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Abstract: We will describe an approach to the theory of fundamental particlesbased on finite-dimensional quantum algebras of observables. We will explain why the unimodularity of the color group suggests an interpretation of the quarklepton symmetry which involves the octonions and leads to the quantum spaces underlying the Jordan algebras of octonionic hermitian 2 Ã— 2 and 3 Ã— 3 matrices as internal geometry for fundamental particles. In the course of this talk, we will remind shortly why the finite-dimensional algebras of observables are the finite-dimensional euclidean Jordan agebras and we will describe their classifications. We will also explain our differential calculus on Jordan algebras and the theory of connections on Jordan modules. It is pointed out that the above theory of connections implies potentially a lot of scalar particles.

Finite quantum geometry, octonions and the theory of fundamental particles

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Octonions and the Standard Model
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General framework

- External geometry: Lorentzian spacetime M
  $\mathcal{C}(M)$ with Poincaré group action and equivariant $\mathcal{C}(M)$-modules.
- Internal geometry : Finite quantum geometry
  $J=$ finite-dimensional algebra of quantum observables with some further structure $\Rightarrow G \subset \mathrm{Aut}(J)$ and equivariant J-modules.
- $\Rightarrow \mathcal{J} = \mathcal{C}(M, J)$, $\mathcal{J}$-modules and connections
  $\Rightarrow$ gauge interactions, etc.
  $\mathcal{J}$ defines an "almost classical quantum geometry".

Internal space for a quark

$E \simeq \mathbb{C}^3$ with (color) $SU(3)$ action

$$\begin{cases} SU(3) \subset U(3) \Rightarrow E \text{ is Hilbert with scalar product } \langle \bullet, \bullet \rangle \\ \text{Unimodularity of } SU(3) \Rightarrow \text{ volume } = \text{3-linear form on } E, vol(\bullet, \bullet, \bullet) \end{cases}$$

$\Rightarrow$ antilinear antisymmetric product x on E

$$vol(Z_1, Z_2, Z_3) = \langle Z_1 \times Z_2, Z_3 \rangle$$

$SU(3)$-basis $=$ Orthonormal basis (e_k) of E such that

$$v(e_1, e_2, e_3) = 1$$

By chosing an origin $SU(3)$-basis $\leftrightarrow SU(3)$
2 products $x : E \times E \to E$ and $\langle , \rangle : E \times E \to \mathbb{C}$

Unital $SU(3)$-algebra

$$SU(3) = \{U \in GL(E) | x \text{ and } \langle , \rangle \text{ are preserved}\}$$

$$\| Z_1 \times Z_2 \|^2 = \| Z_1 \|^2 \| Z_2 \|^2 - |\langle Z_1, Z_2 \rangle|^2$$

add a unit $\Rightarrow \mathbb{C} \oplus E \qquad \mathbb{1} = (1, 0)$

$(1,0)(0,Z) = (0,Z) = (0,Z)(1,0), (z_1, 0)(z_2, 0) = (z_1 z_2, 0)$

$(0, Z_1)(0, Z_2) = (\alpha \langle Z_1, Z_2 \rangle, \beta Z_1 \times Z_2), |\alpha| = |\beta| = 1$

$\Rightarrow \| (0, Z_1) \|^2 \| (0, Z_2) \|^2 = \| (0, Z_1)(0, Z_2) \|^2$

natural to require $\| (z_1, Z_1)(z_2, Z_2) \| = \| (z_1, Z_1) \| \| (z_2, Z_2) \|$

solution :

$(z_1, Z_1)(z_2, Z_2) = (z_1 z_2 - \langle Z_1, Z_2 \rangle, \bar{z}_1 Z_2 + z_2 Z_1 + i Z_1 \times Z_2)$

$\Rightarrow (\bar{z}, -Z)(z, Z) = (z, Z)(\bar{z}, -Z) = \| (z, Z) \|^2 \, \mathbb{1}$

An interpretation of the quark-lepton symmetry

$SU(3)$ is the group of complex-linear automorphisms of $\mathbb{C} \oplus E$ which preserves the above product and E carries the fundamental representation of $SU(3)$ while $\mathbb{C}$ corresponds to the trivial one.

$\Rightarrow E$ being the internal space of a quark, it is "natural" to consider $\mathbb{C}$ as the internal space of the corresponding lepton.

As a real algebra $\mathbb{C} \oplus E$ is 8-dimensional isomorphic to the octonion algebra $\mathbb{O}$.
$SU(3) \subset G_2 = Aut(\mathbb{O})$ is the subgroup preserving i, a given imaginary element of $\mathbb{O}$ with $i^2 = -1$.

