

Title: Correlators in integrable models with Separation of Variables

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Series: Quantum Fields and Strings

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Abstract: I will review recent progress in application of separation of variables method.

In particular I will review the construction for the integrable spin chains with $gl(N)$ symmetry.

By finding, for the first time, the matrix elements of the SoV measure explicitly I will show how to compute various correlation functions and wave function overlaps in a simple determinant form.

General philosophy of application of these methods to the problems related to AdS/CFT, N=4 SYM etc. will be discussed too.



SoV

By N. Gromov

Based on: 2011.08229 1910.13442
1907.03788 1610.08032

With: A. Cavaglia F. Levkovich-Maslyuk
P. Ryan G. Sizov D. Volin



PI 2 Feb
2021



Problem: $\langle \psi | \hat{\sigma} | \Phi \rangle = ?$

- In integrable theories

Important examples:

- Spin chains

⋮

- $N=4$ SYM

related



When a spin-chain is solvable by
Bethe Ansatz \rightarrow can use Bethe
roots



$$\Psi = \sum_{\text{perm}} \dots e^{i u_1 P_1 + i u_2 P_2}$$

\nearrow
coord BA

in terms
 \swarrow of P_i
or u_i

$$\langle \Psi | O | \Phi \rangle = \sum_{\text{perm}} \sum_{\text{perm}} \dots$$

