Title: A new look at symmetries of 3d gravity

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Abstract: I will review the analysis of boundary symmetries in first order 3d gravity, and explain how the study of the boundary current algebra and the Sugawara construction actually leads to two dual notions of diffeomorphism charges. This provides a new understanding of the relationship between the second order and first order formulations, and of the existence of finite distance asymptotic symmetries (as strange as this sounds) in topological theories. This analysis is performed on the most general theory of first order 3d gravity, which also enables to understand the duality between curvature and torsion, as well as the relationship with chiral and massive gravity.

Symmetries and dualities in 3d gravity

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Based on:
MG, Goeller, Merino, [2011.09873], JHEP
MG, Goeller, [2012.05263]

Motivations

Quantum gravity

- Central role played by symmetries, but which ones?
- What are the fundamental degrees of freedom and where do they live?
- Many pieces of answers from different approaches (AdS/CFT, holography, LQG, . . .)

3d gravity

- Topological theory, no local (bulk) degrees of freedom
- Good place to understand classical symmetry algebras and boundary dynamics [Brown, Henneaux | Ashtekar, Bicak, Schmidt | Barnich, Compère, Gomberoff, Gonzáles]
- Reasonably well-understood quantization [Witten | Ponzano, Regge | Barrett | Freidel, Louapre | Noui, Perez]
- Recent work on finite distance holography via spin foams [Dittrich, Goeller, Livine, Riello]
- Interesting features: coupling to point particles, quantum groups, massive gravity, . . . [Deser, Jackiw, Freidel, Dupuis, Girelli, Bais, Meusburger, Schroers, Noui, Perez, . . .]

Our goal: 3d gravity revisited

- Study general first order Lagrangian
- Relate finite distance symmetries to asymptotic ones
- Reveal interesting duality

Preliminaries

Index-free notation

- Independent connection and triad variables with curvature and torsion

$$F = d\omega + \frac{1}{2}[\omega \wedge \omega] \qquad d_\omega e = de + [\omega \wedge e]$$

- Implicit internal indices

$$e \wedge F = \eta_{ij} e^i \wedge F^j \qquad e \wedge [e \wedge e] = \varepsilon_{ijk} e^i \wedge e^j \wedge e^k$$

- E.g. Einstein–Hilbert Lagrangian

$$\sqrt{-g}\, R = 2e \wedge F = \varepsilon^{\mu\nu\rho} \eta_{ij} e^i_\mu F^j_{\nu\rho}$$

Covariant phase space

- Starting point is the pre-symplectic potential

$$\delta L = \text{EOM} \wedge \delta\Phi + d\theta$$

- It determines the symplectic structure

$$\Omega(\delta, \delta) = \int_\Sigma \delta\theta$$

- The Hamiltonian generator of a symmetry δ_α is $\not{\delta}\mathcal{H}(\alpha) = \Omega(\delta_\alpha, \delta)$
- Their Poisson bracket is $\big\{\mathcal{H}(\alpha), \mathcal{H}(\beta)\big\} = \Omega(\delta_\alpha, \delta_\beta)$

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3d gravity revisited

Most general first order theory

- Mielke–Baekler Lagrangian [Mielke, Baekler, Hehl, Blagojevic, Cvetkovic]

$$L = \frac{\sigma_0}{3} e\wedge[e\wedge e] + 2\sigma_1 e\wedge F + \sigma_2 \omega\wedge\left(d\omega + \frac{1}{3}[\omega\wedge\omega]\right) + \sigma_3 e\wedge d_\omega e$$
$$= \text{volume} + \text{Einstein–Hilbert} + \text{Chern–Simons} + \text{torsion}$$

- Variation

$$\delta L = d\Big(2\sigma_1 \delta\omega\wedge e + \sigma_2 \delta\omega\wedge\omega + \sigma_3 \delta e\wedge e\Big)$$
$$+ \delta e\wedge\Big(2\sigma_1 F + 2\sigma_3 d_\omega e + \sigma_0 [e\wedge e]\Big)$$
$$+ \delta\omega\wedge\Big(2\sigma_2 F + 2\sigma_1 d_\omega e + \sigma_3 [e\wedge e]\Big)$$

- Potential θ
- EOMs nicely rewritten as

$$2F + p[e\wedge e] \approx 0 \qquad\qquad 2d_\omega e + q[e\wedge e] \approx 0$$