The 3 generations

6 flavors of quark-lepton
$(u, \nu_e), (d, e), (c, \nu_\mu), (s, \mu), (t, \nu_\tau), (b, \tau)$
grouped in 3 generations, columns of

generations			
quarks $Q = 2/3$ leptons $Q = 0$	u ν_e	c ν_μ	t ν_τ
quarks $Q = -1/3$ leptons $Q = -1$	d e	s μ	b τ

This sort of "triality" combined with the above interpretation of the quark-lepton symmetry suggest to add over each space-time point the finite quantum system corresponding to the exceptional Jordan algebra.

Quantum geometry - I

J (real vector space) quantum analog of a space of real functions.
Squaring $x \mapsto x^2$ for $x \in J$ such that $x.y = \frac{1}{2}((x+y)^2 - x^2 - y^2)$ is bilinear.
J is *power associative* if by defining $x^{n+1} = x.x^n$
(i) $x^r.x^s = x^{r+s}$

J is *formally real* if one has
(ii) $\sum_{k \in I} (x_k)^2 = 0 \Rightarrow x_k = 0, \ \forall k \in I$

Theorem

A finite-dimensional commutative real algebra J which is power associative and formally real is a Jordan algebra, that is one has

$$x^2.(y.x) = (x^2.y).x, \quad \forall x, y \in J.$$

Such a Jordan algebra is also called an *Euclidean* Jordan algebra.

Quantum geometry - II

Condition (i) and (ii) are necessary for spectral theory (with real spectra).

There are various infinite-dimensional extensions of the above theorem $\Rightarrow$ various formulations of "quantum geometry", etc.

In most cases the Jordan algebras which describe quantum geometries are hermitian (real) subspaces of complex $*$-algebras invariant by the anticommutator $x.y = \frac{1}{2}(xy + xy)$.

$\Rightarrow$ In these cases one can use the noncommutative geometric setting.

Properties of finite-dimensional Euclidean Jordan algebras

Let J be a finite-dimensional Euclidean Jordan algebra.
Then J has a unit $\mathbb{1} \in J$ and $\forall x \in J$

$$x = \sum_{r \in I_x} \lambda_r e_r, \;\; e_r e_s = \delta_{rs} e_r \in J, \;\; \lambda_r \in \mathbb{R}$$

with $\mathbb{1} = \sum_{r \in I_x} e_r$, $card(Ix) \leq n(J) \in \mathbb{N}$

$\Rightarrow$ functional calculus with $\mathbb{R}[X]$.

Furthermore J is a direct sum of a finite number of simple ideals.

Finite-dimensional simple Euclidean Jordan algebras

Theorem

A finite-dimensional simple Euclidean Jordan algebra is isomorphic to one of

$c = 1$	$\mathbb{R}$
$c = 2$	$J_2^n = JSpin_{n+1} = \mathbb{R}\mathbb{1} + \mathbb{R}^{n+1}, \gamma^\mu.\gamma^\nu = \delta^{\mu\nu}\mathbb{1},\ n \geq 1$
$c = 3$	$J_3^1 = H_3(\mathbb{R}),\ J_3^2 = H_3(\mathbb{C}),\ J_3^4 = H_3(\mathbb{H}),\ J_3^8 = H_3(\mathbb{O})$
$c = n \geq 4$	$J_n^1 = H_n(\mathbb{R}),\ J_n^2 = H_n(\mathbb{C}),\ J_n^4 = H_n(\mathbb{H})$

These correspond to the "finite quantum spaces" (i.e. "real function's spaces" over the "quantum spaces").

The "octonionic factors" J_2^8 and J_3^8

The above interpretation which connects the quark-lepton symmetry and the unimodularity of the color group points the attention to the factors

$$J_2^8 = H_2(\mathbb{O}) = JSpin_9$$

$$J_3^8 = H_3(\mathbb{O})$$

together with the subgroups of $\mathrm{Aut}(J_2^8) = Spin_9$ and of $\mathrm{Aut}(J_3^8) = F_4$ which preserve the splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ (and act $\mathbb{C}$-linearly on $\mathbb{C}^3$).

