Title: Supersymmetry and RCHO revisited

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Collection: Octonions and the Standard Model

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Abstract: Various links between supersymmetry and the normed division algebras R,C,H,O were found in the 1980s. This talk will focus on the link between K=R,C,H,0 and supersymmetric field theories in a Minkowski spacetime of dimension D=3,4,6,10. The first half will survey the history starting with a 1944/5 paper of Dirac and heading towards the links found in 1986/7 between R,C,H,O and super-Yang-Mills theories. The second half will review a result from 1993 that connects, via a twistor-type transform, the superfield equations of super-Maxwell theory in D=3,4,6,10 to a K-chirality constraint on a K-valued worldline superfield of N=1,2,4,8 worldline supersymmetry. This provide an explicit connection of octonions to the free-field D=10 super-Maxwell theory.

Supersymmetry and $\mathbb{RCHO}$ revisited

Paul K. Townsend

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- $D = 3, 4, 6, 10$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and supersymmetry. Some history: 1945-1990
- Massless particles, Lorentz harmonics and celestial spheres. 1991-2 work of Delduc, Galperin, Howe, Sokatchev, Stelle
- Twistor-like transform for $D = 3, 4, 6, 10$ super-Maxwell equations and $\mathbb{K}$-chirality for $N = 1, 2, 4, 8$ worldline superfields (Galperin, Howe, PKT, 1993)

"Octonions & SM" workshop, 15 Feb '21

Dirac's wartime 'braneworld'

The 1944/5 Proceedings of the Royal Irish Academy includes a paper by Dirac on "Applications of Quaternions to Lorentz Transformations". This paper cites no one and has rarely been cited. It contains the following observations:

- Lorentz group in a 6D Minkowski spacetime is $SL(2;\mathbb{H})$
- $SL(2;\mathbb{H})$ acts by fractional linear transformations on the celestial 4-sphere $\mathbb{HP}^1$.
- Restricting to $\mathrm{Mink}_4 \subset \mathrm{Mink}_6$ one finds that

$$SL(2;\mathbb{H}) \to SL(2;\mathbb{C}) \times U(1)$$

where $SL(2;\mathbb{C})$ acts by fractional linear transformations on the celestial 2-sphere $\mathbb{CP}^1 \subset \mathbb{HP}^1$

Many years later ...

N-extended susy 2D sigma-models. Metric on target manifold M is

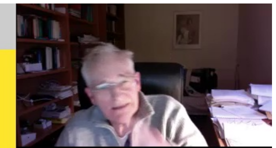
- Real for $N = 1$
- Kähler for $N = 2$ (Zumino, 1979)
- HyperKähler for $N = 4$ (Alvarez-Gaumé & Freedman, 1981)
- Flat for $N = 8$, but $\dim M = 8m$ for integer m

In addition, a superspace formulation of N-extended 4D SYM suggested a connection of $N = 1, 2$ with $\mathbb{C}, \mathbb{H}$ (Ivanov, 1982).

By considering the maximal (Minkowski) spacetime dimension D permitted by susy, one sees that (Kugo, PKT, 1982)

$$D = 2 + \dim \mathbb{K}\,, \qquad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\,.$$

Expected for $\mathbb{RCH}$ (and $\mathbb{O}$?) from $\boxed{Spin(1, 1 + \dim \mathbb{K})) \cong SL(2; \mathbb{K})}$



Lorentz groups in $D = 3, 4, 6, 10$

- $D = 3$: $SL(2;\mathbb{R}) \cong Spin(1,2)$
- $D = 4$: $SL_1(2;\mathbb{C}) \cong Spin(1,3) \times U(1)$
- $D = 6$: $SL(2;\mathbb{H}) \cong Spin(1,5)$
- $D = 10$: $SL(2;\mathbb{O}) \cong Spin(1,9)$ [algebra: (Sudbery, 1983); group: (Tachibana, Imaeda, 1989; Manogue, Schray 1993; Viera 2015)]

Check: $Sl(2;\mathbb{K})$ has $4\dim\mathbb{K} - 1$ generators for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, but

$$\text{(naive) } \dim \mathrm{sl}(2;\mathbb{O}) \neq 4 \times 8 - 1 = 31 \qquad (?)$$

