Title: Spin (8,9,10), Octonions and the Standard Model

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Collection: Octonions and the Standard Model

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Abstract: "I will start by explaining how the (Weyl) spinor representations of the pseudo-orthogonal group Spin(2r+s,s) are the spaces of even and odd polyforms on Cr x Rs. Then, the triality identifies the Majorana-Weyl spinors of Spin(8) with octonions. Combining the two constructions one finds that the groups Spin(8+s,s) all have an octonionic description, with Weyl spinors of this group being a copy of O^(2^s). This also gives an octonionic description of the groups that can be embedded into Spin(8+s,s).

Applying this construction to Spin(10,2) gives an octonionic description of Spin(10). The latter arises as the subgroup of Spin(10,2) that commutes with a certain complex structure on the space of its Weyl spinors O4. This gives a description of Weyl spinors of Spin(10) as O2_C, and an explicit description of the Lie algebra of Spin(10) as that of 2x2 matrices (of a special type) with complex (and octonionic) entries.

It is well known from the SO(10) GUT that fermions of one generation of the SM can be described as components of a single Weyl spinor of Spin(10). Combining this with the previous construction one gets an explanation of why it is natural to identify elementary particles with components of two copies of complexified octonions. I explicitly describe the dictionary that provides this identification.

I also describe how a choice of a unit imaginary octonion induces some natural complex structures on the space of Spin(10) Weyl spinors. For one of these complex structures, its commutant in Spin(10) is SU(2)_L x SU(2)_R x SU(3) x U(1). One thus gets a surprisingly large number of structures seen in the SM from very little input - a choice of a unit imaginary octonion. "

Spin (8, 9, 10), Octonions and the SM

joint with
Niren Bhoja

Over the years, many works by many authors attempted to use octonions to describe elementary particles
(starting with Gunaydin and Gursey '73)

Most often in the setting of Clifford algebras,
or division algebras

These works go some way, exhibiting some of the structures seen in the SM. But then things get lost in the subtleties, like complexification, or more states than desired

It is desirable to have a clean statement

The purpose of this talk is to explain one clean statement that I understand

Logic of the statement:

We know that all fermions of one SM generation

$$\begin{pmatrix} \nu \\ e \end{pmatrix} \quad \begin{pmatrix} u \\ d \end{pmatrix}^r \quad \begin{pmatrix} u \\ d \end{pmatrix}^g \quad \begin{pmatrix} u \\ d \end{pmatrix}^b$$

RH neutrino $\longrightarrow$ $\bar{\nu}$ $\quad \bar{u}^r \quad \bar{u}^g \quad \bar{u}^b$

$\bar{e} \quad \bar{d}^r \quad \bar{d}^g \quad \bar{d}^b$

all Lorentz spinors of the same chirality (e.g. unprimed)

fit into a single Weyl spinor representation of Spin(10)

Spin(10) turns out to have an octonionic description

(*)

$$\mathrm{spin}(10) = \begin{pmatrix} A - ia & L_x + iL_y \\ -L_{\bar{x}} + iL_{\bar{y}} & A' + ia \end{pmatrix} \in \mathrm{End}\,(\mathbb{O}^2_{\mathbb{C}})$$

$x, y \in \mathbb{O}$

$a \in \mathbb{R}$

$A, A' \in so(8)$

representation of the same so(8) Lie algebra matrix on Spin(8) spinors of different chirality

$L_x, L_y$ - operators of left multiplication by an octonion (associative!)

In particular Weyl spinor Spin(10) $= \mathbb{O}^2_{\mathbb{C}}$

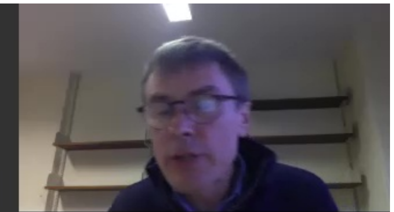
i.e. a pair of complexified octonions

The two statements combined imply that all fermions of one SM generation can be identified with components of two complexified octonions

Explicit dictionary will be provided later

These observations remove the mystery (at least for me) as to why octonions were observed to be relevant

The only mystery that remains is why fermions fit into a Weyl spinor of Spin(10), but we got used to this



## Plan of the rest of the talk

1) Construction that leads to (*)

Octonions appear as a convenient language that allows to describe such complicated Lie algebra as Spin(10) in such simple way as (*)

Without octonions Spin(10) is a mess

2) Explicit dictionary

$\mathbb{O}^2_{\mathbb{C}} \Longleftrightarrow$ elementary particles

3) Observations related to various complex structures on $\mathbb{O}^2_{\mathbb{C}}$ and to how $G_{SM}$ sits inside $\mathrm{Spin}(10)$

# Spinors

How do we describe spinor representations of Spin(r,s) explicitly?