Action of $G_{SM} = SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$ on J_2^8

$Spin_9 = \mathrm{Aut}(J_2^8)$, the subgroup which preserves the splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ is the group G_{SM}. To express this action write

$$\begin{pmatrix} \zeta_1 & x \\ \bar{x} & \zeta_2 \end{pmatrix} \in J_2^8$$

as

$$\begin{pmatrix} \zeta_1 & x \\ \bar{x} & \zeta_2 \end{pmatrix} = \begin{pmatrix} \zeta_1 & z \\ \bar{z} & \zeta_2 \end{pmatrix} + Z \in J_2^2 \oplus \mathbb{C}^3$$

where $x = z + Z \in \mathbb{C} \ \oplus \mathbb{C}^3$ represents $x \in \mathbb{O}$. The action of $(U, V) \in U(3) \times SU(2)$ is then
$H \mapsto VHV^*$, $Z \mapsto UZ$ on $H \oplus Z \in J_2^2 \oplus \mathbb{C}^3$.
This is in fact an action of $SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6 = G_{SM}$.

Action of $SU(3) \times SU(3)/\mathbb{Z}_3$ on J_3^8

$F_4 = Aut(J_3^8)$, the subgroup which preserves the representations of the octonions occurring in the matrix elements of J_3^8 as elements of $\mathbb{C} \oplus \mathbb{C}^3$ is $SU(3) \times SU(3)/\mathbb{Z}_3$. To express this action write

$$\begin{pmatrix} \zeta_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \zeta_2 & x_1 \\ x_2 & \bar{x}_1 & \zeta_3 \end{pmatrix} \in J_3^8$$

as

$$\begin{pmatrix} \zeta_1 & z_3 & \bar{z}_2 \\ \bar{z}_3 & \zeta_2 & z_1 \\ z_2 & \bar{z}_1 & \zeta_3 \end{pmatrix} + (Z_1, Z_2, Z_3) \in J_3^2 \oplus M_3(\mathbb{C})$$

where $x_i = z_i + Z_i \in \mathbb{C} \oplus \mathbb{C}^3$ is the representation of $x_i \in \mathbb{O}$.

The action of $(U, V) \in SU(3) \times SU(3)$ is then
$H \mapsto VHV^*$, $M \mapsto UMV^*$ on $H \oplus M \in J_3^2 \oplus M_3(\mathbb{C})$.

The action of U is the previous action of the color $SU(3)$.

The $\mathbb{Z}_3$-splitting principle

Yokota suggests a simpler formulation (Arxiv: 0909.0431), $i \in \mathbb{C}$ corresponds to $i \in \mathbb{O} \Rightarrow \mathbb{Z}_3 \subset SU(3) \subset G_2 = \mathrm{Aut}(\mathbb{O})$. The $\mathbb{Z}_3$ action on $\mathbb{O}$ is induced by $w \in \mathrm{Aut}(\mathbb{O})$

$$w(z + Z) = z + \omega_1 Z, \quad \omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

One has $w^3 = I$ and this also induces a $\mathbb{Z}_3$-action by automorphism, again denoted w, on J_2^8 (then $w \in Spin_9$) and on J_3^8 (then $w \in F_4$). The corresponding subgroups leaving w invariant are given by

$$(G_2)^w = SU(3)$$

$$(Spin_9)^w = G_{SM} (= SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6)$$

$$(F_4)^w = SU(3) \times SU(3)/\mathbb{Z}_3$$

Exceptional quantum factor

$$J_3^8 = H_3(\mathbb{O}) = \{3 \times 3 \text{ hermitian octonionic matrices}\}$$

- Albert has shown that it cannot be realized as a part stable for the anticommutator of an associative algebra.
- It follows from the theory of Zelmanov that this is the only exceptional factor.

Center

A arbitrary $\mathbb{K}$-algebra; *the center $Z(A)$ of A* is the set of the $z \in A$ such that

$$[x, z] = 0, \ \forall x \in A$$

and

$$[x, y, z] = [x, z, y] = [z, x, y] = 0, \ \forall x, y \in A$$

where $[x, z] = xz - zx$, $[x, y, z] = (xy)z - x(yz)$, $\forall x, y, z \in A$. $Z(A)$ is a commutative associative subalgebra of A.

Lemma

Assume that A is commutative. Then one has :
$z \in Z(A) \Leftrightarrow [x, y, z] = 0, \ \forall x, y \in A.$

Proof.

$[x, z] = 0$ is clear ; $[x, y, z] = -[z, y, x] = 0$ by commutativity and again by commutativity $[x, y, z] - [y, x, z] = 0$ implies $[x, z, y] = 0$. $(\equiv [y, z, x] \equiv -[x, z, y])$.

