEVEN RAYAN

- "WHAT IS... A HYPERPOLIGON"
  NOTICES OF THE AMS, VOL 68 N.1. JANUARY 2021.
- "MODULI SPACE OF GENERALIZED HYPERPOLYGONS"
  THE QUATERLY JOURNAL OF MATHEMATICS
  VOL 00 (2020) 1-25. DECEMBER 2020.

FOR HIGGS BUNDLES - SEE [NOTICES OF THE AMS VOL 67 N.5 MAY 2020]

THE PLAN

1. (GENERALIZED) HYPERPOLYGONS

2. FROM HYPERPOLYGONS TO HIGGS BUNDLES

3. INTEGRABLE SYSTEMS

1. HYPERPOLYGONS

The moduli space of Hyperpolygons appears as the Nakajima variety of star-shaped quivers.

Since we will be generalizing this set up, let's do a quick recap of how this variety is built.

$$\mathrm{Rep}\left(\begin{array}{ccc} R_u & & R_v \\ \bullet & \longrightarrow & \bullet \\ u & & v \end{array}\right) = \underbrace{\mathrm{Hom}(\mathbb{C}^{R_u}, \mathbb{C}^{R_v})}_{V}$$

1. HYPERPOLYGONS

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$$\mathrm{Rep}\left(\begin{tikzcd} \underset{u}{\overset{R_u}{\bullet}} \arrow[r, "x"] & \underset{v}{\overset{R_v}{\bullet}} \arrow[l, bend left, dashed, "y"] \end{tikzcd}\right) = \underbrace{T^*\mathrm{Hom}(\mathbb{C}^{R_u}, \mathbb{C}^{R_v})}_{T^*V} \cong \begin{array}{c} x \\ \mathrm{Hom}(\mathbb{C}^{R_u}, \mathbb{C}^{R_v}) \\ \oplus \\ \mathrm{Hom}(\mathbb{C}^{R_v}, \mathbb{C}^{R_u}) \\ y \end{array}$$

NAKAJIMA QUIVER

To construct moduli spaces, we consider symplectic reduction, since it will be the path to generalized Hyperpolygons.

To each quiver Q we can associate two functions, which can be built node by node, and which give us moment maps:

$$\mu : T^* \mathrm{Rep}(Q) \longrightarrow \bigoplus_{v} \mathfrak{u}(R_v)^*$$

$$\gamma : T^* \mathrm{Rep}(Q) \longrightarrow \bigoplus_{v} \mathfrak{gl}(R_v)^*$$

These moment maps are associated to the complex adjoint action of

$$G = \Big(\prod_{\substack{v \\ \text{non central}}} U(R_v) \times SU(R) \Big) / \pm 1 \quad \text{on } \mathrm{Rep}(Q)$$

Label of central node

for $U(R_v)$ and $\mathfrak{u}(R_v)$ acting by left and right multiplication on the arrow, and $SU(R)$ by multiplication on one side of arrow to/from center.

To each quiver Θ we can associate two functions, which can be built node by node, and which give us moment maps:

$$\mu : T^* \mathrm{Rep}(\Theta) \longrightarrow \bigoplus_{v} \mathfrak{u}(R_v)^*$$

$$\gamma : T^* \mathrm{Rep}(\Theta) \longrightarrow \bigoplus_{v} \mathfrak{gl}(R_v)^*$$

For
$$\begin{tikzcd} u \arrow[r, "x"'] & v \arrow[l, bend right, dashed, "y"'] \end{tikzcd}$$
we have

$$\mu_u(x, y) = \underbrace{x^* x - y y^*}_{\mathfrak{u}(R_u)^*}$$

$$\gamma_u(x, y) = \underbrace{x y}_{\mathfrak{gl}(R_u)^*}$$

The Nakajima quiver variety of a quiver Q is constructed from Q and its moment maps, and it is a Hyperkähler variety. We first consider

$$T^* \mathrm{Rep}(Q) /\!\!/_{\alpha} G \qquad \text{for} \quad G = \prod_{v} GL(R_v, \mathbb{C})$$

To obtain the Nakajima quiver variety we restrict $T^* \mathrm{Rep}(Q)$ to the intersection of two level sets