Problem due to failure of Jacobi identity [Sudbery]:

$$[A,[B,X]] - [B,[A,X]] = [[A,B],X] + E(A,B)X\,,$$

where $E(A,B) \in G_2$. Since $\dim G_2 = 14$, the correct count is

$$\dim[sl(2;\mathbb{O})] = 31 + 14 = 45 \qquad \checkmark$$

Super-Yang-Mills and $\mathbb{RCHO}$

- 1976. Brink, Schwarz & Scherk construct SYM for $D = 2, 4, 6, 10$. An identity cubic in D-dim **commuting** spinors is required. However, $D = 3$ is maximal for the assumed $(1, 1)$ susy, not $D = 2$, so no connection to $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ was suspected.
- 1984. Green & Schwarz find same cubic identity for GS superstring: now $D = 3, 4, 6, 10$.
- 1986. Sierra interprets cubic identity as Jordan identity for 3×3 hermitian matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (as does Schray in 1994)
- 1987. Evans interprets cubic identity in terms of the Adams "trialities" that are satisfied only for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Aside: Supersymmetry and the Magic square

1983. Günaydin, Sierra, PKT relate $D = 5$ Maxwell-Einstein supergravity theories to cubic-norm Jordan algebras, and recover Freudenthal-Tits magic square by dimensional reduction to $D = 4, 3$.

Conformal groups for $D = 3, 4, 6, 10$

Define $Sp^\dagger(2n;\mathbb{K})$ as group preserving skew-hermitian quadratic form

For $n = 2$ we have the conformal isometry groups of Minkowski spacetime in $D = 3, 4, 6, 10$

- $D = 3$. $Sp(4;\mathbb{R}) \equiv Spin(2,3)$
- $D = 4$. $Sp^\dagger(4;\mathbb{C}) \equiv Spin(2,4) \times U(1)$
- $D = 6$. $Sp^\dagger(4;\mathbb{H}) \equiv Spin(2,6)$
- $D = 10$ $Sp^\dagger(4;\mathbb{O}) \equiv Spin(2,10)$ (Chung & Sudbery, 1987)

These are conformal isometries of Mink_d except extra $U(1)$ for $\mathbb{K} = \mathbb{C}$
For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we have $\dim[sp(4;\mathbb{K})] = 6\dim\mathbb{K} + 4$

For $\mathbb{K} = \mathbb{O}$ the add 14 rule again applies:

$\dim sp(4;\mathbb{O}) = 6 \times 8 + 4 + 14 = 66$ ✓

Rotation subgroups of $SL(2;\mathbb{K})$

Define $SO^{\dagger}(n;\mathbb{K})$ as group preserving Hermitian quadratic form. For $n=2$ we get rotation groups for $D=3,4,6,10$:

- $D=3$. $SO(2;\mathbb{R}) \equiv U(1) \cong Spin(2)$
- $D=4$. $SO^{\dagger}(2;\mathbb{C}) \equiv U(2) \cong Spin(3) \times U(1)$
- $D=6$. $SO^{\dagger}(2;\mathbb{H}) \cong Sp_2 \cong Spin(5)$
- $d=10$. $SO^{\dagger}(2;\mathbb{O}) \cong Spin(9)$ (Sudbery, ...)

These are rotation subgroups except extra $U(1)$ for $\mathbb{K}=\mathbb{C}$.

For $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$, we have $\dim[so(2;\mathbb{K})] = 3\dim\mathbb{K} - 2$

For $\mathbb{K}=\mathbb{O}$ apply add 14 rule

$\dim[so(2;\mathbb{O})] = (3\times 8 - 2) + 14 = 36$ ✓

A String/M-theory digression

Fluctuations of planar static M2,D3,M5 branes of string/M-theory yield maximal-susy field theories on a Minkowski worldvolume of dimension $d = 3, 4, 6$ [and $d = 10$ if we count the E_8 SYM on the 11D-spacetime boundaries of "Heterotic M-theory" (Horava, Witten, 1996)].