Derivations

A arbitrary $\mathbb{K}$-algebra ; a linear endomorphism δ of A is a *derivation of* A (into A) if it satisfies

$$\delta(xy) = \delta(x)y + x\delta(y), \quad \forall x, y \in A$$

The space $\mathrm{Der}(A)$ of all derivations of A is a $Z(A)$-module

$$(z\delta)(x) = z\delta(x), \quad \forall z \in Z(A), \forall x \in A$$

$\mathrm{Der}(A)$ is also a Lie algebra

$$[\delta_1, \delta_2](x) = \delta_1(\delta_2(x)) - \delta_2(\delta_1(x)), \forall \delta_1, \delta_2 \in \mathrm{Der}(A), \forall x \in A$$

One has

$$\delta(Z(A)) \subset Z(A), \quad \forall \delta \in \mathrm{Der}(A)$$

and

$$[\delta_1, z\delta_2] = z[\delta_1, \delta_2] + \delta_1(z)\delta_2, \quad \forall \delta_1, \delta_2 \in \mathrm{Der}(A), \quad \forall z \in Z(A)$$

that is $(\mathrm{Der}(A), Z(A))$ is a *Lie Rinehart algebra*

Categories of algebras

$\mathbb{K}$ a fixed field ; all vector spaces, algebras are over $\mathbb{K}$

A category of algebras is a category $\mathcal{C}$ such that its objects are algebras and its morphisms are algebra-homomorphisms.

$\mathcal{C}_{\mathbf{Alg}}$ = category of all algebras and all algebra-homomorphisms

$\mathcal{C}_{\mathbf{Alg}_1}$ = category of unital algebras and unital algebra-homomorphisms

$\mathcal{C}_{\mathbf{Lie}}$ = category of Lie algebras

$\mathcal{C}_{\mathbf{Jord}}$ = category of Jordan algebras

$\mathcal{C}_{\mathbf{Jord}_1}$ = category of unital Jordan algebras

$\mathcal{C}_{A}$ = category of associative algebras

$\mathcal{C}_{A_1}$ = category of unital associative algebras

$\mathcal{C}_{A_Z}$ = category of all associative algebras but morphisms sending centers into centers.

$\mathcal{C}_{\mathbf{Com}}$ = category of commutative algebras, etc.

Bimodules

$\mathcal{C}$ a category of algebras
$A \in \mathcal{C}$ an object, M a vector space such that there are

$$A \otimes M \to M, a \otimes m \mapsto am \text{ and } M \otimes A \to M, m \otimes a \mapsto ma$$

define the product $(A \oplus M) \otimes (A \oplus M) \to A \oplus M$

$$(a \oplus m) \otimes (a' \oplus m') \mapsto aa' \oplus (am' + ma')$$

M is an *A-bimodule for $\mathcal{C}$* if

1. $A \oplus M \in \mathcal{C}$
2. $A \to A \oplus M$ is a morphism of $\mathcal{C}$
3. $A \oplus M \to A$ is a morphism of $\mathcal{C}$

Examples : Bimodules for the above categories (exercise !)

Jordan (bi)-modules I

J Jordan algebra, M vector space with

$$J \otimes M \to M, \quad x \otimes \Phi \mapsto x\Phi$$
$$M \otimes J \to M, \quad \Phi \otimes x \mapsto \Phi x$$

such that the null-split extension $J \oplus M$

$$(x \oplus \Phi)(x' \oplus \Phi') = (xx' \oplus x\Phi' + \Phi x')$$

is again a Jordan algebra then M is a *Jordan bimodule*

$$\Leftrightarrow \begin{cases} (i) \ \ x\Phi = \Phi x \\ (ii) \ x(x^2\Phi) = x^2(x\Phi) \\ (iii) \ (x^2y)\Phi - x^2(y\Phi) = 2((xy)(x\Phi) - x(y(x\Phi))) \end{cases}$$

If J has a unit $\mathbb{1} \in J$, M is *unital* if
$(iiii)$ $\mathbb{1}\Phi = \Phi$

In view of (i), a Jordan bimodule is simply called a *Jordan module*.