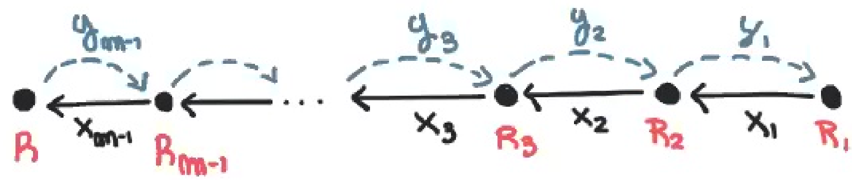
compact!

$$\mu^{-1}(\alpha) \cap \gamma^{-1}(0) / G \qquad G = \prod_{v} U(R_v)/\pm 1$$

$$\alpha \in Z(g^*)$$

From work of Hitchin-Karlhede-Lindström-Roček, the quotient inherits quaternionically commuting complex structures I, J, K and Riemannian metric g.

Our interest is on (generalized) Hyperpolygons. To build them, we start with an A-Type quiver



The corresponding Nakajima quiver variety is

$$T^*\mathcal{F}_{\underline{R}}$$

Partial flag variety

$$\underline{R}\subseteq[R]=\{1,2,\dots,R\}$$

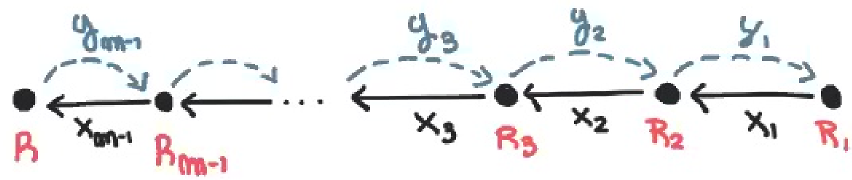
$$\#(\underline{R})=m$$

$$G=\prod_{\underline{R}/\{R\}}\mathcal{U}(R_i)/\pm 1$$

$$Z(g^*)=\bigoplus_{i=1}^{m-1}u(R_i)^*$$

$$(\alpha,0,0\dots0)\quad \alpha\in\mathbb{R}$$

Our interest is on (generalized) Hyperpolygons. To build them, we start with an A-Type quiver



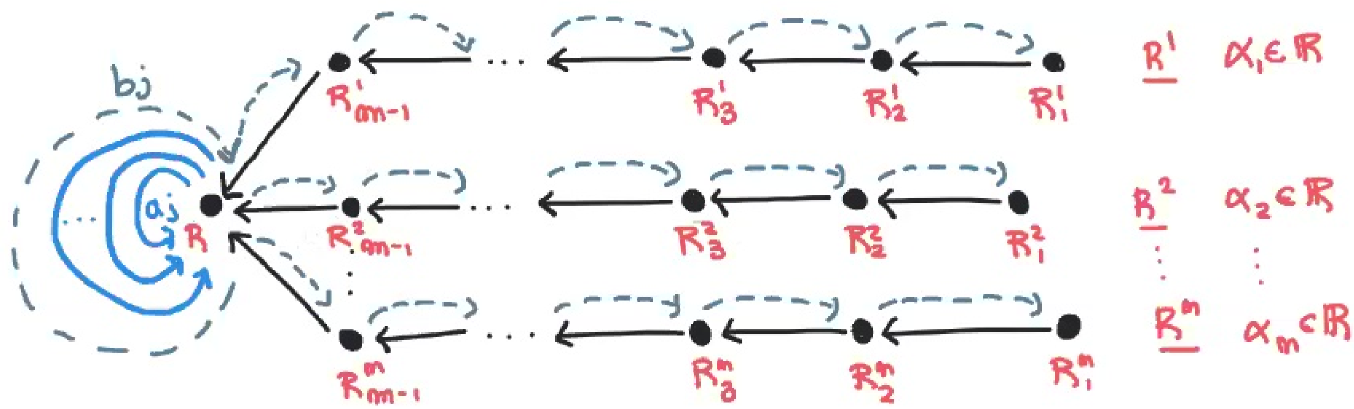
The corresponding Nakajima quiver variety is

$T^*\mathcal{F}_{\underline{R}}$ and it has a $\mathcal{U}(R)$ action w/ complex moment map

$$\gamma_m(x_{m-1}, y_{m-1}) = x_{m-1}y_{m-1}$$

and we can use this to understand star-shaped quivers.