- "Internal" curvature p and torsion q defined by $\sigma_0 = p\sigma_1 + q\sigma_3$ and $\sigma_3 = p\sigma_2 + q\sigma_1$
- Torsion equation solved by $\omega = \Gamma - qe/2$
- Second order EOM

$$R + \lambda[e\wedge e] = 0 \qquad\qquad \lambda = p + \frac{q^2}{4}$$

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Symmetries

- Lorentz

$$\delta^j_\alpha e = [e, \alpha] \qquad \delta^j_\alpha \omega = d_\omega \alpha$$

- Translations

$$\delta^t_\phi e = d_\omega \phi + q[e, \phi] \qquad \delta^t_\phi \omega = p[e, \phi]$$

- Diffeomorphisms $\delta^d_\xi = \mathcal{L}_\xi$ can be rewritten on-shell as field-dependent gauge transformations

$$\begin{aligned}\delta^d_\xi e &= d(\xi\lrcorner e) + \xi\lrcorner(de)\\ &= \delta^j_{\xi\lrcorner\omega} e + \delta^t_{\xi\lrcorner e} e + \frac{1}{2}\xi\lrcorner(2d_\omega e + q[e\wedge e])\\ \delta^d_\xi \omega &= \delta^j_{\xi\lrcorner\omega}\omega + \delta^t_{\xi\lrcorner e}\omega + \frac{1}{2}\xi\lrcorner(2F + p[e\wedge e])\end{aligned}$$

- Commutation relations

$$[\delta^j_\alpha, \delta^t_\phi] = \delta^t_{[\alpha,\phi]} \qquad [\delta^j_\alpha, \delta^j_\beta] = \delta^j_{[\alpha,\beta]} \qquad [\delta^t_\phi, \delta^t_\chi] = p\,\delta^j_{[\phi,\chi]} + q\,\delta^t_{[\phi,\chi]}$$

- Introduce the 6-dimensional algebra

$$[J_i, T_j] = \varepsilon_{ij}{}^k T_k \qquad [J_i, J_j] = \varepsilon_{ij}{}^k J_k \qquad [T_i, T_j] = \varepsilon_{ij}{}^k (pJ_k + qT_k)$$

- Casimirs

$$\mathcal{C}_1 = pJ^2 + T^2 \qquad \mathcal{C}_2 = JT + TJ - qJ^2$$

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Charges and algebra

- Contraction with symplectic structure gives e.g. $\delta\mathcal{J}(\alpha) = \Omega(\delta_\alpha, \delta)$
- Lorentz and translation charges

$$\mathcal{J}(\alpha) \approx 2\oint_S \alpha(\sigma_1 e + \sigma_2 \omega) \qquad \mathcal{T}(\phi) \approx 2\oint_S \phi(\sigma_1 \omega + \sigma_3 e)$$

- Infinite-dimensional centrally-extended current algebra

$$\{\mathcal{J}(\alpha), \mathcal{T}(\phi)\} = \mathcal{T}([\alpha, \phi]) - 2\sigma_1 \oint_S \alpha d\phi$$

$$\{\mathcal{J}(\alpha), \mathcal{J}(\beta)\} = \mathcal{J}([\alpha, \beta]) - 2\sigma_2 \oint_S \alpha d\beta$$

$$\{\mathcal{T}(\phi), \mathcal{T}(\chi)\} = p\mathcal{J}([\phi, \chi]) + q\mathcal{T}([\phi, \chi]) - 2\sigma_3 \oint_S \phi d\chi$$

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Diffeomorphisms

- In terms of the field-dependent gauge transformations we have

$$\not\delta\mathcal{D}(\xi) = \not\delta\mathcal{J}(\xi \lrcorner\, \omega) + \not\delta\mathcal{T}(\xi \lrcorner\, e)$$

- Calculation gives

$$\not\delta\mathcal{D}(\xi) \approx \oint_S \delta\big(2\sigma_1(\xi \lrcorner\, \omega)e + \sigma_2(\xi \lrcorner\, \omega)\omega + \sigma_3(\xi \lrcorner\, e)e\big) - \xi \lrcorner\, \theta$$

