Title: Entangled subspaces and generic local state discrimination with pre-shared entanglement

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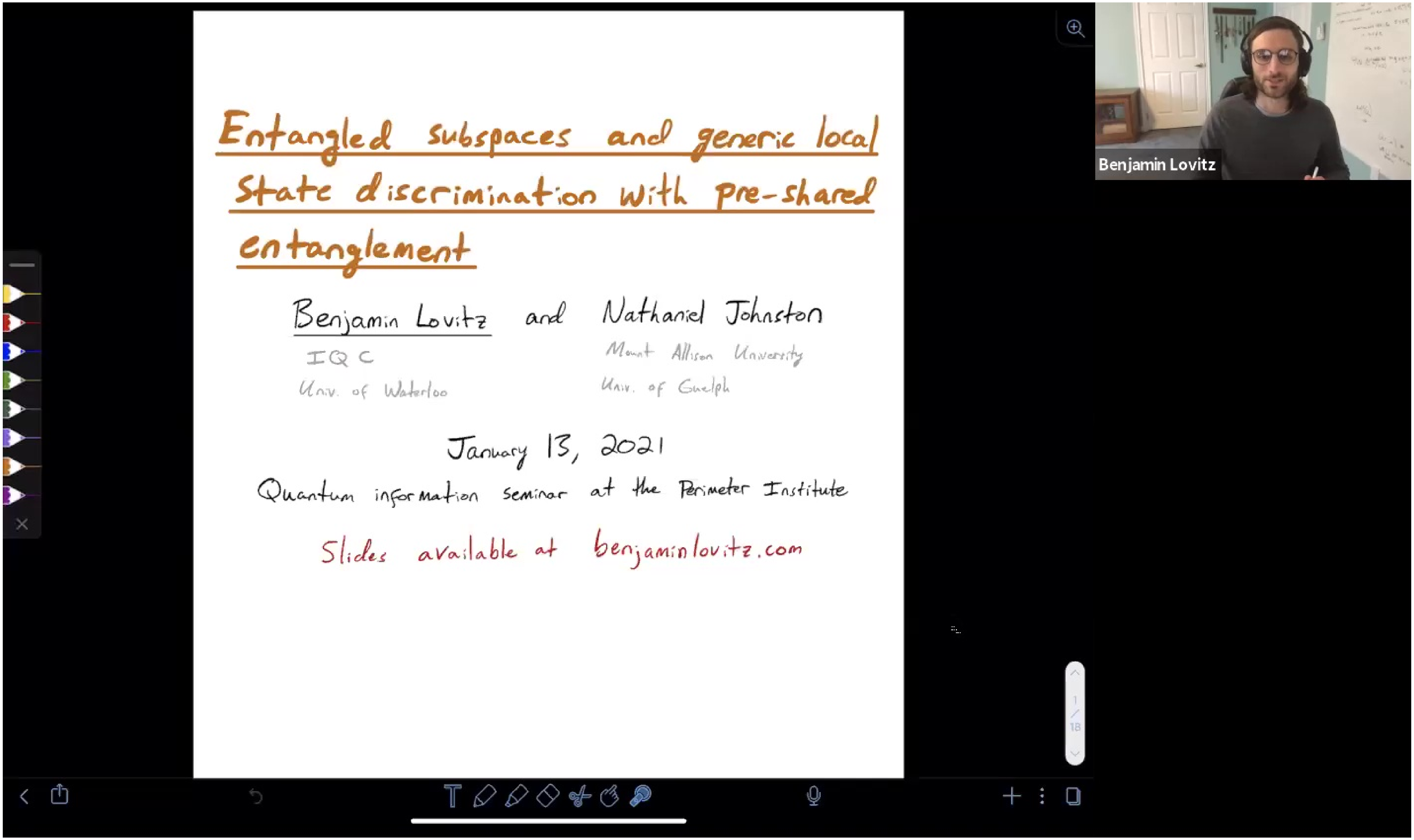
Abstract: Walgate and Scott have determined the maximum number of generic pure quantum states in multipartite space that can be unambiguously discriminated by an LOCC measurement [Journal of Physics A: Mathematical and Theoretical, 41:375305, 08 2008]. In this work, we determine this number in a more general setting in which the local parties have access to pre-shared entanglement in the form of a resource state. We find that, for an arbitrary pure resource state, this number is equal to the Krull dimension of (the closure of) the set of pure states obtainable from the resource state by SLOCC. This dimension is known for several resource states, for example the GHZ state.

Local state discrimination is closely related to the topic of entangled subspaces, which we study in its own right. We introduce r-entangled subspaces, which naturally generalize previously studied spaces to higher multipartite entanglement. We use algebraic geometric methods to determine the maximum dimension of an r-entangled subspace, and present novel explicit constructions of such spaces. We obtain similar results for symmetric and antisymmetric r-entangled subspaces, which correspond to entangled subspaces of bosonic and fermionic systems, respectively.