Usual textbook way – weights. You act on the highest weight state with roots, and build the representation space.

Universal method. But for large reps the complexity gets out of hand very quickly.

Another textbook way. $\gamma$- matrices generating a Clifford algebra are tensor products of lower-dim $\gamma$- matrices. This is already much more efficient for Spin(r,s) as one can prove lots of important facts about Clifford algebras this way.

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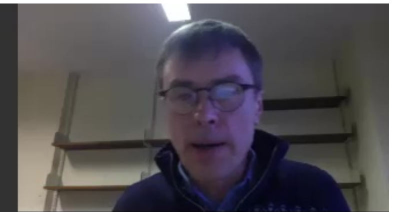

I find it difficult to work with complicated tensor products. The method I am presenting is related, but is much more efficient for practical comps.

I am after analogs of statements

$$\mathrm{Spin}(1,3) \approx SL(2,\mathbb{C}) \qquad \text{Spinors are } \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}^2$$

$$\mathrm{Spin}(1,5) \approx SL(2,\mathbb{H}) \qquad \text{Spinors are } \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{H}^2$$

Many such statements are in the literature, but the literature stops in some # of dims. How do you go further?

## Explicit construction of spinors of Spin$(r,s)$ using polyforms

The general case is a combination of two methods, both in the literature

**Spin$(2n)$**  Relies on a choice of a complex structure in $\mathbb{R}^{2n}$ $\Longleftrightarrow$ pure spinors

$$\mathbb{R}^{2n} \sim \mathbb{C}^n$$

(Weyl) spinors are (even, odd) polyforms in $\mathbb{C}^n$

E.g. $\Lambda^+(\mathbb{C}^2) = \alpha + \beta\, dz^1 dz^2$

Introduce creation-annihilation operators

$i=1,\dots,n$

$$a_i = dz^i \wedge \dots \qquad \text{creation operator}$$

$$a_i^\dagger = i_{\partial/\partial z^i} \qquad \text{annihilation}$$

Satisfy $\quad a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}$

Define

$$\gamma_i = a_i + a_i^\dagger$$

$$\gamma_{i+n} = i(a_i - a_i^\dagger)$$

Generate $\mathrm{Cliff}(2n)$

Very explicit description. Any computation is possible.

<u>Spin$(n,n)$</u>

Relies on a choice of a $J^2=+1$ $G(J\cdot,J\cdot)=-G(\cdot,\cdot)$
structure on $\mathbb{R}^{n,n}$ (null eigenspaces)

(Weyl) spinors are (even, odd) polyforms on $\mathbb{R}^n$

$$\gamma_i = a_i + a_i^{\dagger}$$

$$\gamma_{i+n} = a_i - a_i^{\dagger}$$ ← square to $-1$

Generate Cliff$(n,n)$

# General case Spin$(2n+r, r)$

Spinors are polyforms in $\mathbb{C}^n \otimes \mathbb{R}^r$

n complex and r real coordinates

Cliff$(2n+r, r)$ is generated by

$2n$ $\gamma$-matrices $\quad a+a^{\dagger}$, $\; i(a-a^{\dagger})$ $\quad$ creation-annihilation of complex coordinates

$2r$ $\gamma$-matrices $\quad a+a^{\dagger}$, $\; a-a^{\dagger}$ $\quad$ creation-annihilation of the real coords