Many interesting cases are not
solvable by BA

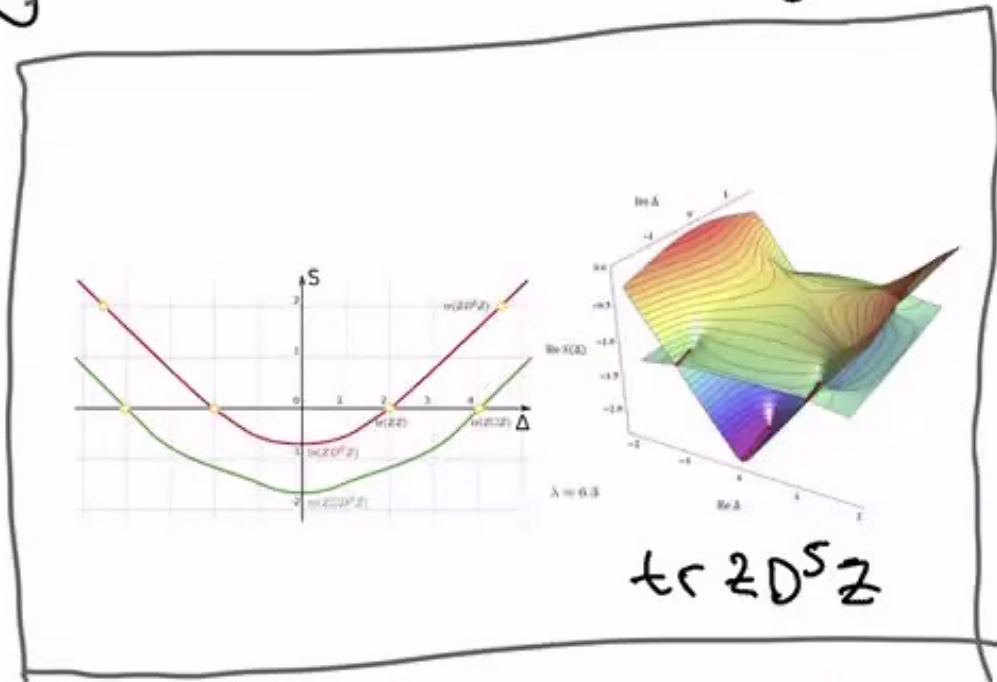
including $N=4$ SYM* ↓ next slide

Universal Language for int. models
Q-functions

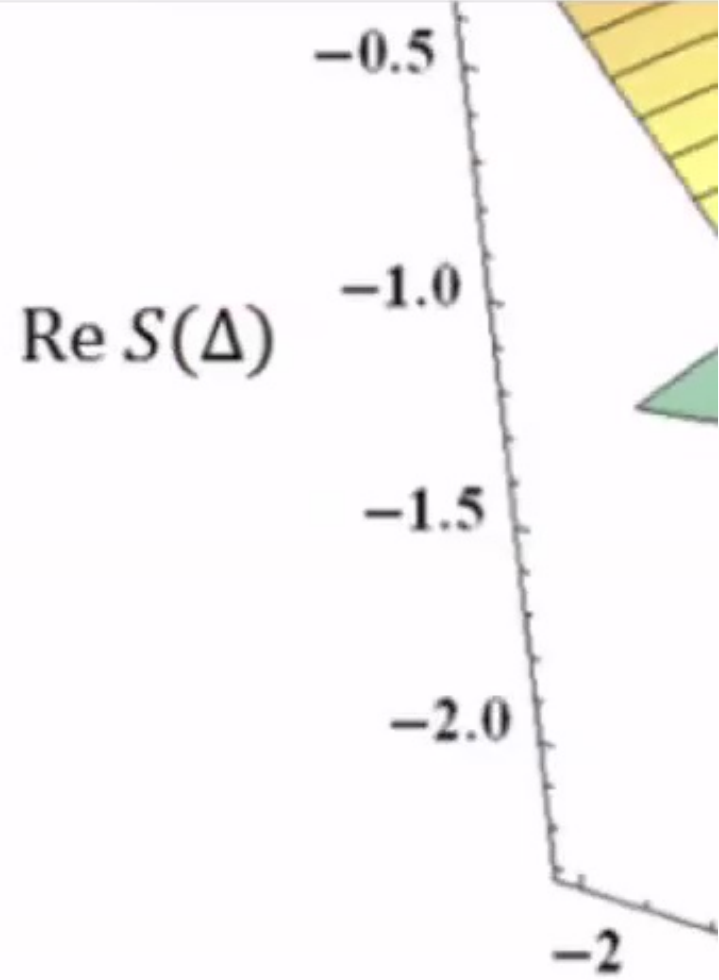
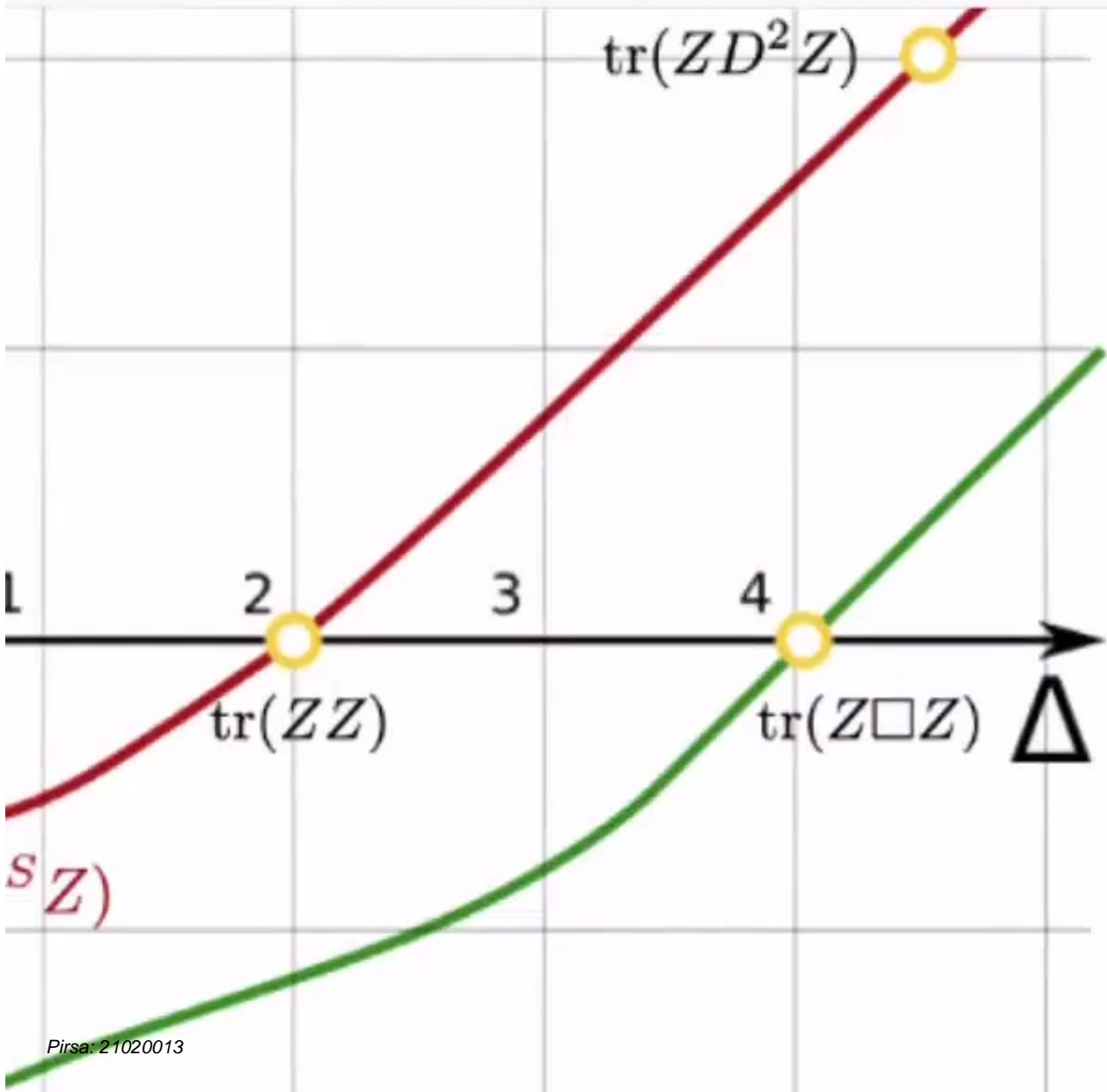
\exists BA \Rightarrow Q's are Baxter poly.
 $Q = \prod (u - u_i)$

\nexists BA \Rightarrow Q's functions of u
can be computed (e.g. $N=4$ SYM)

In $N=4$ SYM at weak coupling
 one can approximate Q 's by z -polynomials
 But only up to the "wrapping order" in λ



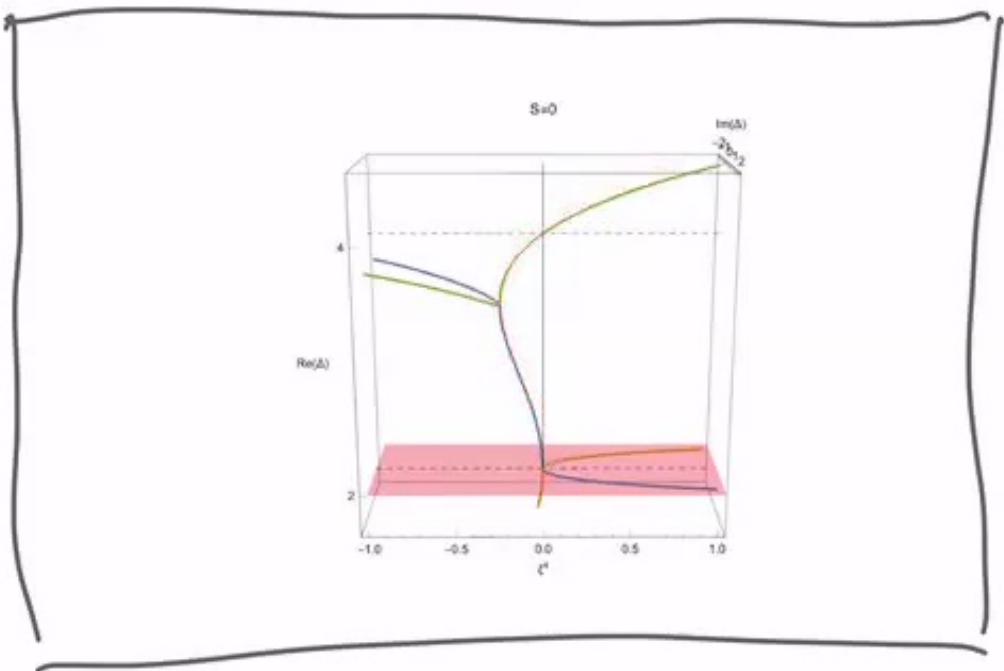
Riemann surface of "twist-2" ops



10.6.2



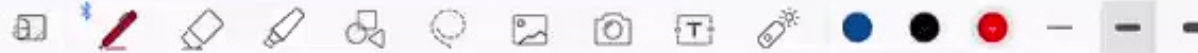
example from Fishnet



no magnons
↘

many magnons
↙

$\text{tr}(\varphi_1^2)$ and $\text{tr}(\varphi_1^2 \varphi_2 \varphi_2^+)$
non-perturbative "mixing"



Goal: develop tools for computing
 $\langle \psi | \hat{O} | \Phi \rangle$ in terms of Q-functions