- Contains integrable and non-integrable piece
- Integrability can be achieved with either
  - tangent vector fields $\xi^\mu = (\xi^t, \xi^r, \xi^\varphi) = (0, 0, \xi^\varphi)$
  - boundary conditions on the dynamical fields

BTZ black hole

- Line element

$$ds^2 = -\left(-m + \lambda r^2 + \frac{j^2}{4r^2}\right)dt^2 + \left(-m + \lambda r^2 + \frac{j^2}{4r^2}\right)^{-1} dr^2 + r^2\left(d\varphi - \frac{j}{2r^2}dt\right)^2$$

- Vector field $\xi^\mu = (\xi^t, 0, \xi^\varphi)$ gives

$$\mathcal{D}(\xi) = \xi^t(c_1 m - \lambda c_2 j) + \xi^\varphi(c_2 m - c_1 j) \qquad c_1 = 2\sigma_1 - q\sigma_2 \qquad c_2 = 2\sigma_2$$

- Entropy

$$S = 4\pi^2(c_1 r_+ - c_2 r_- \sqrt{\lambda})$$

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Dual diffeomorphisms

- Focus now on tangential vector fields ξ
- Diffeos are a particular integrable combination of field-dependent gauge transformations
- Are there other such integrable generators? Yes, the dual diffeomorphisms

$$\mathcal{D}^*(\xi) = p\mathcal{J}(\xi \lrcorner e) + q\mathcal{T}(\xi \lrcorner e) + \mathcal{T}(\xi \lrcorner \omega)$$

- Inaccessible from Chern–Simons theory since there we can only write $\mathcal{D}(\xi) = \mathcal{F}(\xi \lrcorner A)$
- Transformations

$$\delta^{d^*}_{\xi} \begin{pmatrix} e \\ \omega \end{pmatrix} \approx \begin{pmatrix} \pounds_\xi \omega + q \pounds_\xi e \\ p \pounds_\xi e \end{pmatrix}$$

- Surprising but true!

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Sugawara construction

- Virasoro (or Witt) algebra from current algebra
- Build quadratic generators starting from the linear ones $(\mathcal{J}^i_n, \mathcal{T}^i_n)$ in Fourier
- Consider the building blocks

$$\mathcal{Q}^1_n = 2\sum_{k\in\mathbb{Z}} \mathcal{J}^i_{n+k}\mathcal{T}^i_{-k} \qquad \mathcal{Q}^2_n = \sum_{k\in\mathbb{Z}} \mathcal{T}^i_{n+k}\mathcal{T}^i_{-k} \qquad \mathcal{Q}^3_n = \sum_{k\in\mathbb{Z}} \mathcal{J}^i_{n+k}\mathcal{J}^i_{-k}$$

- With this we can reconstruct

$$\begin{aligned}\mathcal{D}_n &= \sigma_1\mathcal{Q}^1_n - \sigma_2\mathcal{Q}^2_n - \sigma_3\mathcal{Q}^3_n \\ \mathcal{D}^*_n &= p\,\sigma_1\mathcal{Q}^3_n - p\,\sigma_2\mathcal{Q}^1_n + (\sigma_1 - q\,\sigma_2)\mathcal{Q}^2_n\end{aligned}$$

- These generators also uniquely arise from the Casimirs as

$$\begin{aligned}\mathcal{D} &= \sigma_2\widetilde{\mathcal{C}}_1 - \sigma_1\widetilde{\mathcal{C}}_2 \\ \mathcal{D}^* &= (\sigma_1 - q\,\sigma_2)\widetilde{\mathcal{C}}_1 - p\,\sigma_2\widetilde{\mathcal{C}}_2\end{aligned}$$

- The dual diffeomorphisms are therefore as natural as the “usual” diffeomorphisms

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Finite distance symmetry algebra

- Introduce $\mathcal{A} = \mathcal{D}^* - q\mathcal{D}/2$
- $\mathfrak{witt} \oplus \mathfrak{witt}$ algebra

$$\begin{aligned}
\{\mathcal{A}(\xi), \mathcal{D}(\zeta)\} &= -\mathcal{A}([\xi, \zeta]) \\
\{\mathcal{D}(\xi), \mathcal{D}(\zeta)\} &= -\mathcal{D}([\xi, \zeta]) \\
\{\mathcal{A}(\xi), \mathcal{A}(\zeta)\} &= -\lambda\mathcal{D}([\xi, \zeta])
\end{aligned}$$