Example: $\quad \mathrm{Spin}(3,1) = \mathrm{Spin}(2+1,1)$

need 1 complex coordinate $\quad$ 1 real

$z \qquad\qquad t$

$$\Delta^{+}(\mathbb{C}\otimes\mathbb{R}) = \alpha + \beta\, dz\, dt$$

wedge product assumed, but not written

$$\Delta^{-}(\mathbb{C}\otimes\mathbb{R}) = \gamma\, dt + \delta\, dz$$

$$\gamma_1 \quad (a_z + a_z^{\dagger})(\alpha + \beta\, dz\, dt) = \alpha\, dz + \beta\, dt$$

$$\gamma_2 \quad i(a_z - a_z^{\dagger})(\alpha + \beta\, dz\, dt) = i(\alpha\, dz - \beta\, dt)$$

$$\gamma_3 \quad (a_t + a_t^{\dagger})(\alpha + \beta\, dz\, dt) = \alpha\, dt - \beta\, dz$$

$$\gamma_0 \quad (a_t - a_t^{\dagger})(\alpha + \beta\, dz\, dt) = \alpha\, dt + \beta\, dz$$

$$(a_t + a_z^+)(\gamma dt + \delta dz) = \gamma dz dt + \delta$$

$$i(a_t - a_z^+)(\gamma dt + \delta dz) = i(\gamma dz dt - \delta)$$

$$(a_t + a_t^+)(\gamma dt + \delta dz) = -\delta dz dt + \gamma$$

$$(a_t - a_t^+)(\gamma dt + \delta dz) = -\delta dz dt - \gamma$$

So, if $\alpha + \beta dz dt = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\quad \gamma dt + \delta dz = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$

then

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix} \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$\sigma_0 = id$

$\sigma_i$ - Pauli matrices

# Reality conditions

Very important notion of Majorana spinors.

Define

complex conjugation

$$\Gamma_A = \left( \prod \text{ all real } \gamma\text{-matrices} \right) *$$

$$\Gamma_B = \left( \prod \text{ all imaginary } \gamma\text{-matrices} \right) *$$

Can check that $\Gamma_A, \Gamma_B$ either commute or anti-commute with all $\gamma$-matrices

$\Rightarrow$ Commute with all $[\gamma, \gamma]$ Lie algebra

Can check that $\Gamma_A^2, \Gamma_B^2 = \pm \mathbb{1}$

depends on $\Gamma$s

Majorana condition: If $\exists$ an anti-linear $\tilde{\Gamma}$ : $\tilde{\Gamma}^2 = +1$, and $\Gamma$ commutes or anti-commutes with all $\gamma$s, can impose $\tilde{\Gamma}(\text{Spinor}) = \text{Spinor}$

E.g. Spin(3,1) — Product of all real $\gamma$s does the job

$\tilde{\Gamma} = (\gamma_1\gamma_3\gamma_0)*$

$$(\gamma_1\gamma_3\gamma_0)*\,(\gamma_1\gamma_3\gamma_0)* = \gamma_3\gamma_0\gamma_3\gamma_0 = -\gamma_0^2 = \mathbb{1}$$

$$\tilde{\Gamma}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}-\bar{\beta}\\ \bar{\alpha}\end{pmatrix}$$

and so $\begin{pmatrix}\alpha\\ \beta\\ -\bar{\beta}\\ \bar{\alpha}\end{pmatrix}$ is a Majorana (Dirac) spinor

# Invariant inner product

This also has a simple description with polyforms

$$\langle \Psi_1, \Psi_2 \rangle = \left( \overbrace{\text{polyform } \Psi_1} \wedge \text{polyform } \Psi_2 \right) \Big|_{\text{restrict to top form}}$$

reversal of ordering

$$\langle \alpha_1 + \beta_1 dz\,dt ,\ \alpha_2 + \beta_2 dz\,dt \rangle$$

$$= (\alpha_1 + \beta_1\, dt\,dz) \wedge (\alpha_2 + \beta_2\, dz\,dt)\Big|$$

$$= \alpha_1\beta_2 - \alpha_2\beta_1$$

$dz\,dt$ as top form

$$\langle \gamma_1 dt + \delta_1 dz ,\ \gamma_2 dt + \delta_2 dz \rangle$$

$$= (\gamma_1 dt + \delta_1 dz) \wedge (\gamma_2 dt + \delta_2 dz)\Big|$$

$$= \delta_1\gamma_2 - \delta_2\gamma_1$$

Both very familiar expressions

$$(\alpha_1\ \ \beta_1)\begin{pmatrix} 0 & 1 \\ \end{pmatrix}\begin{pmatrix} \alpha_2 \\ \end{pmatrix}$$

## Spin(8) and octonions

The complexity of the above method grows very quickly, and it would be no better than any other method.