Our interest is on (generalized) Hyperpolygons, so consider now star-shaped quivers: interlacing n-many A-star quivers:

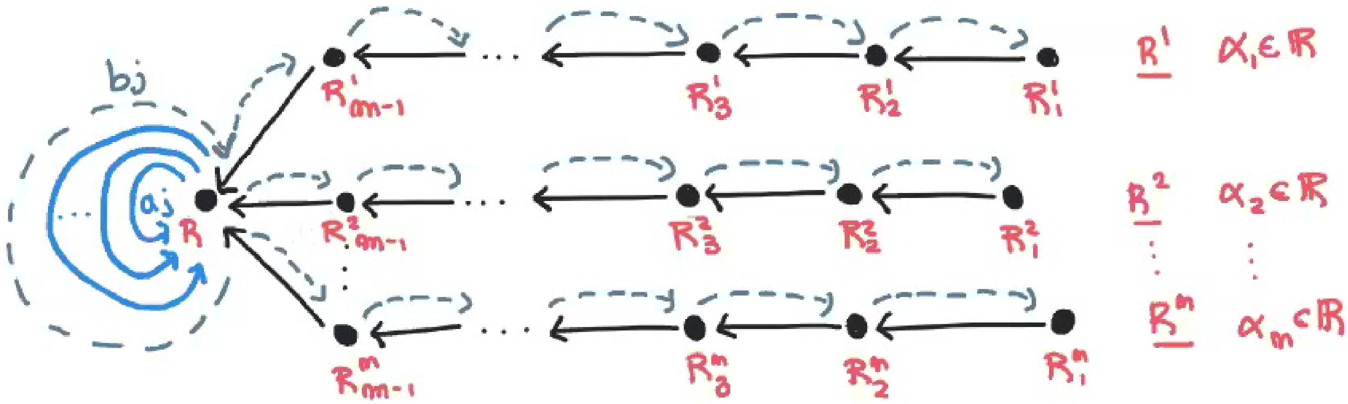


To build the space of generalized Hyperpolygons, we need first to add g loops to obtain a comet-shaped quiver.

We will consider its Nakajima quiver with the dotted arrows

At the central node R we restrict to the SU(R) action.

Our interest is on (generalized) Hyperpolygons, so consider now star-shaped quivers: interlocking n-many A-star quivers:

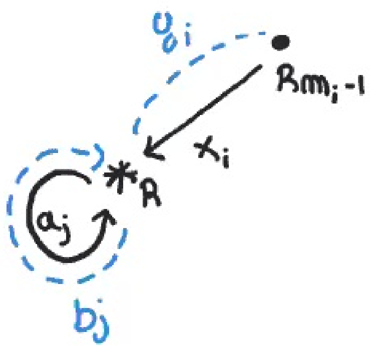


The Nakajima quiver variety of the above comet Q_n is

$$T^*\mathcal{F}_{\underline{R}^1} \times \cdots \times T^*\mathcal{F}_{\underline{R}^n} \times T^*\mathfrak{sl}(R,\mathbb{C})^g \,/\!/\!/_{0}\, SU(R)$$

$$\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$$

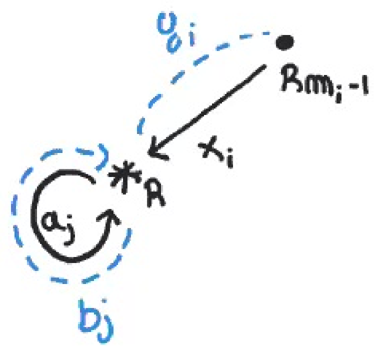
To build the appropriate moment maps, it is useful to consider how the loops appear. At the central node * we can express the moment maps as follows:



$$\mu_*(x,y,a,b) = \sum_{i=1}^{m} \left(x_i x_i^* - y_i^* y_i\right)_0 + \sum_{j=1}^{g} [a_j, a_j^*] + [b_j, b_j^*] = 0$$

$$\gamma_*(x,y,a,b) = \sum_{i=1}^{m} \left(x_i y_i\right)_0 + \sum_{j=1}^{g} [a_j, b_j] = 0$$