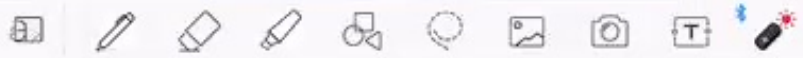
- SoV — „Sep. of Variables“

r, φ, θ

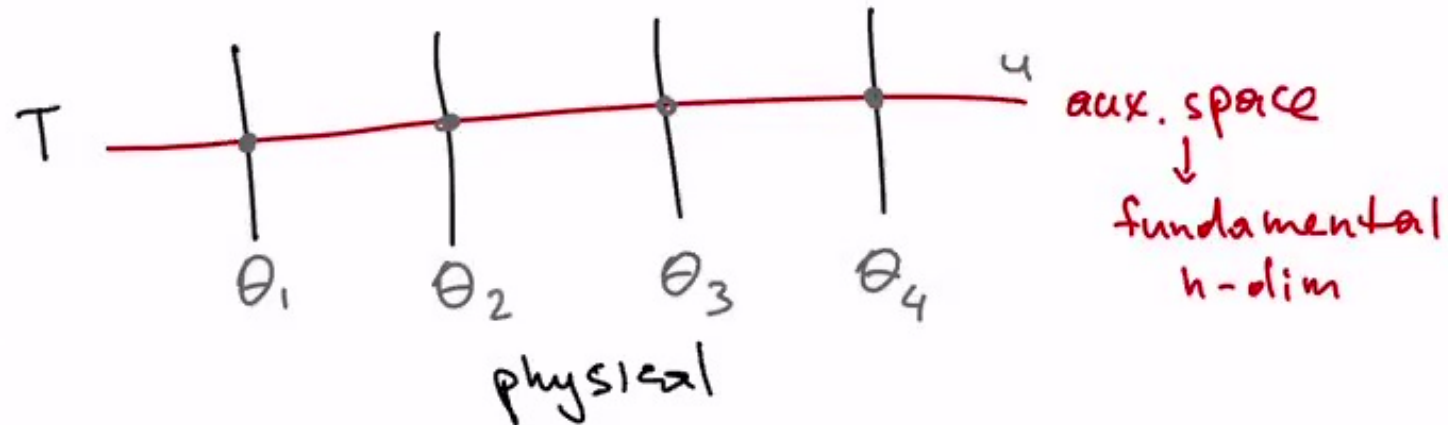
$$\psi(r, \varphi, \theta) = \underline{\underline{R(r)}} \Phi(\varphi) \Theta(\theta)$$

$r^2 \cos \theta$

- how to build those nice coords
- building blocks
- me



Set-up: $sl(n)$ spin-chain



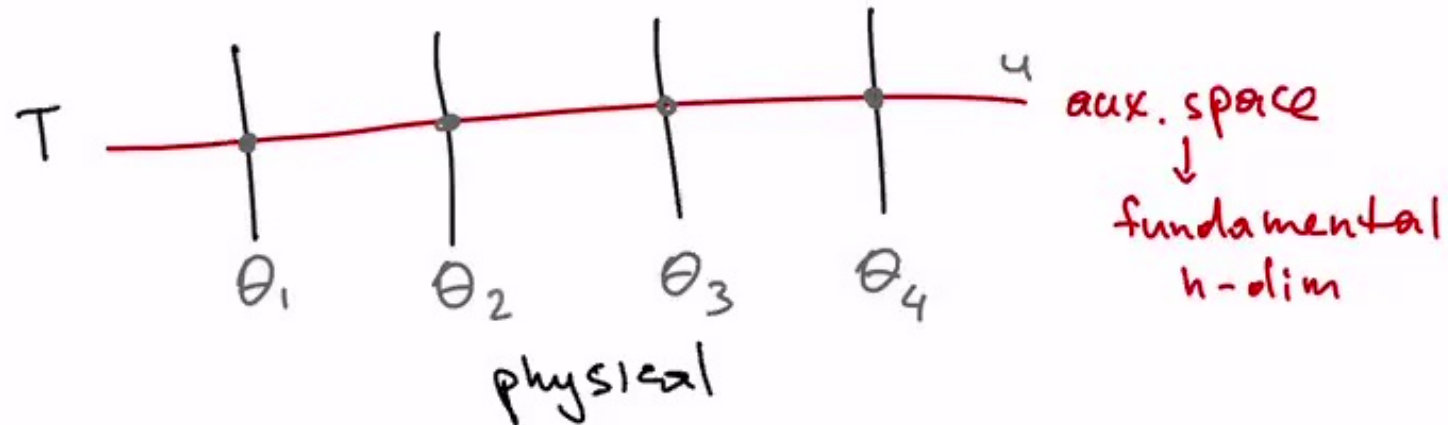
$$a \text{---} \bullet \text{---} b = u \sigma_{ab} + i E_{ba}$$