But one can use the two special cases

(Weyl) Spinors of Spin(4) are quaternions $\mathbb{H}$

(Majorana-Weyl) Spinors of Spin(8) are octonions $\mathbb{O}$

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This implies $\mathrm{Spin}(4+r, r)$ has quaternionic description

$\mathrm{Spin}(8+r, r)$ has octonionic description

$\mathrm{Spin}(2+r, r)$
$\mathrm{Spin}(6+r, r)$ remain complex

## Majorana-Weyl spinors of Spin(8)

$$\Gamma = \left(\prod 4 \text{ imaginary } \gamma\text{'s}\right) * \qquad\qquad \Gamma^2 = +1$$

Real even and odd polyforms are

$$S_- = \text{odd} = u_1 e^1 + \bar{u}_1 e^{234} + u_2 e^2 + \bar{u}_2 e^{314} + u_3 e^3 + \bar{u}_3 e^{124} + i\bar{u}_4 e^4 + i u_4 e^{123}$$

$$S_+ = \text{even} = w_1 e^{41} + \bar{w}_1 e^{23} + w_2 e^{42} + \bar{w}_2 e^{31} + w_3 e^{43} + \bar{w}_3 e^{12} + i\bar{w}_4 + i w_4 e^{4123}$$

Majorana condition halved the dimensions $\mathbb{C}^8 \to \mathbb{C}^4$

The parametrisation by $u, w \in \mathbb{C}^4$ is chosen to become convenient later

**Triality map** We always have $V \times S_+ \to S_-$

Clifford multiplication

For Spin(8) $V, S_+, S_-$ are all copies of $\mathbb{R}^8$.

$V, S_+, S_-$ can be identified with $\mathbb{O}$ so that

$V \times S_+ \to S_-$ is octonionic multiplication

The parametrisation of $S_+, S_-$ by $\mathbb{C}^4$ is done with this identification in mind

# Octonionic model for Spin(8)

One thus finds that Majorana-Weyl $S_\pm \simeq \mathbb{O}$ and Cliff(8) is generated by

$$\Gamma_x \begin{pmatrix} S_+ \\ S_- \end{pmatrix} = \begin{pmatrix} 0 & L_x \\ L_{\bar{x}} & 0 \end{pmatrix} \begin{pmatrix} S_- \\ S_+ \end{pmatrix}$$

$x \in \mathbb{O}$

$\bar{x}$ - octonionic conjugation

$L_x$ - left multiplication by $x$

$$L_x L_{\bar{x}} = |x|^2 \cdot \mathbb{1}$$

Octonions are not associative, but

$L_x$ are associative operators

This octonionic description of $\mathrm{Spin}(8)$, together with the general construction of the spinors shows that $\mathrm{Spin}(8+r, r)$ will have an octonionic description

This gives a great economy of description – allows to store information about spinors in octonionic "Bytes" rather than complex "Bits"

One can get an octonionic model for many interesting groups by restricting to subgroups of $\mathrm{Spin}(8+r, r)$

## Octonionic model for Spin(9,1) and Spin(9)

The cleanest octonionic model for $G$ arises by thinking about it as $G \subset \mathrm{Spin}(8+r,r)$

For Spin(9) this means going to Spin(9,1)

$$\Delta^+(\mathbb{C}^4 \times \mathbb{R}) = \Delta^+(\mathbb{C}^4) + \Delta^-(\mathbb{C}^4)\, dx$$

$$\Delta^-(\mathbb{C}^4 \times \mathbb{R}) = \Delta^-(\mathbb{C}^4) + \Delta^+(\mathbb{C}^4)\, dx$$

q octonions          p octonions

$$\Delta^+(\mathbb{C}^4),\ \Delta^-(\mathbb{C}^4) \approx \mathbb{O}$$

But are Weyl spinors of Spin(8) of different chirality, so important to distinguish

$$\Psi^+ = q + p\,dx = \begin{pmatrix} p \\ q \end{pmatrix} \qquad \Psi^- = \tilde{p} + \tilde{q}\,dx = \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}$$