Jordan (bi)-modules II

J, M being as before, set $L_x\Phi = x\Phi$ then (ii) reads

$$[L_x, L_{x^2}] = 0$$

while (iii) reads

$$L_{x^2y} - L_{x^2}L_y - 2L_{xy}L_x + 2L_xL_yL_x = 0$$

which is equivalent to

$$\begin{cases} L_{x^3} - 3L_{x^2}L_x + 2L_x^3 = 0 \\ [[L_x, L_y], L_z] + L_{[x,z,y]} = 0 \end{cases}$$

where $[x, z, y] = (xz)y - x(zy)$ is the associator. Condition $(iiii)$ reads

$$L_{\mathbb{1}} = \mathbb{1} (= I_M)$$

Free J-modules and free $Z(J)$-modules I

J a Jordan algebra is canonically a J-module which is unital whenever J has a unit.

Lemma

Let J be a Jordan algebra, E and F be vector spaces and let $\varphi : J \otimes E \to J \otimes F$ be a J-module homomorphism. Then one has

$$\varphi(Z(J) \otimes E) \subset Z(J) \otimes F$$

Proof.

Choose basis (e_α) and (f_λ) for E and F. One has $\varphi(z \otimes e_\alpha) = m_\alpha^\lambda \otimes f_\lambda$ for $z \in Z(J)$ and some $m_\alpha^\lambda \in J$. On the other hand one has $(xy)z = x(yz)$ for any $x, y \in J$
$\Rightarrow \varphi((xy)z \otimes e_\alpha) = (xy)\varphi(z \otimes e_\alpha) = x\varphi(yz \otimes e_\alpha) = x(y\varphi(z \otimes e_\alpha))$
$\Leftrightarrow [x, y, m_\alpha^\lambda] = 0.$ □

Free J-modules and free $Z(J)$-modules II

Proposition

Let J be a unital Jordan algebra. Then $J \otimes E \mapsto Z(J) \otimes E$ and $(\varphi : J \otimes E \to J \otimes F) \mapsto (\varphi \restriction Z(J) \otimes E : Z(J) \otimes E \to Z(J) \otimes F)$ is an isomorphism between the category of free unital J-modules and the category of free unital $Z(J)$-modules.

Indeed from the above lemma $\varphi \restriction (Z(J) \otimes E)$ is a $Z(J)$-module homomorphism of $Z(J) \otimes E$ into $Z(J) \otimes F$.
Conversely any $Z(J)$-module homomorphim $\varphi_0 : Z(J) \otimes E \to Z(J) \otimes F$ extends uniquely by setting $x\varphi_0(\mathbb{1} \otimes E) = \varphi(x \otimes E) \in J \otimes F$ as a J-module homomorphism.

Clifford algebras as JSpin-modules

$C\ell_{n+1}$ is a unital module over $J_2^n = JSpin_{n+1}$ via

$$L_\gamma(A) = \frac{1}{2}(\gamma A + A\gamma)$$

Canonical isomorphism of $\mathbb{Z}_2$-graded vector space (PBW)

$$\Gamma : \wedge\mathbb{R}^{n+1} \to C\ell_{n+1},\ \ \omega_{i_1}\dots{}_{i_p} \mapsto \Gamma(\omega) = \omega_{i_1}\dots{}_{i_p}\,\gamma^{i_1}\dots\gamma^{i_p}$$

$$\Rightarrow C\ell_{n+1} = \oplus_{p=0}^{n+1}\Gamma^p \text{ with } \Gamma^p = \Gamma(\wedge^p\mathbb{R}^{n+1})$$

Proposition

For any integer $p \leq \frac{1}{2}n$, $\Gamma^{2p} \oplus \Gamma^{2p+1}$ is an irreducible J_2^n-submodule of $C\ell_{n+1}$ and if $n+1 = 2m$ then $\Gamma^{2m} \simeq \mathbb{R}$ is also an irreducible submodule of $C\ell_{n+1} = C\ell_{2m}$.

The decomposition of $C\ell_{n+1}$ into irreducible J_2^n-modules follows.

J_3^8-modules

Any Jordan algebra J is canonically a J-module which is unital whenever J has a unit.

The list of the unital irreducible Jordan modules over the finite-dimensional Euclidean Jordan algebras is given in [Jacobson]. In the case of the exceptional algebra one has the following proposition

Proposition

Any unital irreducible J_3^8-module is isomorphic to J_3^8 (as module).

In particular, any finite unital module over J_3^8 is of the form $J_3^8 \otimes E$ for some finite-dimensional real vector space E. Thus the complexified $J_3^8 \otimes \mathbb{C}$ of J_3^8 is a free J_3^8-module.