To build the appropriate moment maps, it is useful to consider how the loops appear. At the central node $*$ we can express the moment maps as follows:



$$\mu_*(x, y, a, b) = \sum_{i=1}^{m} (x_i x_i^* - y_i^* y_i)_0 + \sum_{j=1}^{g} [a_j, a_j^*] + [b_j, b_j^*] = 0 = \gamma_*(x, y, a, b) = \sum_{i=1}^{m} (x_i y_i)_0 + \sum_{j=1}^{g} [a_j, b_j]$$

the quotient space

$$\mathcal{X}^{g}_{\underline{R}^1, \underline{R}^2 \dots \underline{R}^n}(\alpha) = \frac{\mu_*^{-1}(0) \cap \gamma_*^{-1}(0)}{SU(R)}$$

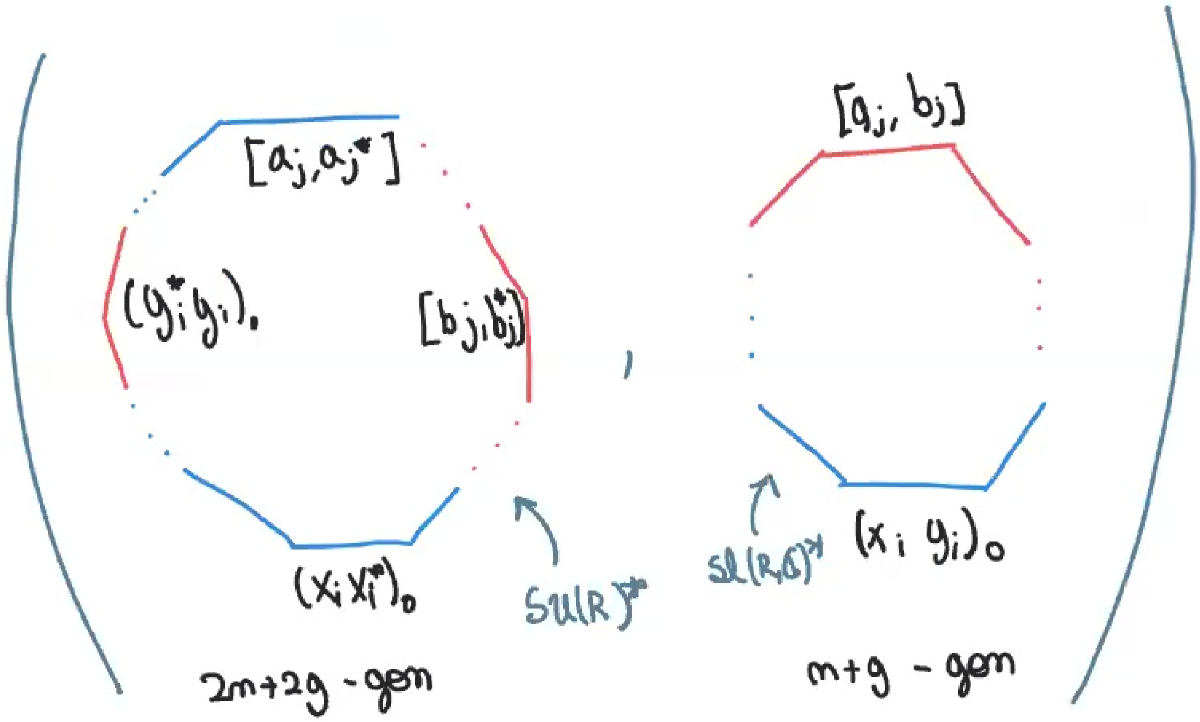
Eg. When all Flags are complete

$mn(R-1) + 2(g-1)(R^2-1)$

is a hyperkähler variety of complex dimension

$$2\left(\sum_{i=1}^{m} \dim \mathcal{F}_{\underline{R}^i} + (g-1)(R^2-1)\right)$$

We can think of the objects in $\mathcal{X}^g_{\underline{R}^1,\dots,\underline{R}^m}(\alpha)$ in the following way:



pairs of polygons with sides determined by the quantities in the moment maps.