- In the flat limit $\lambda(p, q) = 0$ this is centreless $\mathfrak{bms}_3$
- Asymptotic symmetry algebras recovered at finite distance
- No central extensions here because the vector fields are tangent
- Central extensions can be introduced with either
  - twisted Sugawara construction, which amounts to $\xi^r \propto \partial_\varphi \xi^\varphi(\varphi)$
  - more general vector fields and "boundary conditions"

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Asymptotic symmetries

- With $\partial_u(\mathcal{M},\mathcal{N})= (\lambda\mathcal{N}',\mathcal{M}')$, a family of metrics solution to $R_{\mu\nu}+2\lambda g_{\mu\nu}=0$ is

$$ds^2=\Big(\mathcal{M}(u,\varphi)-\lambda r^2\Big)du^2-2du\,dr+\mathcal{N}(u,\varphi)du\,d\varphi+r^2d\varphi^2$$

- Preserved by the vector fields

$$\xi^u=f \qquad \xi^r=f''-rg'-\mathcal{N}\frac{f'}{2r} \qquad \xi^\varphi=g-\frac{f'}{r}$$

- Parameters changing as

$$\delta^d_\xi\mathcal{M}=g\mathcal{M}'+2\mathcal{M}g'-2g'''+\lambda\big(2\mathcal{N}f'+f\mathcal{N}'\big)$$
$$\delta^d_\xi\mathcal{N}=f\mathcal{M}'+2\mathcal{M}f'-2f'''+2\mathcal{N}g'+g\mathcal{N}'$$

- Diffeomorphism charge (in first order variables)[1]

$$\mathcal{D}_B(\xi)=\oint_S f(c_1\mathcal{M}+\lambda c_2\mathcal{N})+g(c_1\mathcal{N}+c_2\mathcal{M})+\mathcal{O}(r^{-1})$$

- Exact $\mathcal{O}(r^{-1})$ contribution can be removed using the Kosman derivative $\mathcal{K}_\xi=\mathcal{L}_\xi+\delta^j_\rho$ with [Jacobson, Mohd, Prabhu, De Paoli, Oliveri, Speziale]

$$\mathcal{K}_{\xi_{\text{Killing}}}e=0 \qquad \rho^i(\xi,e)=-\frac{1}{2}\varepsilon^i{}_{jk}g^{\mu\nu}e^j_\mu\mathcal{L}_\xi e^k_\nu$$

[1] Recall $c_1=2\sigma_1-q\sigma_2$ and $c_2=2\sigma_2$

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Asymptotic symmetry algebra

- Write $\mathcal{D}_B(\xi) = \mathcal{E}(f) + \mathcal{L}(g) + \mathcal{O}(r^{-1})$
- Centrally-extended double Virasoro, reducing to $\mathfrak{bms}_3$ in the flat limit

$$\{\mathcal{E}(f), \mathcal{L}(g)\} = -\mathcal{E}([f,g]) + c_1 \oint_S f g'''$$

$$\{\mathcal{L}(g_1), \mathcal{L}(g_2)\} = -\mathcal{L}([g_1,g_2]) + c_2 \oint_S g_1 g_2'''$$

$$\{\mathcal{E}(f_1), \mathcal{E}(f_2)\} = -\lambda \left(\mathcal{L}([f_1,f_2]) - c_2 \oint_S f_1 f_2''' \right)$$

- Virasoro central charges $c^{\pm} = 6\big(c_1/\sqrt{|\lambda|} \pm c_2\big)$
- Can actually be defined at any finite distance r in the bulk: symplectic symmetries [Compère, Donnay, Lambert, Schulgin, Mao, Seraj, Sheikh-Jabbari]
- $\mathcal{D}_B(\xi)$ algebra has two factors and central extensions because ξ contributes in two directions
- Before with ξ^φ we needed $\mathcal{D}$ and $\mathcal{A}$ to get the same (centreless) algebra
- Suggests a relationship $\mathcal{D} \sim \mathcal{L}$ and $\mathcal{A} \sim \mathcal{E}$

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Relating finite distance and asymptotic charges