J_3^8-modules for 2 families by generation

$$J^u = \begin{pmatrix} \alpha_1 & \nu_\tau + t & \bar{\nu}_\mu - c \\ \bar{\nu}_\tau - t & \alpha_2 & \nu_e + u \\ \nu_\mu + c & \bar{\nu}_e - u & \alpha_3 \end{pmatrix}$$

$$J^d = \begin{pmatrix} \beta_1 & \tau + b & \bar{\mu} - s \\ \bar{\tau} - b & \beta_2 & e + d \\ \mu + s & \bar{e} - d & \beta_3 \end{pmatrix}$$

or with the previous representation

$$J^u = \begin{pmatrix} \alpha_1 & \nu_\tau & \bar{\nu}_\mu \\ \bar{\nu}_\tau & \alpha_2 & \nu_e \\ \nu_\mu & \bar{\nu}_e & \alpha_3 \end{pmatrix} + (u, c, t)$$

$$J^d = \begin{pmatrix} \beta_1 & \tau & \bar{\mu} \\ \bar{\tau} & \beta_2 & e \\ \mu & \bar{e} & \beta_3 \end{pmatrix} + (d, s, b)$$

α_i, β_j new Majorana particles $\Rightarrow$ OK for the cancellation of anomalies !

Problem with the $U(1) \times SU(2)$-symmetry

$$q = (z_1, z_2) = z_1 + z_2 j \in \mathbb{H}$$

The subgroup of $\mathrm{Aut}(\mathbb{H})$ which preserves i is $U(1)$

$$z_1 + z_2 j \mapsto z_1 + e^{i\theta} z_2 j$$

$$\begin{pmatrix} \xi_1 & q \\ \bar{q} & \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & z_1 \\ \bar{z}_1 & \xi_2 \end{pmatrix} + z_2 \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \in J_2^4$$

Subgroup of $\mathrm{Aut}(J_2^4)$ which preserves $\cdots = U(1) \times SU(2)$

$$\begin{pmatrix} \xi_1 & z_1 \\ \bar{z}_1 & \xi_2 \end{pmatrix} + z_2 \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \mapsto U \begin{pmatrix} \xi_1 & z_1 \\ \bar{z}_1 & \xi_2 \end{pmatrix} U^* + e^{i\theta} z_2 \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}$$

as

$$U \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} U^* = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}$$

Triality in J_3^8 and the 3 generations

Two ways to describe the underlying triality of J_3^8:

W1 - this triality corresponds to the 3 octonions of the matrix of an element of J_3^8,

W2 - this triality corresponds to the 3 canonical subalgebras of hermitian 2×2 matrices of J_3^8 corresponding themselves to the 3 octonions of W1.

W1 and W2 are equivalent but lead naturally to 2 conceptually different interpretations. In fact $J_2^8 = JSpin_9$ corresponds to a complete generation.

$J_2^8 = JSpin_9$ for one generation

1. $Aut(J_2^8) = Spin_9$
   $G_{SM} = SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$ is ($\simeq$) the subgroup of $Spin_9$ which preserves the splitting $\mathbb{C} \oplus \mathbb{C}^3$ of $\mathbb{O}$ and acts $\mathbb{C}$-linearly on $\mathbb{C}^3$.

2. The $*$-algebra $C\ell_9^c = M_{16}(\mathbb{C}) \oplus M_{16}(\mathbb{C})$ is generated by the relations

$$\begin{cases} \frac{1}{2}(xy + yx) = x \circ y, & \forall x, y \in J_2^8 \\ x^* = x, & \forall x \in J_2^8 \\ \mathbb{1} = \mathbb{1}_{J_2^8} & \end{cases}$$

3. J_2^8 is strongly special.

The correspondence "triality-generation" in J_3^8

$P^2 = P$, primitive= pure state of J_3^8
$\leftrightarrow J_2^8(P) = (\mathbb{1} - P)J_3^8(\mathbb{1} - P) \simeq JSpin_9$
$Aut(J_2^8(P))$= subgroup of F_4 which preserves $P \simeq Spin_9$

P_i diagonal $\leftrightarrow J_2^8(P_i) \leftrightarrow$ generation $i \quad (i \in \{1, 2, 3\})$

$$Aut(J_2^8(P_i)) \cap \frac{SU(3)_c \times SU(3)}{\mathbb{Z}_3} = G_i \simeq \frac{SU(3)_c \times SU(2) \times U(1)}{\mathbb{Z}_6}$$

Each $J_2^8(P_i)$ with the identification $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ has automorphism group $G_i \subset F_4$ isomorphic to the standard model group for one generation