Corresponding planar static supergravity solutions have near-horizon 'AdS× S' geometries (Gibbons, PKT, 1993). Note sequence of super-isometry groups (Arvanitakis, Barns-Graham, PKT, 2017):

- $d = 3$. M2-brane : $OSp(8|4; \mathbb{R})$
- $d = 4$. D3-brane : $(OSp)^\dagger(4|4; \mathbb{C})$
- $d = 6$. M5-brane : $(OSp)^\dagger(2|4; \mathbb{H})$
- $d = 10$. HM9-'brane': $(OSp)^\dagger(1|4; \mathbb{O})$?

  [This has been interpreted as "D=11 de Sitter-like superalgebra' (Hasiewicz, Lukierski, 1984)]

Lorentz (co)vectors and $\mathbb{RCHO}$

For spacetime dimension $D = 3, 4, 6, 10$ we can represent position in Minkowski spacetime by a 2×2 Hermitian matrix $\mathbb{X}$ over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$:

$$\mathbb{X} = \begin{pmatrix} -X^0 + X^1 & \mathbf{X} \\ \overline{\mathbf{X}} & -X^0 - X^1 \end{pmatrix} \qquad (\mathbf{X} \in \mathbb{K}) .$$

For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, Lorentz transformation of D-vector is

$$\mathbb{X} \to \mathbb{L}\mathbb{X}\mathbb{L}^\dagger , \qquad \boxed{\det(\mathbb{L}\mathbb{L}^\dagger) = 1} \quad \Rightarrow \quad \mathbb{L} \in SL(2; \mathbb{K})$$

For co-vector, e.g. particle momentum, we have

$$\mathbb{P} \to (\mathbb{L}^\dagger)^{-1}\mathbb{P}\mathbb{L}^{-1}$$

Trace reversal [Schray]

If hermitian $\mathbb{M}$ is a vector then $\widetilde{\mathbb{M}} := \mathbb{M} - (\mathrm{tr}\mathbb{M})\,\mathbb{I}_2$ is a covector. And

$$\widetilde{\widetilde{\mathbb{M}}} = \mathbb{M} , \qquad \mathbb{M}\widetilde{\mathbb{M}} = -(\det \mathbb{M})\mathbb{I}_2$$

The (super)particle and $\mathbb{RCHO}$

Action for particle of mass m in $D = 3, 4, 6, 10$ is

$$S = \int dt \left\{ \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{X}}\mathbb{P}) + \frac{1}{2} e \left(\det \mathbb{P} - m^2\right) \right\}$$

The Casalbuoni-Brink-Schwarz superparticle for $D = 3, 4, 6, 10$ requires

$$d\mathbb{X} \to d\mathbb{X} + (\Theta d\Theta^\dagger - \Theta^\dagger d\Theta)\,.$$

for Grassmann-odd $Sl(2;\mathbb{K})$ spinor Θ.

➸ $D = 6, 10$ constructions for $m^2 = 0$ using $\mathbb{H}$, $\mathbb{O}$ spinors were given in 1988 (Kimura, Oda & Oda, Kimura, Nakamura) and again in 1994 (Schray); Schray also found a simple presentation of the 'cubic identity', discussed further in Baez, Huerta, 2009.

Solving the mass-shell constraint

For massless particle we have $\det \mathbb{P} = 0$. We may solve this in terms of an $Sl(2;\mathbb{K})$ spinor U since

$$\mathbb{P} = UU^\dagger \quad \Rightarrow \quad \det \mathbb{P} = 0$$

Whereas U has $2\dim\mathbb{K}$ components, the null D-vector $\mathbb{P}$ has only $D-1 = \dim\mathbb{K}+1$, so U is subject to gauge transformation, which is right-multiplication by a unit-norm element of $\mathbb{K}$. These are the groups

$$Z_2\,, \qquad U(1)\,, \qquad SU(2)\,, \qquad S^7 \quad [?(Sohnius, 1984;\ \ Cederwall, 1992)]$$

For massive particle $\mathbb{P} = \mathbb{U}\mathbb{U}^\dagger$ for invertible matrix $\mathbb{U}$

Now gauge invariance is $\mathbb{U} \to \mathbb{U}\mathbb{R}$ for $\mathbb{R} \in SO^\dagger(2;\mathbb{K})$.