← sl_n generator in some rep

sk... sk

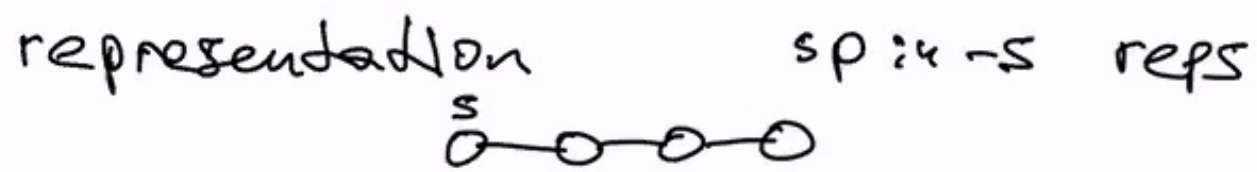


Set-up: $sl(n)$ spin-chain



$a \text{---} \bullet \text{---} b = u \sum_{ab} + i E_{ba}$

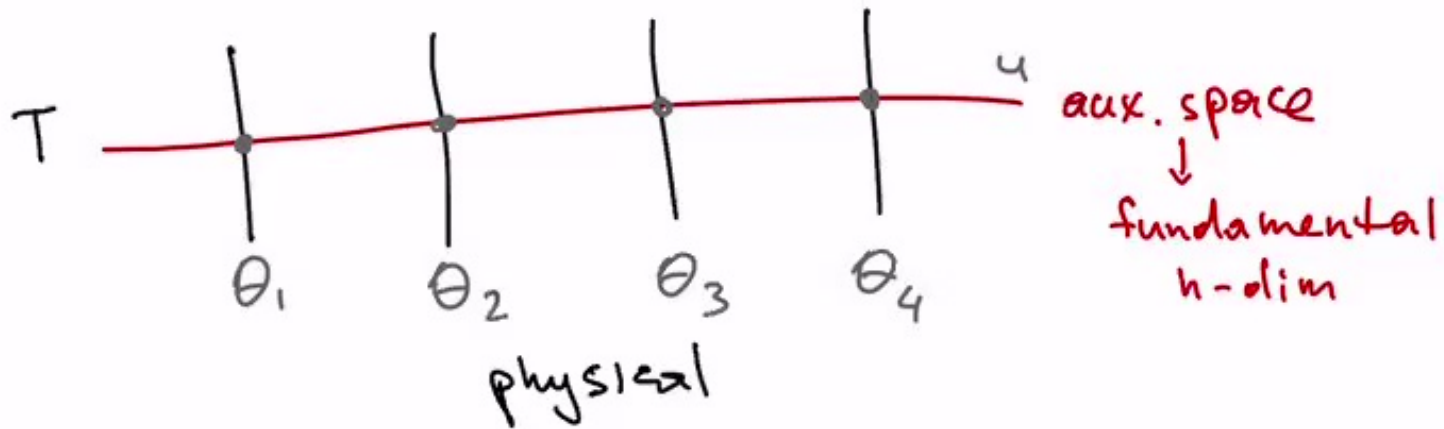
← sl_n generator in some rep



sk... sk...

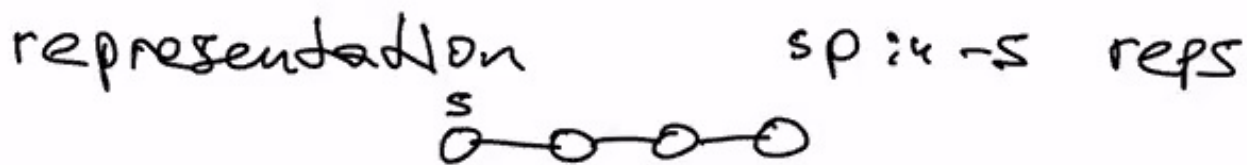


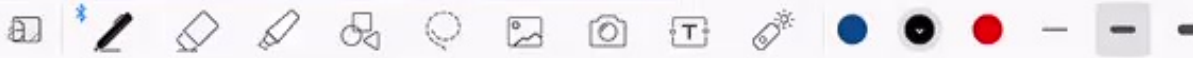
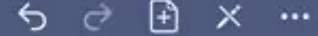
Set-up: $sl(n)$ spin-chain



$a \text{---} | \text{---} b = u \delta_{ab} + i E_{ba}$

↖ sl_n generator in some rep





Sklyanin SOV

$sl(2)$
 $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

poly deg - n • construct P_α, X_β ?
 ↪ simple comb. of A, B, C, D

• classically $p \leftarrow$ quasimom
 i.e. $e^p \leftarrow$ eigenvalue
 of T

if $B = 0 \Rightarrow A$ and D are $e^{\pm p}$

$$A(x_\alpha) = e^{P_\alpha} \quad B(x_\alpha) = 0$$

$$[P_\alpha, X_\beta] = -i \delta_{\alpha\beta}$$

Algebraic BAE: $|\Psi\rangle = B(u_1) \dots B(u_M) |\Omega\rangle$
↑
dir up

$$\langle \bar{x} | B(u) = \prod_{\alpha=1}^L (u - x_{\alpha}) \langle \bar{x} |$$

$$\langle \bar{x} | \Psi \rangle = \prod_{\alpha=1}^L \prod_{i=1}^M (u_i - x_{\alpha}) \langle \bar{x} | \Omega \rangle$$

$$= \pm \prod_{\alpha=1}^L Q(x_{\alpha})$$

norm. of $\langle \bar{x} |$

↑
wave function is factorized

Algebraic BAE: $|\Psi\rangle = B(u_1) \dots B(u_M) |\Omega\rangle$
↑
dir up

$$\langle \bar{x} | B(u) = \prod_{\alpha=1}^L (u - x_{\alpha}) \langle \bar{x} |$$

$$\langle \bar{x} | \Psi \rangle = \prod_{\alpha=1}^L \prod_{i=1}^M (u_i - x_{\alpha}) \langle \bar{x} | \Omega \rangle$$

$$= \pm \prod_{\alpha=1}^L Q(x_{\alpha})$$

||
norm. of $\langle \bar{x} |$

↑
wave function is factorized

$$\Psi = R \Theta \Phi$$



Sklyonin's magic recipe

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{23} & T_{34} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

can try the same
 $T_{12} = 0 \quad T_{13} = 0 \quad e^P = T_{11}$
 too constraining

$$T' = K^T T K \quad K = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

find α : $T'_{13} = 0$

$$T'_{12} = 0 \Leftrightarrow B(\alpha) = T_{23}^2 T_{12} - T_{23} T_{22} T_{13}$$

$$B(\alpha) = 0 \quad T'_{11} = e^{P\alpha} + T_{13} T_{11} T_{23} - T_{13} T_{21} T_{13}$$

Algebraic BAE: $|\Psi\rangle = B(u_1) \dots B(u_M) |\Omega\rangle$
↑
dir up

$$\langle \vec{x} | \hat{B}(u) = \prod_{\alpha=1}^L (u - x_{\alpha}) \langle \vec{x} |$$

$$\langle \vec{x} | \Psi \rangle = \prod_{\alpha=1}^L \prod_{i=1}^M (u_i - x_{\alpha}) \langle \vec{x} | \Omega \rangle$$

$$= \pm \prod_{\alpha=1}^L Q(x_{\alpha})$$

norm. of $\langle \vec{x} |$

wave function is factorized

$$K T K^{-1} = T \quad \Psi = R \Theta \Phi$$

$$B^{\text{good}} = B + \alpha A + \alpha C$$



States in SOV

in algebraic nested BAE

$$\psi = \sum_{\text{permutations}} \dots T_{12}(u_i) T_{13}(v_j) |\mathcal{R}\rangle$$

not clear SOV works!