- Evaluating $\mathcal{D}$ and $\mathcal{A}$ in Bondi gauge on $\xi_{\parallel}=(0,0,h)$ reveals that

$$\mathcal{D}(\xi_{\parallel})=\mathcal{L}(h) \qquad \mathcal{A}(\xi_{\parallel})=\mathcal{E}(h)$$

- The u component of $\mathcal{D}_\mathsf{B}$ can be written as the φ component of the dual diffeomorphism $\mathcal{A}$
- In terms of symmetry action on the fields, one can indeed check that $\delta^\mathsf{a}_{(0,0,h)}=\delta^\mathsf{d}_{(h,0,0)}$
- Akin to a "change of slicing" where the u direction is made tangential and thereby integrable
- WIP to study this in more general (non-Dirichlet) Bondi and Fefferman–Graham gauges [Alessio, Barnich, Ciambelli, Mao, Marteau, Petropoulos, Ruzziconi, Adami, Sheikh-Jabbari, Taghiloo, Yavartanoo, Zwikel, . . .]

3d gravity revisited

Second order equations of motion

- Decompose connection as $\omega = \gamma + k$
- EOMs

$$2F + p[e \wedge e] \approx 0 \qquad\qquad 2d_\omega e + q[e \wedge e] \approx 0$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$2R + 2d_\gamma k + [k \wedge k] + p[e \wedge e] \approx 0 \qquad 2d_\gamma e + 2[k \wedge e] + q[e \wedge e] \approx 0$$

- Torsion equation solved by

$$k \approx -\frac{q}{2} e - T \qquad\qquad [T \wedge e] = d_\gamma e \Rightarrow T = E(d_\gamma e)$$

- Second order EOM

$$2R - 2d_\gamma T + [T \wedge T] + \lambda[e \wedge e] \approx 0$$

On-shell Lagrangians

- Riemannian metric sector with $\gamma = \Gamma$

$$L_{\mathrm{TMG}}[e] = c_1 \Big(L_{\mathrm{HP}}[e] + \lambda L_{\mathrm{V}}[e] \Big) + c_2 L_{\mathrm{CS}}[\Gamma]$$

- Teleparallel sector with $\gamma = 0$

$$L_{\mathrm{TTMG}}[e] = c_1 \Big(e \wedge dT + \lambda L_{\mathrm{V}}[e] \Big) + c_2 L_{\mathrm{CS}}[T]$$

3d massive gravity

Massive gravity

- Complicated in 4d (Boulware–Deser ghost) but can be done (requires background fields) [Ogievetsky, Polubarinov | de Rham, Gabadadze, Tolley]
- May diff-invariant models can be written in 3d: TMG, NMG, GMG, MMG, GMMG, EMG, ... [Deser, Jackiw, Templeton, Bergshoeff, Hohm, Routh, Townsend, Merbis, Zhang, ...]
- In 3d, a U(1) Chern–Simons term gives a gauge-invariant mass to photons
- Similarly, a gravitational Chern–Simons term gives rise to a massive propagating mode

$$L_{\text{TMG}}[e] = c_1 L_{\text{HP}}[e] + c_2 L_{\text{CS}}[\Gamma]$$

- This suggests that there exists a teleparallel topologically massive gravity dual described by

$$L_{\text{TTMG}}[e] = c_1 e \wedge dT + c_2 L_{\text{CS}}[T]$$

Simple massive gravity in 1st order variables [MG, Noui, '18, '19]

- Unleash a massive mode by breaking Lorentz invariance of

$$L = e \wedge d\omega + \frac{\mu_0}{3} e \wedge [e \wedge e] + \mu_1 \omega \wedge [e \wedge e] + \mu_2 e \wedge [\omega \wedge \omega] + \frac{\mu_3}{3} \omega \wedge [\omega \wedge \omega]$$

- Topological theory when $\mu_0 \mu_3 = \mu_1 \mu_2$, and with a single degree of freedom otherwise
- Same massive theory obtained by "detuning" any of the derivatives in initial Lagrangian

Perspectives

We have shown that

- Interesting general new look at classical 3d gravity
- Rich symmetry structures are accessible already at finite distance

Lots of interesting prospects

- Link with second order metric formulation
- Quantization of the model and fate of the $(\mathfrak{p}, \mathfrak{q})$ duality
- Coupling to point particles
- Holography
- Boundary dynamics
- Quantum groups
- More general 3d gauges
- Extension to 4d
- TTMG

Thanks!