$\gamma$- matrices $\in \mathrm{End}(\mathbb{O}^4)$

If only need Lie algebra enough to work with $\mathrm{End}(\mathbb{O}^2)$
as $[\gamma,\gamma]$ block-diagonal

$$\mathrm{spin}(9,1) = \begin{pmatrix} A+r & L_x + L_y \\ -L_{\bar{x}} + L_{\bar{y}} & A'-r \end{pmatrix}$$

$A, A' \in \mathrm{spin}(8)$
$x, y \in \mathbb{O}$
$r \in \mathbb{R}$

From this get spin(9) as
subalgebra of transformations not touching
the negative definite direction

$$\mathrm{Spin}(9) = \begin{pmatrix} A & L_x \\ -L_{\bar{x}} & A' \end{pmatrix}$$

# Octonionic model for Spin(10,2) and Spin(10)

need 2 real coordinates in this case

$$\Psi^+ = p_1 dx_1 + p_2 dx_2 + q_1 + q_2 dx_1 dx_2$$

$\gamma \in \mathrm{End}(\mathbb{O}^8)$ but $[\gamma, \gamma]$ block diagonal, so on a single chirality Weyl spinor $\in \mathrm{End}(\mathbb{O}^4)$

$$\mathrm{Spin}(10,2) = \begin{pmatrix} A+t-s & a-b-c+d & L_{x_1}+L_{y_1} & -L_{x_2}+L_{y_2} \\ -a+b-c+d & A-(t-s) & L_{x_2}+L_{y_2} & L_{x_1}-L_{y_1} \\ -L_{\bar{x}_1}+L_{\bar{y}_1} & -L_{\bar{x}_2}+L_{\bar{y}_2} & A'+(t+s) & -a-b+c+d \\ L_{\bar{x}_2}+L_{\bar{y}_2} & -L_{\bar{x}_1}-L_{\bar{y}_1} & a+b+c+d & A'-(t+s) \end{pmatrix}$$

$A, A' \in \mathrm{spin}(8)$ $\quad a, b, c, d, t, s \in \mathbb{R}$

$x_1, y_1, x_2, y_2 \in \mathbb{O}$

To get Spin(10) we take a complex structure

$J^2 = -\mathbb{1}$

$J = \gamma\gamma$ product of 2 $\gamma$'s that correspond to the negative directions

What commutes with it is Spin(10) $\oplus$ spin(2)

$c, d = 0 \quad s, t = 0 \quad y_1, y_2 = 0 \quad B$ generates Spin(2)

$$\mathrm{spin}(10) = \begin{pmatrix} A & a & L_{x_1} & -L_{x_2} \\ -a & A & L_{x_2} & L_{x_1} \\ -L_{\bar{x}_1} & -L_{\bar{x}_2} & A' & -a \\ L_{\bar{x}_2} & -L_{\bar{x}_1} & a & A' \end{pmatrix}$$

$$\mathrm{Spin}(10) = \begin{pmatrix} A - ia & L_{x_1} + i L_{x_2} \\ -L_{\bar{x}_1} + i L_{\bar{x}_2} & A' + ia \end{pmatrix}$$

$A, A' \in \mathrm{spin}(8)$

$x_1, x_2 \in \mathbb{O}$

$a \in \mathbb{R}$

It acts on $\begin{pmatrix} p_1 + i p_2 \\ q_1 + i q_2 \end{pmatrix} \in \mathbb{O}^2_{\mathbb{C}} = \mathbb{C}^{16}$

This is an explicit octonionic description of spin(10) acting on one of its Weyl spinor representations $\mathbb{O}^2_{\mathbb{C}} = \mathbb{C}^{16}$

# Octonions and elementary particles

We now recall that fermions of one SM generation fit into a single Weyl spinor rep of Spin(10)

Seeing how this happens in the octonionic model just described relates particles to octonions

The most straightforward way is via Pati-Salam

$$SO(4) \times SO(6) \subset Spin(10)$$

$$SO(4) = SU(2) \times SU(2) \qquad SO(6) = SU(4)$$

another SU(2) acting on the RH particles

leptons as the fourth color

$SU(2)_L \updownarrow \begin{pmatrix} \nu \\ e \end{pmatrix} \quad \begin{pmatrix} u \\ d \end{pmatrix}^r \quad \begin{pmatrix} u \\ d \end{pmatrix}^g \quad \begin{pmatrix} u \\ d \end{pmatrix}^b$