But if you believe in magic then

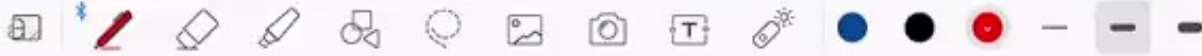
$$\psi = B^0(u_1) \dots B^0(u_n) |\mathcal{R}\rangle$$

now carrying

$$\langle x | B = \prod_{\alpha=1}^L (u - x_{\alpha,a}) \langle x |$$

$a=0,1,2$

$$\langle x | \psi \rangle = \prod_{\alpha} Q(x_{\alpha,a})$$



$$\langle \psi | \psi \rangle = \int Q(x, a)$$

$$E = \frac{1}{\int \psi^2 dx}$$

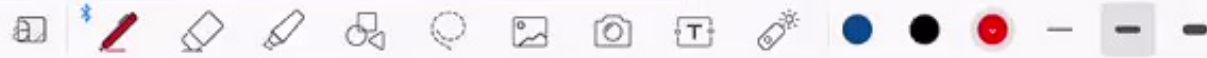
$$\langle x | \psi \rangle = \int Q(x, a)$$

are we happy?

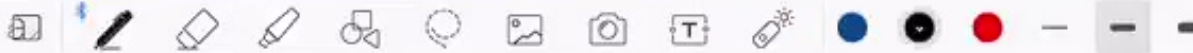
NOT (yet) happy

• $\langle \phi | y \rangle = \int \dots$ ← smile ?

• $\mu^{-1} = \langle x | y \rangle$ ← nice



- $|y\rangle$ is right eigenstate of B ?
 - $\langle x|y\rangle = \delta_{xy}$ ← smile
 - $\langle \phi|y\rangle$ is not a product
- $|y\rangle$ is $\langle x|^\dagger$
 - $\langle \phi|y\rangle$ works = \square
 - $\langle x|y\rangle =$ ugly \circ



Functional SoV

Idea: assuming SoV work, consider

$$\langle \psi^A | \psi^B \rangle = \delta_{AB} \quad \text{eigenstates of } T$$

$$\sum_{x,y} \langle \psi^A | y \rangle_{\mu} \langle x | \psi^B \rangle = \sum_{x,y} \Pi Q^A_{\mu} \Pi Q^B = \delta_{AB}$$

$N^L \times N^L - N^L \text{ ident.}$

so we would have $N^2 - N$ orthogonality relations on Q 's

 $S(2)$ case

$$E_{12} = \partial_x \quad E_{21} = -x^2 \partial_x - 2sx$$

$$E_{11} = -x \partial_x - s \quad E_{22} = x \partial_x + s$$

scalar product: monomials are \perp , but

$$e_n = x^n \sqrt{\frac{\Gamma(n+2s)}{\Gamma(n+1)\Gamma(2s)}} \quad \|e_n\| = 1$$

BAE

$$\frac{Q_\theta(u_k - is)}{Q_\theta(u_k + is)} = - \frac{Q(u_k + i)}{Q(u_k - i)} \quad k=1 \dots M$$

$\prod_{i=1}^M (u - \theta_i)$ $Q(u - u_i)$

$$T(u) = Q_\theta^{[-2s]} \frac{Q^{--}}{Q} + Q_\theta^{[+2s]} \frac{Q^{++}}{Q}$$

$$Q_\theta(u_k + is) \quad Q(u_k - i)$$

$$T(u) = \underbrace{Q_\theta^{-1}}_{[-2s]} \frac{Q^-}{Q} + \underbrace{Q_\theta^{-1}}_{[+2s]} \frac{Q^{++}}{Q}$$

$$\hat{O}^A = Q_\theta^{[2s]} D^2 - T^A + Q_\theta^{[-2s]} D^{-2}$$

$$\hat{O} Q = 0$$

$$((f \ g)) \text{ s.t. } ((f \ \hat{O} g)) = ((g \ \hat{O} f))$$

$$\int_{-\infty}^{\infty} \mu_\alpha(u) f(u) g(u)$$

$$\mu_\alpha = \text{periodic} \cdot \prod_{\beta=1}^L \frac{\Gamma(s - iu + i(\Theta_\beta))}{\Gamma(\dots)}$$

$$\frac{1}{1 - e^{2\pi i(u - \Theta_\alpha - is)}} \quad \alpha = 1, \dots, L$$

$$\left((Q^A (\hat{\sigma}^A - \hat{\sigma}^B) Q^B) \right)_\alpha = 0$$

$$\parallel$$

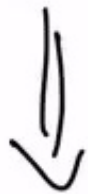
$$\left((Q^A (T^A - T^B) Q^B) \right)_\alpha$$

$$(T^A)$$

e.v. of cons. \int
 \downarrow
 $I^A_\beta u^\beta$

$$\parallel$$

$$\left((Q^A u^\beta Q^B) \right)_\alpha \left(\begin{pmatrix} I^A_\beta \\ I^B_\beta \end{pmatrix} \right) = 0$$



L equations

$$\det \left((Q^A u^\beta Q^B) \right)_\alpha = 0 \quad \text{if } A \neq B$$

α, β
 $L \neq L$

$$x_i = a_i + i c + i n, \quad n_i \geq 0$$



↓

$$\det \left((Q^A \cup Q^B) \right)_{\alpha} = 0 \quad \text{if } A \neq B$$

α, β
 $L \times L$

$$x_{\alpha} = \theta_{\alpha} + i s + i n_{\alpha} \quad n_{\alpha} \geq 0$$

$$M_{n_1 \dots n_L} = \frac{\Delta(\{x^{\alpha}\})}{\Delta(\{\theta_{\alpha}\})} \prod_{\alpha=1}^L \frac{\Gamma_{\alpha, n_{\alpha}}}{\Gamma_{\alpha, 0}}$$

Pochhammer Γ

$$\Gamma_{\alpha, n} = -\frac{1}{2\pi} \prod_{\beta=1}^L (n+1 - i\theta_{\alpha} + i\theta_{\beta})_{2s-1}$$

$$\sum_{n_1 \dots n_L} \prod_{\alpha} Q^A(x_{\alpha}) \prod_{\beta} Q^B(x_{\beta}) \underbrace{M_{s,0} = 0}_{\text{Sata det.}} \begin{matrix} ? \\ \parallel \\ \langle x|y \rangle \end{matrix}$$

α, β
 $L \times L$

$$x_\alpha = \theta_\alpha + i s + i n_\alpha \quad n_\alpha \geq 0$$

$$M_{n_1, \dots, n_L} = \frac{\Delta(\{x^\alpha\})}{\Delta(\{\theta_\alpha\})} \prod_{\alpha=1}^L \frac{\Gamma_{\alpha, n_\alpha}}{\Gamma_{\alpha, 0}}$$