The space $\mathcal{X}^{g}_{\underline{R}^1,\dots,\underline{R}^n}(\alpha)$ is the moduli space of (generalized) hyperpolygons of length α.

Hyperpolygons where first studied by Konno as a hyperkähler extension of the usual polygon space, and later by Harada-Proudfoot as a Nakajima quiver variety. They are closely related to Higgs bundles, as first seen by Godinho-Mandini for rank 2 and later extended to any rank by Fisher-Rayan.

With Steven Rayan we extended this to the setting of comet-shaped quivers, leading to the space of generalized hyperpolygons and we show they are also closely related to Higgs bundles and integrable systems.

2. GENERALIZED HYPERPOLYGONS AND HIGGS BUNDLES

Nakajima quiver varieties can be seen as finite-dim analogs of Hitchin systems. Recall we have two moment map equations:

$$\mu_*(x, y, a, b) = \sum_{i=1}^{m} (x_i x_i^* - y_i^* y_i)_0 + \sum_{j=1}^{g} [a_j, a_j^*] + [b_j, b_j^*] = 0 = \gamma_*(x, y, a, b) = \sum_{i=1}^{m} (x_i y_i)_0 + \sum_{j=1}^{g} [a_j, b_j]$$

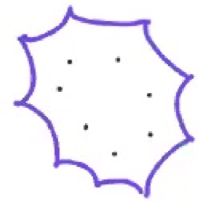
2. Generalized Hyperpolygons and Higgs Bundles

Nakajima quiver varieties can be seen as finite-dim analogs of Hitchin systems. Recall we have two moment map equations:

$$F(A) + \phi \wedge \phi^* = 0 \qquad \left\{ \left(\sum_{i=1}^{m} (x_i x_i^*)_0 + \sum_{j=1}^{g} [a_j, a_j^*] \right) + \left(-\sum_{i=1}^{m} (y_i^* y_i)_0 + [b_j, b_j^*] \right) = 0 \right.$$

$$\partial_A \phi \text{ holomorphic} \qquad \left\{ \sum_{i=1}^{m} (x_i y_i)_0 = -\sum_{j=1}^{g} [a_j, b_j] \right.$$

We can associate a corresponding Higgs Field by considering a punctured surface:

Let $D=\sum_i z_i$ in a 4g-gon  $\subset \mathcal{H}$ ($\mathbb{C}$ if $g=0,1$)

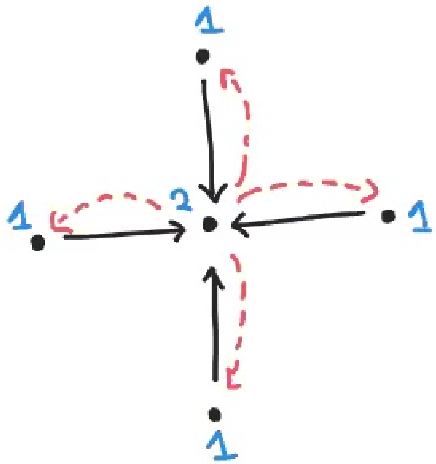
Then, the Higgs Field can be defined as

$$\phi(z)=\sum_{i=1}^{m}\frac{x_i y_i}{z-g_z(z_i)}\,dz$$

For the trivial rank R bundle on

$$X=\mathcal{H}/\Gamma \qquad (\mathbb{C}/\Lambda \text{ for } g=1, \quad \mathbb{P}^1 \ g=0)$$

Let's take a look at an example in the case of hyperpolygons (not generalized ⇒ no higher genus)



affine D_4 Dynkin diagram

each flag is of the form

$\underline{R}^i = (1,2) = [2]$

We consider $X_{[2],[2],[2],[2]}(\underline{\alpha})$ ← the Nakajima quiver variety for the affine Dynkin diagram D_4

- this is a K_3 surface with complete ALE metric
- Embeds into the Hitchin system on $\mathbb{P}^1 / \{z_1, z_2, z_3, z_4\}$. The associated Higgs bundles are parabolic of Rank 2 on $\mathbb{P}^1$ with 4 tame singularities.
- the embedding is not one of hyperkähler varieties: the difference between this hyperpolygon space and the parab. Higgs space is the Hitchin section.