Entangled subspaces and generic local State discrimination with pre-shared entanglement
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Quantum information seminar at the Perimeter Institute
Slides available at benjaminlovitz.com
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A pinch of algebraic geometry

From linear algebra to algebraic geometry

Fundamental objects of linear algebra: linear spaces.

Proposition: Every linear subspace $S \subseteq \mathbb{C}^N$ is equal to the kernel of a linear map.

Proof: Let $S^\perp$ be the orthogonal complement of S, and let $\{|v_1\rangle, \dots, |v_k\rangle\}$ be a basis for $S^\perp$. Then $S = \ker(T)$, where $T = \begin{pmatrix} \langle v_1| \\ | \\ \langle v_k| \end{pmatrix}$. $\blacksquare$

Notation: $\ker(T) = V(\langle v_1|, \dots, \langle v_k|) \subseteq \mathbb{C}^N$

= "The common zero locus of the linear functionals $\langle v_1|, \dots, \langle v_k|$"

More generally, for polynomials $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_N]$,

$$V(f_1, \dots, f_k) \underset{\text{def}}{=} \{|u\rangle \in \mathbb{C}^N \mid f_i(|u\rangle) = 0 \ \forall i = 1, \dots, k\}.$$

$$\subseteq \mathbb{C}^N$$

Fundamental object of algebraic geometry: Affine varieties

Benjamin Lovitz

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$$\subseteq \mathbb{C}^N$$

Fundamental object of algebraic geometry: Affine Varieties

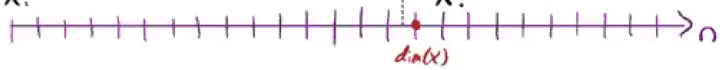
- A quasi-affine variety is the set difference of two affine varieties
- A constructible set is the union of quasi-affine varieties
- A morphism of algebraic varieties is a polynomial map between algebraic varieties.

A useful characterization of dimension

Let $X \subseteq \mathbb{C}^N$ be a constructible set that forms a cone over $\mathbb{C}$. We say a linear subspace $W \subseteq \mathbb{C}^N$ *trivially intersects* X if $X \cap W = \{0\}$.

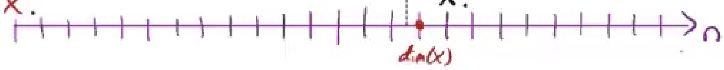
Theorem / Definition: $\dim(X) = \dim(\bar{X})$, and

Almost all linear subspaces of dimension $N-n$ non-trivially intersect X. | Almost all linear subspaces of dimension $N-n$ trivially intersect $\bar{X}$.



Furthermore,

All linear subspaces of dimension $N-n$ non-trivially intersect $\bar{X}$. | Almost all linear subspaces of dimension $N-n$ trivially intersect $\bar{X}$.



Exercise: Prove this when $X \subseteq \mathbb{C}^N$ is a linear subspace.

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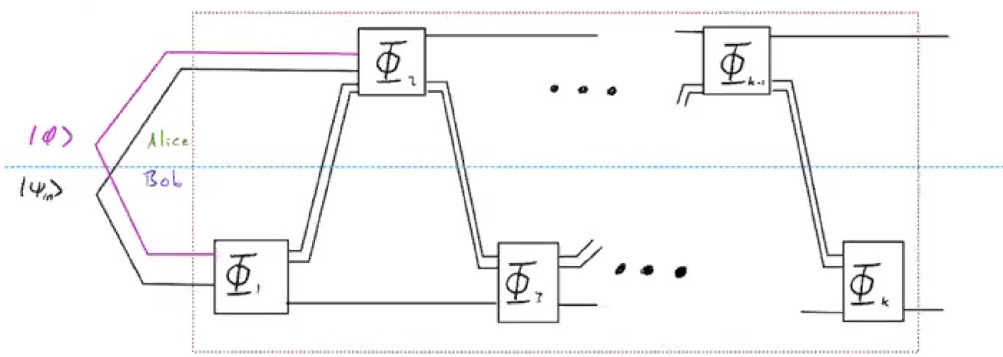
Resource states for two-party LOCC

Question: Suppose Alice and Bob wish to perform some task on an input state $|\psi_{in}\rangle \in A \otimes B$, and are allowed to use pre-shared entanglement in the form of a resource state $|\varphi\rangle \in \mathbb{C}^r \otimes \mathbb{C}^r$. What is the most useful state $|\varphi\rangle$ for them to share?



LOCC Channel

Answer: The state $|\varphi\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} |ii\rangle$ is best, as it can be converted into any other state in $\mathbb{C}^r \otimes \mathbb{C}^r$ by LOCC (Nielsen's theorem).