Celestial spheres as $SL(2;\mathbb{K})$ coset spaces

For any D, the Lorentz algebra $spin(1, D-1)$ has the decomposition

$$h_- \oplus h_0 \oplus h_+ \,; \qquad h_0 = spin(D-2) \oplus so(1,1)\,,$$

where the $(D-2)$-vectors $h_\pm$ are raising/lowering operators for $SO(1,1)$ weight. The subalgebra $h_B = h_0 \oplus h_+$ generates a maximal parabolic (Borel) subgroup H_B of $Spin(1, D-1)$, and the celestial $(D-2)$-sphere is the coset space (Galperin, Howe, Stelle, 1991).

$$Spin(1, D-1)/H_B\,, \qquad H_B = [Spin(D-2) \times SO(1,1)] \ltimes H_+\,.$$

➽ For $D = 3, 4, 6, 10$ we may write $\mathbb{L} \in SL(2;\mathbb{K})$ as a pair of spinors:

$$\mathbb{L} = (U^+, U^-)\,, \qquad [\text{equivalently, } (\mathbb{L}^\dagger)^{-1} = (V_+, V_-)]$$

where (U^+, V_-) have weight $+1$ (and the others -1)

Integrating over a celestial sphere

Lorentz invariant measure on celestial sphere involves only (U^+, V_-) because only these spinors are H_+-inert. Because $\boxed{(V_-)^\dagger U^+ \equiv 0}$,

$$\omega^{++} = V_-^\dagger \, dU^+$$

is H_B-invariant except for its non-zero weight. It can be used to construct a measure $d\mu$ of weight $2\dim \mathbb{K}$, so an integral is H_B-invariant if the integrand has weight $-2\dim \mathbb{K}$ (Delduc, Galperin, Sokatchev 1992).

- For $D = 3, 4$ the spinors U^+ and V_- are in equivalent Lorentz reps.

Celstial spheres as projective lines

H_B acts on U^+ as right-multiplication by a non-zero element of $\mathbb{K}$. This tells us that $SL(2; \mathbb{K})/H_B = \mathbb{KP}^1$

Solving the 3D wave-equation

A positive-energy solution to $\Box\phi(x) = 0$ can be written as

$$\phi(x) = \int d^3p\, e^{ip\cdot x}\delta(p^2)\tilde{\varphi}(p) = \int \frac{d^2\mathbf{p}}{2p^0}\, e^{ip\cdot x}\tilde{\varphi}(p) \qquad (p^0 = |\mathbf{p}|).$$

Set $p = \Omega\bar{u}^+\gamma u^+$ and use $dp_1 \wedge dp_2 = 2p^0 d\Omega \wedge \bar{u}^+ du^+$ to get

$$\phi(x) = \int_{S^1} \bar{u}^+ du^+ \left[\int_0^\infty d\Omega\, e^{i\Omega x^{++}}\tilde{\varphi}(p)\right] = \int_{S^1} d\mu\; f(x^{++}, u^+)$$

with $x^{++} = (\bar{u}^+ \not{x} u^+)$. By construction $f(\lambda^2 x^{++}, \lambda u^+) = \lambda^{-2} f(x^{++}, u^+)$

➸ $\Box\phi = \int_{S^1} d\mu\, (u^+\gamma u)^2\, \partial^2_{++} f = 0 \qquad \left(\partial_{++} = \frac{\partial}{\partial x^{++}}\right)$

$= 0$ (because $\bar{u}^+\gamma u^+$ is null)

4D and the twistor transform

A positive-energy solution to $\Box\phi(x) = 0$ can be written as

$$\phi(x) = \int_{S^2} d\mu\; f(x^{++}, u^+) \qquad x^{++} = x^{\alpha\dot\beta} u^+_\alpha \bar u^+_{\dot\beta}$$

where u^+ is complex 2-cpt Weyl spinor, and f has weight -4 since

$$d\mu = (\varepsilon^{\alpha\beta} u^+_\alpha du^+_\beta) \wedge (\varepsilon^{\dot\alpha\dot\beta} \bar u^+_{\dot\alpha} d\bar u^+_{\dot\beta}) = |u_1|^4 dz \wedge d\bar z \qquad (z = u_2/u_1)$$

$\Rightarrow\; d\mu\, f = dz \wedge d\bar z\, F$ for zero-weight function $F = |u_1|^4 f$.