3. INTEGRABLE SYSTEMS

Given the correspondence between Higgs bundles and (generalized) Hyperpolygons, we look into the appearance of integrable systems.

(back in '94 Nakajima mentioned that one would expect Nakajima quiver varieties to be completely integrable Hamiltonian systems)

Fisher-Rayan showed this is the case for

$$\mathcal{X}^0_{[1,R],\dots[1,R]}(\underline{\alpha}) \quad R \leq 3 \text{ and arbitrary } n$$

by embeding the space into certain space of tame parabolic Higgs bundles on the punctured sphere.

We can now extend this to generalized Hyperpolygons whose comets have complete or minimal arms.

Recall that to construct $\mathcal{X}^{g}_{\underline{R}^1,\dots,\underline{R}^n}(\alpha)$ we considered a quotient and a restriction via two moment maps.

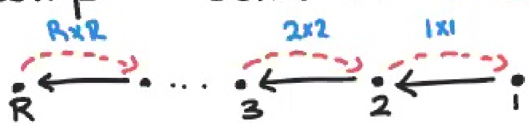
$$\underbrace{T^*F_{\underline{R}^1}\times T^*F_{\underline{R}^2}\times\dots\times T^*F_{\underline{R}^n}}_{\text{GELFAND-TSETLIN-TYPE}}\times\underbrace{T^*\mathfrak{sl}(R,\mathbb{C})^g}_{\text{LIE-POISSON TYPE}} \longrightarrow \text{all these have are hyperkähler varieties}$$

$$\downarrow$$

$$\mathcal{X}^{g}_{\underline{R}^1,\underline{R}^2,\dots,\underline{R}^n}(\alpha) \longrightarrow \text{inherits its symplectic structure.}$$

Do these integrable systems descend with the symplectic structure?

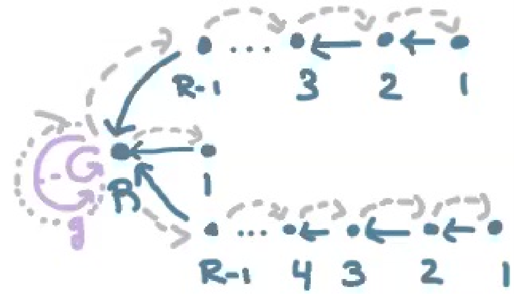
To study this, first assume C complete arms in the comet and $n-C$ minimal arms (this is, with lables $(1,R)$).
The complete arms each $K\times K$ block contributes K invariants.



Hence, we have the following invariants: (for c the number of complete arms in the comet)

- $c\left(1+2+\ldots+(R-1)\right) = \frac{c\,(R-1)R}{2}$

natural invariants from c complete flag which come from the Gelfand-Tsetlin integrable systems



- $(m-c)(R-1)$ from the minimal flags; and
- $g\,(R^2-1)$ From the loops $b_j \in sl(R,\mathbb{C})$

For each R, m, g, c, the Hyperkähler reduction by $SU(R)$ fixes $N(R,m,g,c)$ of the invariants, and we show that

$$\frac{c\,(R-1)R}{2} + (m-c)(R-1) + g\,(R^2-1) - N(R,m,g,c) = \frac{1}{2}\dim \mathcal{X}^{g}_{\underline{R}^1,\ldots,\underline{R}^2}(\alpha)$$

Theorem (w/ Rayan): the space $\mathcal{X}^{g}_{\underline{R}^1,\ldots,\underline{R}^m}(\alpha)$ is a completely integrable Hamiltonian system of Gelfand-Tsetlin Type.

In particular, this shows the existence of

sub-integrable systems

in meromorphic Hitchin systems that do not see the complex geometry of the algebraic curve.

Moreover, through the Hyperkähler structure, we may ask about the appearance of Lagrangians or complex holomorphic subspaces of those sub-integrable systems.

THANKS!