The extended electroweak symmetry $SU(3)_{ew}$

$$J_i = J_2^8(P_i), \;\; Aut(J_i) \simeq Spin_9$$

$$SU(3)_c \times SU(3)/\mathbb{Z}_3 \subset F_4 = \mathrm{Aut}(J_3^8)$$

$$Aut(J_i) \subset F_4$$

$$SU(3)_c \times SU(3)/\mathbb{Z}_3 \cap Aut(J_i) = G_i$$

$$G_i \simeq SU(3)_c \times SU(2) \times U(1)/\mathbb{Z}_6$$

$\Rightarrow$ The second $SU(3)$ project onto the electroweak symmetry for each generation .
This $SU(3)$ will be called extented electroweak symmetry and denoted by $SU(3)_{ew}$.
Internal symmetry $SU(3)_c \times SU(3)_{ew}/\mathbb{Z}_3 \subset F_4$

Differential graded Jordan algebras

$\Omega = \oplus_{n\in\mathbb{N}}\Omega^n$ which is a Jordan superalgebra (for $\mathbb{N}/2\mathbb{N}$)

$$ab = (-1)^{|a||b|} ba \text{ for } a \in \Omega^{|a|}, b \in \Omega^{|b|}$$

and graded Jordan identity

$$(-1)^{|a||c|}[L_{ab}, L_c]\text{gr} + (-1)^{|b||a|}[L_{bc}, L_a]\text{gr} + (-1)^{|c||b|}[L_{ca}, L_b]\text{gr} = 0$$

with a differential d

$$d^2 = 0$$

$$d\Omega^n \subset \Omega^{n+1}$$

$$d(ab) = d(a)b + (-1)^{|a|} a d(b)$$

Model for algebras of differential forms on quantum spaces.
Differential calculus over J = differential graded Jordan algebra Ω with $\Omega^0 = J$.

Derivation-based differential calculus

J unital Jordan algebra with center $Z(J)$

$$\Omega^n_{\mathrm{Der}}(J) = \mathrm{Hom}_{Z(J)}(\wedge^n_{Z(J)}\mathrm{Der}(J), J)$$

$\Omega_{\mathrm{Der}}(J) = \oplus_n \Omega^n_{\mathrm{Der}}(J)$ is canonically a differential graded Jordan algebra with

$$\begin{aligned} d\omega(X_0, \cdots, X_n) &= \textstyle\sum_{0 \leq k \leq n} (-1)^k X_k \, \omega(X_0, \overset{k}{\overset{\vee}{\cdots}}, X_n) \\ &+ \textstyle\sum_{0 \leq r < s \leq n} (-1)^{r+s} \, \omega([X_r, X_s], X_0, \overset{r}{\overset{\vee}{\cdots}} \cdots \overset{s}{\overset{\vee}{\cdots}}, X_n) \end{aligned}$$

referred to as the *derivation-based differential calculus over J*.

Universal property for J_3^8

Theorem

Any homomorphism φ of unital Jordan algebra of J_3^8 into the Jordan subalgebra Ω^0 of a unital differential graded Jordan algebra $\Omega = \bigoplus \Omega^n$ has a unique extension $\tilde{\varphi} : \Omega_{Der}(J_3^8) \to \Omega$ as a homomorphism of differential graded Jordan algebras.

$\Omega_{\mathrm{Der}}(J_3^8) = J_3^8 \otimes \wedge \mathfrak{f}_4^*$ with the Chevalley-Eilenberg differential.

Derivation-based connections I

J= unital Jordan algebra, center=$Z(J)$, M= unital J-module.
A *derivation-based connection on M is a linear mapping* $X \mapsto \nabla_X$ of $\mathrm{Der}(J)$ into $\mathcal{L}(M)$ such that for $x \in J$ and $z \in Z(J)$

$$\begin{cases} \nabla_X(xm) = X(x)m + x\nabla_X(m) \\ \\ \nabla_{zX}(m) = z\nabla_X(m) \end{cases}$$

curvature of ∇

$$R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

$$\begin{cases} R_{X,Y}(xm) = xR_{X,Y}(m) \\ \\ R_{zX,Y}(m) = zR_{X,Y}(m) \end{cases}$$

$\mathfrak{g} \subset \mathrm{Der}(J)$, Lie subalgebra and $Z(J)$-submodule
$\Rightarrow$ *derivation-based* $\mathfrak{g}$*-connection on M* (by restriction).