➽ As S^2 is symplectic manifold we can use the Duistermaat-Heckman theorem to localise contributions to points on S^2, which can be expressed as contour integrals $\rightarrow$ Penrose's twistor transform solution.

Solving 3D super-Maxwell equations

The 3D superMaxwell equations are equivalent to the following equation for anticommuting spinor superfield $\Psi(x,\theta)$:

$$D_\alpha \Psi^\alpha = 0\,, \qquad \{D_\alpha, D_\beta\} = \frac{\partial}{\partial x^{\alpha\beta}}$$

Solutions to this equation can be written as

$$\Psi_\alpha = \int_{S^1} d\mu\ u^+_\alpha\ \psi(x^{++},\theta^+,u^+)\,, \qquad (\theta^+ = \bar\theta u^+)$$

where ψ is any (anticommuting) worldline superfield of weight -3:

$$D_\alpha \Psi^\alpha = \int_{S^1} d\mu\ (\varepsilon^{\alpha\beta} u^+_\beta u^+_\alpha) D_+ \psi(x^{++},\theta^+,u^+)\,, \qquad \{D_+, D_+\} = \frac{\partial}{\partial x^{++}}$$

$$= 0 \qquad \text{(because } u^+ \text{ is commuting spinor)}$$

Solving 4D super-Maxwell

Now we have $D_\alpha \Psi^\alpha = 0$ for anticommuting complex Weyl-spinor superfield $\Psi(x, \theta, \bar{\theta})$. As for 3D, solutions can be expressed as

$$\Psi_\alpha = \int_{S^2} d\mu \; u^+_\alpha \; \psi(x^{++}, \theta^+, \bar{\theta}^+ u^+) \qquad (\theta^+ = \theta^\alpha u^+_\alpha, \; \bar{\theta}^+ = \bar{\theta}^{\dot{\alpha}} \bar{u}^+_{\dot{\alpha}})]$$

where ψ is now an $N = 2$ worldline superfield of weight -5. However, it is now constrained by the chirality condition on Ψ:

$$0 = \bar{D}_{\dot{\alpha}} \Psi_\alpha = \int_{S^2} d\mu \; u^+_\alpha \bar{u}^+_{\dot{\alpha}} \; \bar{D}_+ \psi \qquad \{D_+, \bar{D}_+\} = \partial_{++}$$

$\boxed{\Rightarrow \;\; \bar{D}_+ \psi = 0}$ i.e. ψ must be a chiral $N = 2$ worldline superfield.

Solving 6D super-Maxwell I

Use $SL(2;\mathbb{H}) \cong SU^*(4)$ to replace U^+ (V_-) by 4-cpt chiral 'symplectic-Majorana' spinors:

$$U^+ \;\to\; u_\alpha^{+A}, \qquad V_- \;\to\; v^\alpha_{-\dot A} \qquad (\alpha = 1,2,3,4;\; A, \dot A = 1,2)$$

The 6D Maxwell antichiral spinor superfield $\Psi^\alpha_j(x,\theta)$ satisfies

$$(i) \quad D^i_\alpha \Psi^\alpha_j = 0\,, \quad (ii) \quad D^{(i}_\alpha \Psi^{j)\beta} = 0 \qquad D^i_\alpha, D^j_\beta\} = \varepsilon^{ij}\partial_{\alpha\beta}$$

Using i $v^\alpha_{\dot A} u^A_\alpha \equiv 0$, solve (i) by

$$\Psi^\alpha_j = \int_{S^4} d\mu \; v^\alpha_{-\dot A} \; \psi^{\dot A}_j(x^{++}, \theta^+, u^+)\,, \qquad \left[(\theta^+)^{iA} = \theta^{\alpha i} u^A_\alpha\right]$$

for $[SU(2) \times SU(2)]$-vector $N = 4$ worldline superfield ψ of weight -9.

10D preliminaries

Use 16-cpt chiral spinors of $Spin(1, D-1)$. Write $L \in Spin(1, D-1)$ as

$$L = (u_\alpha^{+A} u_\alpha^{-\dot{A}}) \quad \to \quad (L^{-1})^T = (v^\alpha_{+\dot{A}}, v^\alpha_{-A})$$

where the 16 spinors are separated into two $Spin(8)$ 8-plets of of opposite chirality and $SO(1,1)$ weight. As before, we can use only the positive-weight spinors $(u_\alpha^{+A}, v^\alpha_{-\dot{A}})$ for integration over the celestial sphere.