Pochhammer Γ

$$\Gamma_{\alpha, n} = -\frac{1}{2\pi} \prod_{\beta=1}^L (n+1 - i\theta_\alpha + i\theta_\beta)_{2s-1}$$

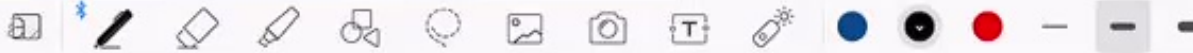
$$\sum_{n_1, \dots, n_L} \underbrace{\prod_{\alpha} Q^A(x_\alpha)}_{\langle x | \psi \rangle} \underbrace{\prod_{\beta} Q^B(x_\beta)}_{\langle \phi | y \rangle} \underbrace{\mu_{s,0} v^0}_{\begin{matrix} \text{solve int.} \\ \downarrow \\ ? \parallel \\ \langle x | y \rangle \end{matrix}}$$

that works!

$$\det_{L \times L} (Q^A Q^B u^\beta) = \langle \phi | \psi \rangle$$

$$|\psi\rangle = \prod_{\beta} (v_\beta) | \phi \rangle$$

$$\langle \phi | \psi \rangle = \sum \mu \prod Q^A \prod Q^B$$



SL(n) - new features

TQ-relation

$$\hat{O}Q_1 = \dots D^3 Q_1 + \tau_2 D Q_1 + \tau_1 D^{-1} Q_1 + \dots D^{-3} Q_1$$

$$\hat{O}^+ = \dots D^{-3} + \tau_2 D^{-1} + \tau_1 D + \dots D^3$$

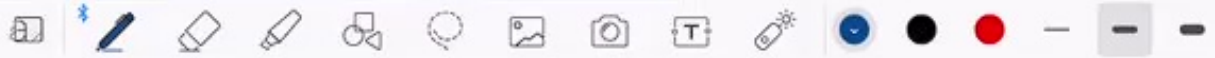
$$\hat{O}^+ Q_1 \neq 0 \quad \hat{O}^+ Q_{12} = 0 \quad \leftarrow \text{aux. roots } Q$$

$$\hat{O}^+ Q_{13} = 0$$

(()) same as sl_2

$$\det \left(\left(Q_1^A \quad u_{1 \dots L}^{\beta-1} D^{\pm} \quad Q_{1,2 \text{ or } 3} \right) \right)_d$$

$2L \times 2L$



4 13 -

(()) same as sl_2

$$\det_{2L \times 2L} \left(Q_1^A \prod_{i=1}^{\beta-1} u_i D^{\pm} Q_{1,2 \text{ or } 3} \right)_{\alpha} = 0$$

sl_n orthogonality

$$\det_{(a,L)(b,\beta)} \left(Q_1^A \prod_{i=1}^{\beta-1} u_i D^{3-2b} Q_{1,a+1}^B \right)_{\alpha} = 0$$

$2L \times 2L \text{ det}$

$\int \rightarrow \Sigma$

$$\Sigma \underbrace{M \prod Q_1^A}_{\text{measure } \langle \chi | \psi \rangle} \prod \left(\begin{array}{cc} Q_{12}^B & Q_{13}^B \\ Q_{12}^B & Q_{13}^B \end{array} \right)$$

4 13 - 13

(()) same as sl_2

$$\det_{2L \times 2L} \left(Q_1^A \prod_{i=1}^{\beta-1} D_i^\pm Q_{1,2 \text{ or } 3} \right) = 0$$

sl_n orthogonality

$$\det_{(a,L)(b,\beta)} \left(Q_1^A \prod_{i=1}^{\beta-1} D_i^{3-2b} Q_{1,a+1}^B \right) = 0$$

$2L \times 2L$ det

$\int \rightarrow \Sigma$

$$\Sigma \underbrace{M \prod Q_1^A}_{\text{measure } \langle \chi | \psi \rangle} \prod \left(\begin{matrix} Q_{12}^B & Q_{13}^B \\ Q_{12}^B & Q_{13}^B \end{matrix} \right)_{\langle \phi | \psi \rangle}$$



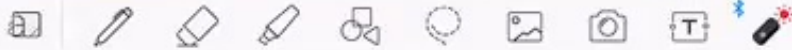
extracting $\langle x|y \rangle^{-1}$

$$M_{y,x} |_{m_{\alpha,a}=n_{\alpha,a}-\sigma_{\alpha,a}+\sigma_{\alpha,a}^0} = (-1)^{|\sigma|} \frac{\Delta_1 \Delta_2}{\Delta_0^2} \prod \frac{r_{\alpha,n_{\alpha,1}} r_{\alpha,n_{\alpha,2}}}{r_{\alpha,0}^2}$$

For example $sl(3) \quad L=2$:

$$M_{2000}^{1100} = \frac{\theta_{12} + 2i}{\theta_{12}} \frac{r_{12}}{r_{10}} = + \frac{s(2s+1)(2s-i\theta_{12})(-i\theta_{12}+2s+1)}{\theta_{12}(\theta_{12}+i)}$$