Derivation-based connections II

$\Omega_{\mathrm{Der}}(M) = \mathrm{Hom}_{Z(J)}(\wedge \mathrm{Der}(J), M)$, ∇ linear endomorphism of $\Omega_{\mathrm{Der}}(M)$ such that

$$\begin{cases} \nabla(\Omega^n_{\mathrm{Der}}(M)) \subset \Omega^{n+1}_{\mathrm{Der}}(M) \\ \\ \nabla(\omega\Phi) = d(\omega)\Phi + (-1)^m \omega \nabla(\Phi) \end{cases}$$

for any $m, n \in \mathbb{N}$, $\omega \in \Omega^m_{\mathrm{Der}}(J)$ and $\Phi \in \Omega_{\mathrm{Der}}(M)$.
$\Rightarrow$ *curvature* ∇^2

$$\nabla^2(\omega\Phi) = \omega\nabla^2(\Phi)$$

Let ∇ be such a connection and define $\nabla_X(m)$ as in I by

$$\nabla_X(m) = \nabla(m)(X)$$

for $m \in M = \Omega^0_{\mathrm{Der}}(M)$, $X \in \mathrm{Der}(J)$
Conversely, ∇ as in I $\Rightarrow$ ∇ as here with

$$\begin{aligned} \nabla(\Phi)(X_0, \cdots, X_n) &= \textstyle\sum_{p=0}^{n} (-1)^p \nabla_{X_p}(\Phi(X_0, \overset{p}{\overset{\vee}{\cdots}}, X_n)) \\ &+ \textstyle\sum_{0\leq r<s\leq n} (-1)^{r+s} \, \Phi([X_r, X_s], X_0, \overset{r}{\overset{\vee}{\cdots}} \overset{s}{\overset{\vee}{\cdots}}, X_n) \end{aligned}$$

General connection

$\Omega = \oplus \Omega^n$ = differential graded Jordan algebra, $\Gamma = \oplus \Gamma^n$ graded module over Ω.

A connection on Γ, is a linear endomorphism of Γ satisfying

$$\begin{cases} \nabla(\Gamma^n) \subset \Gamma^{n+1} \\ \nabla(\omega\Phi) = d(\omega)\Phi + (-1)^m \omega\nabla(\Phi) \end{cases}$$

for $\omega \in \Omega^n$, $\Phi \in \Gamma \Rightarrow$

$$\nabla^2(\omega\Phi) = \omega\nabla^2(\Phi)$$

∇^2 homogeneous Ω-module homomorphism of degree 2 is *the curvature of* ∇.

$$\nabla\nabla^2 = \nabla^2\nabla$$

is *the Bianchi identity of* ∇.

Connections on free modules

J unital Jordan algebra, $M = J \otimes E$ free J-module, Ω differential calculus over J such that Ω is generated by $J = \Omega^0$ as differential graded Jordan algebra.
$\nabla : \Omega \otimes E \to \Omega \otimes E$ connection induced by $\nabla : J \otimes E \to \Omega^1 \otimes E$.

Proposition

1. $\overset{0}{\nabla} = d \otimes I_E : J \otimes E \to \Omega^1 \otimes E$ *defines a flat connection on* M *which is gauge invariant whenever the center of* J *is trivial.*
2. *Any other* Ω*-connection* ∇ *on* M *is defined by* $\nabla = \overset{0}{\nabla} + A : J \otimes E \to \Omega^1 \otimes E$ *where* A *is a* J*-module homomorphism of* $J \otimes E$ *into* $\Omega^1 \otimes E$.
3. *If* $\Omega = \Omega_{Der}$ *(i.e. for derivation-based connections) one has* $(\nabla^2)(X, Y) = R_{X,Y} =$ $XA_Y - YA_X + [A_X, A_Y] - A_{[X,Y]}, \ \forall X, Y \in Der(J)$.

Connections on Clifford algebras as JSpin-modules

Any $X \in \mathrm{Der}(J_2^n) = \mathfrak{so}(n+1)$ has an extension as derivation of the Clifford algebra $C\ell_{n+1} \Rightarrow$

$$\overset{0}{\nabla}_X \Gamma = X\Gamma$$

defines a derivation-based connection which is flat

$$\overset{0}{R}_{XY} = [\overset{0}{\nabla}_X, \overset{0}{\nabla}_Y] - \overset{0}{\nabla}_{[X,Y]} \equiv 0$$

Any other connection (for Ω_{Der}) is of the form

$$\nabla_X = \overset{0}{\nabla}_X + A_X$$

where A_X is a J_2^n-module endormorphism of $C\ell_{n+1}$ which depends linearly of $X \in \mathfrak{so}(n+1)$.

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