➽ It is essential that the 10-vector $\bar{u}^{+A}\gamma u^{+A}$ (summed on A) is null. It is null (by cubic identity) because $(u_\alpha^+, u_\alpha^-) \in Spin(1,9)$ imposes (Galperin, Howe, Stelle 1991 & Delduc, Sokatchev, 1992)

$$\boxed{\bar{u}^{+A}\gamma u^{+B} = \frac{1}{8}\delta^{AB}(\bar{u}^{+C}\gamma u^{+C})}$$

Solving 6D super-Maxwell II

To solve (ii) we need to impose the following worldline superfield constraint:

$$(ii)' \quad D^{(i}_{+A}\psi^{j)\dot{A}} = 0 \qquad\qquad \{D^i_{+A}, D^j_{+B}\} = \varepsilon^{ij}\varepsilon_{AB}\partial_{++}$$

There are four components of $\psi^j_{\dot{A}}$ and four supercovariant derivatives:

$$\psi^j_{\dot{A}} = \delta^i_{\dot{A}}\psi_* + i(\boldsymbol{\sigma})_{\dot{A}}{}^i \cdot \boldsymbol{\psi}\,, \qquad D^i_{+A} = \delta^i_A D_* - i(\boldsymbol{\tau})_A{}^i \cdot \mathbf{D}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are two triplets of Pauli-matrices. In terms of the quaternionic worldline superfield $\psi = \psi_* + \mathbf{i}\cdot\boldsymbol{\psi}$, the equations $(ii)'$ can be written as

$$\mathbf{D}\psi + \mathbf{i}\cdot D_*\psi = 0\,, \qquad \mathbf{i} = (i, j, k)$$

➽ Generalizes $N = 2$ chirality condition $(D_2 + iD_1)\psi = 0$ for complex ψ

Solving 10D super-Maxwell

The 10D super-Maxwell equations for chiral spinor superfield Ψ are

$$(i)\ D_\alpha \Psi^\alpha = 0\,, \qquad (ii)\ [\gamma^{abcd} D]_\alpha \Psi^\alpha = 0$$

We can solve (i) by

$$\Psi^\alpha = \int_{S^8} d\mu\ v^\alpha_{-\dot{A}}\, \psi^{\dot{A}}(x^{++}, \theta^+, u^+) \qquad (\theta^+)^A = \theta^\alpha u_\alpha^{+A}$$

Now have an 8-plet of $N = 8$ worldline superfields; equivalently an octonionic-valued superfield ψ on an octonionic worldline superspace with supercovariant derivatives $D = D_* + \mathbf{e} \cdot \mathbf{D}$, where $\mathbf{e}$ are the 7 imaginary units. Then (ii) is solved by imposing (Galperin, Howe, PKT 1992)

$$\mathbf{D}\psi + \mathbf{e} \cdot D_* \psi = 0\,.$$

An octonionic analog of the $N = 2$ chirality condition.

Last thoughts

For a superparticle in a target space of dimension $D = 2 + \dim \mathbb{K}$, the number of worldline susy charges is

$$N = \dim \mathbb{K}$$

while the number of susy charges of the field theory in $D = 2 + \dim \mathbb{K}$ is

$$N_s = 2 \dim \mathbb{K}$$

But the susy field theories describe fluctuations of BPS on worldvolumes of dimension $d = 2 + \dim \mathbb{K}$, and these are sources in supergravity theories for which the number of supersymmetry charges is

$$N_S = 4 \dim \mathbb{K}$$

and $\mathbb{K} = \mathbb{O}$ yields 32 supersymmetry charges.

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- $d = 6$. M5-brane : $(OSp)^{\dagger}(2|4;\mathbb{H})$
- $d = 10$. HM9-'brane': $(OSp)^{\dagger}(1|4;\mathbb{O})$?

  [This has been interpreted as "D=11 de Sitter-like superalgebra' (Hasiewicz, Lukierski, 1984)]