$$\theta_{12} = \theta_1 - \theta_2$$

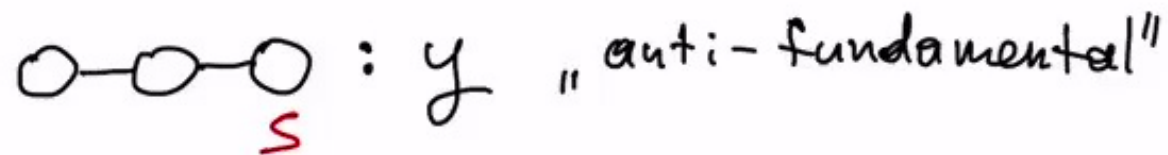


Dual SoV states

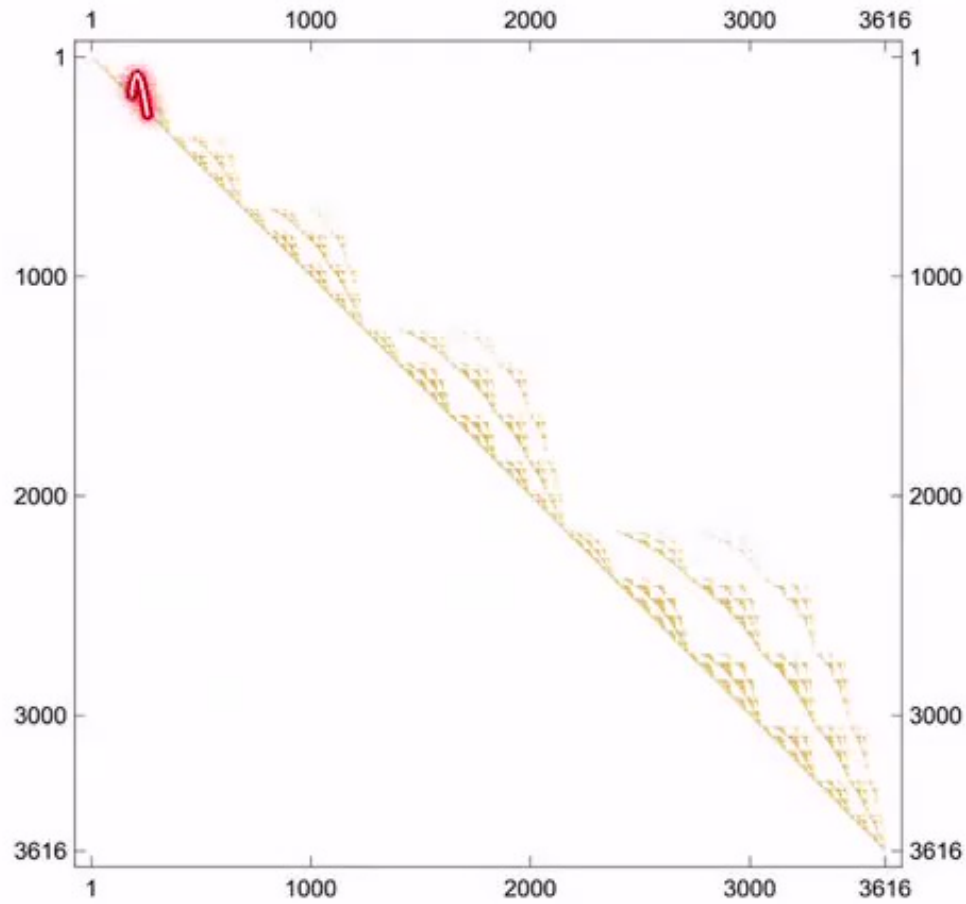
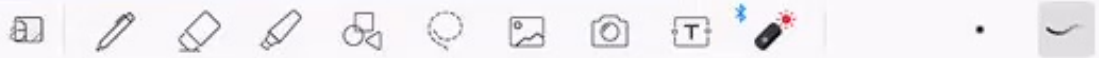
$$\langle \psi | y \rangle = \begin{pmatrix} Q_{12}^+(x_{d1}) & Q_{13}^+(x_{d1}) \\ Q_{12}^-(x_{d2}) & Q_{13}^-(x_{d2}) \end{pmatrix}$$



"fundamental"



"anti-fundamental"



$S_1(4) \quad L=3$



$$B = T_{23} T_{12}^- T_{23} - T_{23} T_{22}^- T_{13} + \dots$$

$$\langle x | \hat{B} = B \langle x |$$

$$C \approx T_{23} T_{12} T_{23}^{++} - T_{23} T_{22} T_{13}^{++} + \dots$$

$$\hat{C} |y\rangle = |y\rangle C$$

$$\langle x | \psi \rangle = \prod Q_i$$

$$\langle \psi | y \rangle = \prod \begin{pmatrix} Q_{12} & Q_{13} \\ Q_{12} & Q_{13} \end{pmatrix}$$



examples $L=2$ $s(13)$

$\langle x |$

$$\langle 0, 0; 1, 1 | = -\frac{ix_1x_2}{2is - \theta_{12}} + \frac{(2s+1)x_2^2}{2s} + \frac{\theta_{12}y_2}{2s(\theta_{12} - 2is)}$$

$$\langle 1, 1; 0, 0 | = \frac{1}{(\theta_{12} + 2is)^2} \left(x_1x_2(i\theta_{12}(4s+1) + 2s) + \frac{\theta_{12}^2(2s+1)x_1^2}{2s} - 2s(2s+1)x_2^2 + \frac{\theta_{12}^2y_1}{2s} + i\theta_{12}y_2 \right)$$

$$\langle 0, 0; 2, 0 | = x_2^2$$

$$\langle 1, 0; 1, 0 | = \frac{1}{\theta_{12} + 2is} (i(2s+1)x_2^2 + (\theta_{12} - i)x_1x_2)$$

$$\langle 2, 0; 0, 0 | = i(2s+1)x_2^2 + (\theta_{12} - i)x_1x_2.$$

$|y\rangle$

$$|0, 0; 1, 1\rangle = \frac{\theta_{12}y_2 - 2is y_1}{\theta_{12} - 2is}$$

$$|1, 1; 0, 0\rangle = y_1$$

$$|0, 0; 2, 0\rangle = \frac{1}{\theta_{12} - 2is} \left(\frac{\theta_{12}(\theta_{12} - i)x_2^2}{\theta_{12} - i(2s+1)} + \frac{4\theta_{12}s x_1x_2}{i\theta_{12} + 2s + 1} + \frac{2s(2s+1)x_1^2}{2is + i - \theta_{12}} + 2is y_1 - \theta_{12}y_2 \right)$$

$$|1, 0; 1, 0\rangle = \frac{-i(2s+1)x_1^2 + (\theta_{12} + i)x_1x_2 + iy_1 - iy_2}{\theta_{12} - 2is}$$

$$|2, 0; 0, 0\rangle = x_1^2 - y_1.$$

$\langle x | y \rangle^{-1}$

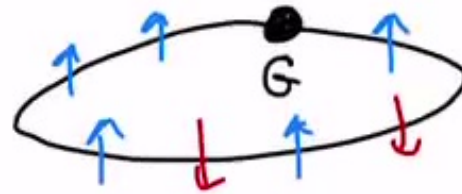
$$\begin{pmatrix} -\frac{4s^2(2s+i\theta_{12})^2}{\theta_{12}^2} & 0 & \frac{s(2s+1)(2s+i\theta_{12})(i\theta_{12}+2s+1)}{\theta_{12}(\theta_{12}-i)} & \frac{4s^2(\theta_{12}^2+4s^2)}{\theta_{12}^2(\theta_{12}^2+1)} & 0 \\ 0 & -\frac{4s^2(2s-i\theta_{12})^2}{\theta_{12}^2} & 0 & \frac{4s^2(\theta_{12}^2+4s^2)}{\theta_{12}^2(\theta_{12}^2+1)} & \frac{s(2s+1)(2s-i\theta_{12})(-i\theta_{12}+2s+1)}{\theta_{12}(\theta_{12}+i)} \\ 0 & 0 & -\frac{s(2s+1)(2s+i\theta_{12})(i\theta_{12}+2s+1)}{\theta_{12}(\theta_{12}-i)} & 0 & 0 \\ 0 & 0 & 0 & \frac{4s^2(\theta_{12}^2+4s^2)}{\theta_{12}^2+1} & 0 \\ 0 & 0 & 0 & 0 & -\frac{s(2s+1)(2s-i\theta_{12})(-i\theta_{12}+2s+1)}{\theta_{12}(\theta_{12}+i)} \end{pmatrix}$$

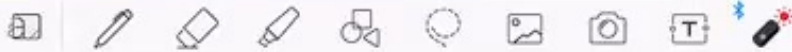
coincides with the prediction!