SU(4) fundamental

All Lorentz spinors of the same chirality

$SU(2)_R \updownarrow \begin{pmatrix} \bar{\nu} \\ \bar{e} \end{pmatrix} \quad \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}^r \quad \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}^g \quad \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}^b$

SU(4) anti-fundamental

A $SO(4)\times SO(6)$ subgroup of $SO(10)$

is most conveniently selected by choosing

a complex structure on its Weyl spinors $\mathbb{O}^2_{\mathbb{C}}$

Let us choose a unit imaginary octonion $u \in \mathrm{Im}\,\mathbb{O}$

$|u|^2 = 1$

Note that there is $S^6 \subset \mathrm{Im}\,\mathbb{O}$ of such choices

Define $J_u = \begin{pmatrix} L_u & 0 \\ 0 & L_u \end{pmatrix} \in \mathrm{End}\,(\mathbb{O}^2)$

Acting on $\mathbb{O}$, $L_u$ is a complex structure $L_u^2 = -\mathbb{1}$

Identifies $\mathbb{O} = \mathbb{C}^4$

Can be extended to $\mathbb{O}_{\mathbb{C}}$ via $L_u(p_1 + i p_2) = L_u p_1 + i L_u p_2$

**Lemma:** The commutant of $J_u$ in $spin(10)$ is $spin(6) \times spin(4)$

**Proof:** Computing the commutator and setting the result to zero one gets the following eqs

$$[A, L_u] = [A', L_u] = 0$$

These taken together reduce $so(8)$ to $so(6)$ + rotations in the $1,u$ plane in $\mathbb{O}$

$$[L_{x_1}, L_u] = [L_{x_2}, L_u] = 0 \implies x_1, x_2 \in \mathrm{Span}(1,u) \subset \mathbb{O}$$

So, we get

$a, b \in \mathbb{R}$ $\quad$ $x_1, x_2 \in \mathbb{C} = \mathrm{Span}(1, u)$

$$\text{commutant } J_u = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} -ia - bL_u & L_{x_1} + iL_{x_2} \\ -L_{\bar{x}_1} + iL_{\bar{x}_2} & ia + bL_u \end{pmatrix}$$

$$\underset{so(6)}{\cap} \qquad\qquad \underset{so(4)}{\cap}$$

we used $A = A'$ when restricted to $so(6)$ that rotates in the $\mathrm{Span}(1, u)^{\perp}$

rotates in the $1, u, \underline{x_1, x_2}$ plane

two coordinates that were added to go from $SO(8)$ to $SO(10)$

Can write the $so(4)$ part even more explicitly by

$$x_1 = \alpha + \gamma u \qquad x_2 = \beta + \delta u$$

$$so(4) = \begin{pmatrix} -ia & \alpha+i\beta \\ -\alpha+i\beta & ia \end{pmatrix} + \begin{pmatrix} -b & \gamma+i\delta \\ \gamma-i\delta & b \end{pmatrix} L_u$$

anti-Hermitian    Hermitian

End of proof

The irreducibles of this $so(4)$ are eigenspaces of $L_u$ !

$L_u$ identifies $\mathbb{O} = \mathbb{C}^4 + \bar{\mathbb{C}}^4$

$(1,0)$    $(0,1)$

$\mathbb{O} = \mathbb{O}^+ + \mathbb{O}^-$

$p = p^+ + p^-$

$q = q^+ + q^-$

$L_u p^+ = -ip^+$

$L_u q^+ = -iq^+$

We can now write the Dictionary

$$\begin{pmatrix}\nu\\ e\end{pmatrix}\begin{pmatrix}u\\ d\end{pmatrix} \qquad \begin{pmatrix}p_1^+ + ip_2^+\\ q_1^+ + iq_2^+\end{pmatrix} \in \begin{pmatrix}\mathbb{O}_{\mathbb{C}}^+\\ \mathbb{O}_{\mathbb{C}}^+\end{pmatrix} \qquad SU(2)_L \text{ spinor}$$