Observation

$\langle x |$ and $| y \rangle$ do not depend on twist
for quasi-periodic case





Can express a large number of correlators in det form!

$$\langle \Phi_{G'} | C(u) B(v) | \Psi_G \rangle \quad Q \rightarrow Q(u-v)$$

Another class

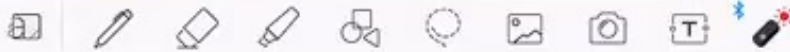
$$\langle \Phi | \delta I | \Phi \rangle$$

$$\langle \langle Q | \delta \hat{O} | Q \rangle \rangle_\alpha = 0$$

\Downarrow

$$\delta I = \frac{\det}{\det}$$

for $\delta \theta_i$
can get local spin vars!



Can express a large number of correlators in det form!

$$\langle \Phi_{G'} | C(u) B(v) | \Psi_G \rangle$$

$$Q \rightarrow Q(u-v)$$

$$Q_{12} \rightarrow \dots$$

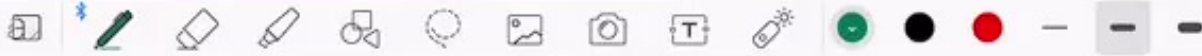
Another class

$$\langle \Phi | \delta I | \Phi \rangle$$

$$\left((Q \delta \hat{O} Q) \right)_a = 0$$

$$\delta I = \frac{\det}{\det}$$

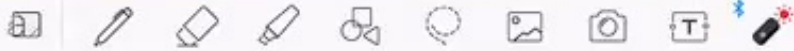
for $\delta \theta_i$
can get local spin vars!



Conclusions

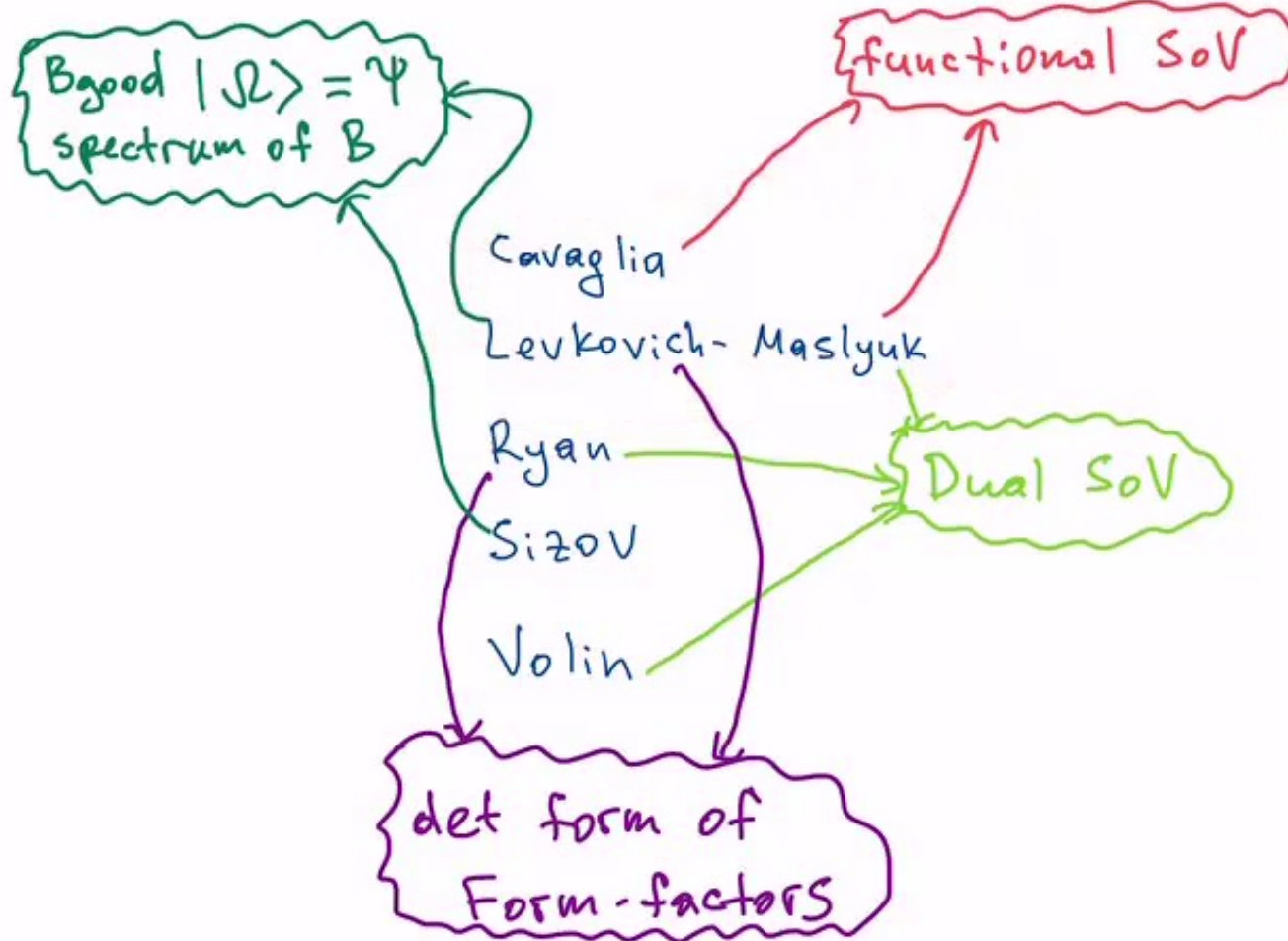
- SoV basis and dual SoV basis
- Measure simple and explicit
- Can compute nontrivial overlaps and form factors
- Can we compute any \hat{O} ?





pi2

Credits





Thank
you!