fundamental of $SU(4)$

$$\begin{pmatrix}\bar{\nu}\\ \bar{e}\end{pmatrix}\begin{pmatrix}\bar{u}\\ \bar{d}\end{pmatrix} \qquad \begin{pmatrix}p_1^- + ip_2^-\\ q_1^- + iq_2^-\end{pmatrix} \in \begin{pmatrix}\mathbb{O}_{\mathbb{C}}^-\\ \mathbb{O}_{\mathbb{C}}^-\end{pmatrix} \qquad SU(2)_R \text{ spinor}$$

anti-fundamental of $SU(4)$

$$SO(4) = SU(2)_L \times SU(2)_R$$

Can get even more similarity with the SM if change the complex structure slightly

**Lemma:** The commutant of $J_u' = \begin{pmatrix} R_u & 0 \\ 0 & R_u \end{pmatrix}$ in Spin(10)

is $SU(3) \times U(1) \times SU(2)_L \times SU(2)_R$

$R_u$ - right multiplication by $u$ $\quad R_u^2 = -\mathbb{1}$

**Proof:** $[A, R_u] = 0 \quad [A', R_u] = 0$

imply $A \in su(3) \oplus u(1) \oplus$ rotations in $1, u$ plane

$[L_{x_1}, R_u] = 0 \quad [L_{x_2}, R_u = 0]$

$\Rightarrow x_1, x_2 \in \mathrm{Span}(1, u)$

Compare observation in 1912.11282

End of proof

Complex structure $R_u$ selects a preferred copy of $\mathbb{C}$

$\mathbb{C} = \mathrm{Span}(1, u) \subset \mathbb{O}$, not just identifies

$\mathbb{O} = \mathbb{C}^4$

# Conclusions

- Method of constructing spinor reps of $Spin(2n+r, r)$ in $\Delta^{\pm}(\mathbb{C}^n \times \mathbb{R}^r)$

- $Spin(8+r, r)$ are all "octonionic", together with related groups

  In particular $Spin(10) \subset Spin(10,2)$

- Weyl spinor of $Spin(10) = \mathbb{O}^2_{\mathbb{C}}$

  Can identify elementary particles with components of $\mathbb{O}^2_{\mathbb{C}}$, but only after a unit imaginary octonion $u \in \mathrm{Im}\,\mathbb{O}$ is selected

Leptons are eigenstates of both $L_u, R_u$,
while the rest of $\mathbb{O}$ describes quarks

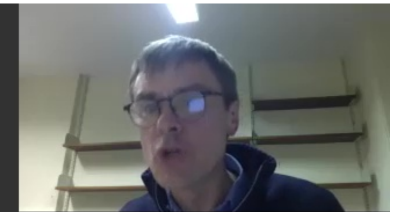
Left and right particles are $(1,0)$ and $(0,1)$
eigenstates of $L_u$ (or $R_u$) respectively

- The commutant of $J'_u = \begin{pmatrix} R_u & 0 \\ 0 & R_u \end{pmatrix}$ is

$$SU(2)_L \times SU(2)_R \times SU(3) \times U(1)$$

same conclusion
(independently)
I. Todorov - private
comm.

Left-Right symmetric extension of the SM also Furey, Hughes
Compare Latham 2006.16265



# Outlook

- Surprisingly large amount of structure seen in the SM comes by choosing a unit imaginary octonion, and the associated complex structure

Observation in Bryant 2011.05568

A generic Weyl spinor of Spin(10) is in the orbit of

$$\begin{pmatrix} \alpha + i\beta u \\ 0 + i0 \end{pmatrix} \qquad \alpha,\beta > 0$$

So, a Weyl spinor of Spin(10) selects a unit imaginary octonion

This feels important!

- Further step can only be taken by bringing in the Lorentz spin.

Hope to talk about $SO(11,3) \supset SO(10)\times SO(1,3)$ in another talk

- Complex structure as the Higgs field. Easy to break $Spin(10) \to SU(3)\times U(1)\times SU(2)\times SU(2)$. Just needs to add the kinetic term for $J \in End$ ( Spinor of $Spin(10)$)

$$([\omega, J])^2$$ — can generate large masses for the rest of $Spin(10)$

$\omega$: $Spin(10)$ connection