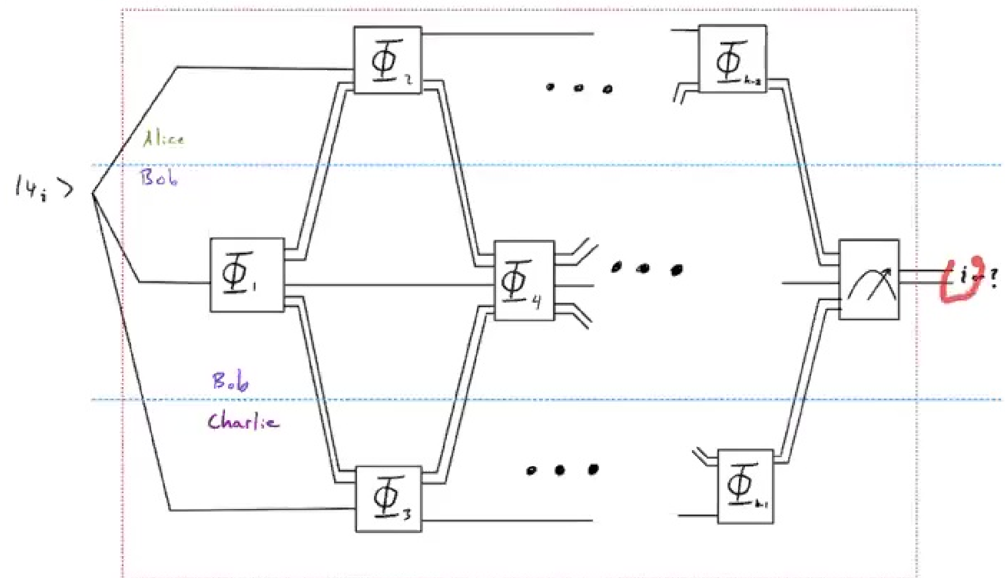
Local unambiguous state discrimination (LUSD)

Definition: A set of states $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\} \subseteq A \otimes B \otimes C$

Alice Bob Charlie

$\mathbb{C}$-vector spaces

is locally (unambiguously) discriminable if there exists an LOCC measurement with $n+1$ outcomes $\{1,2,3,\ldots,n,?\}$ that, when applied to $|\psi_i\rangle$, outputs either i or $?$, with non-zero probability to output i.



LOCC Measurement

Lemma [Chefles 2004]: $\{|\psi_1\rangle, \dots, |\psi_n\rangle\} \subseteq A\otimes B\otimes C$ is locally discriminable $\iff$ there exists $\{|\mu_1\rangle, \dots, |\mu_n\rangle\} \subseteq Seg(A\times B\times C)$ such that $(\langle\mu_i|\psi_j\rangle \neq 0 \iff i=j)$.

$= \{|x\rangle\otimes|y\rangle\otimes|z\rangle \in A\otimes B\otimes C\}$
Product states

Proof sketch:

$\Longrightarrow$: $\{|\psi_1\rangle, \dots, |\psi_n\rangle\}$ locally discriminable $\Longrightarrow$ it is discriminable by a separable POVM $\{M_1, \dots, M_n, M_?\}$, i.e. $\langle\psi_i|M_j|\psi_i\rangle \neq 0 \iff i=j$. For each M_j, choose $|\mu_j\rangle$ any product state in $Im(M_j)$.

$\Longleftarrow$: Use $\{|\mu_1\rangle\langle\mu_1|, \dots, |\mu_n\rangle\langle\mu_n|\}$ in an LOCC Measurement. ∎

The SLOCC Image

Definition: The SLOCC image of a state $|\varphi\rangle \in \mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$ in a space $\underbrace{A \otimes B \otimes C}_{\mathbb{C}\text{-vector spaces}}$, denoted $\mathrm{Im}_{A\otimes B\otimes C}(|\varphi\rangle)$, is the set of pure states $|\mu\rangle \in A \otimes B \otimes C$ for which there exists an LOCC channel that sends $|\varphi\rangle$ to $|\mu\rangle$ with non-zero probability.

(The "S" in SLOCC stands for stochastic)

Fact [Dür, Vidal, Cirac 2000]:

$$\mathrm{Im}_{A\otimes B\otimes C}(|\varphi\rangle) = \left\{ (M_A \otimes M_B \otimes M_C)|\varphi\rangle \;\middle|\; \begin{array}{l} M_A : \mathbb{C}^r \to A \text{ linear,} \\ \text{ditto for } M_B, M_C \end{array} \right\}.$$

Example: The SLOCC image of a bipartite Schmidt-rank-r state is the set of pure states of Schmidt rank at most r.

Example: The SLOCC image of a trivial resource state is the set of product states $= \mathrm{Seg}(A \times B \times C) = \{ |\psi\rangle \otimes |\varphi\rangle \otimes |\mu\rangle \mid |\psi\rangle \in A, |\varphi\rangle \in B, |\mu\rangle \in C \}$

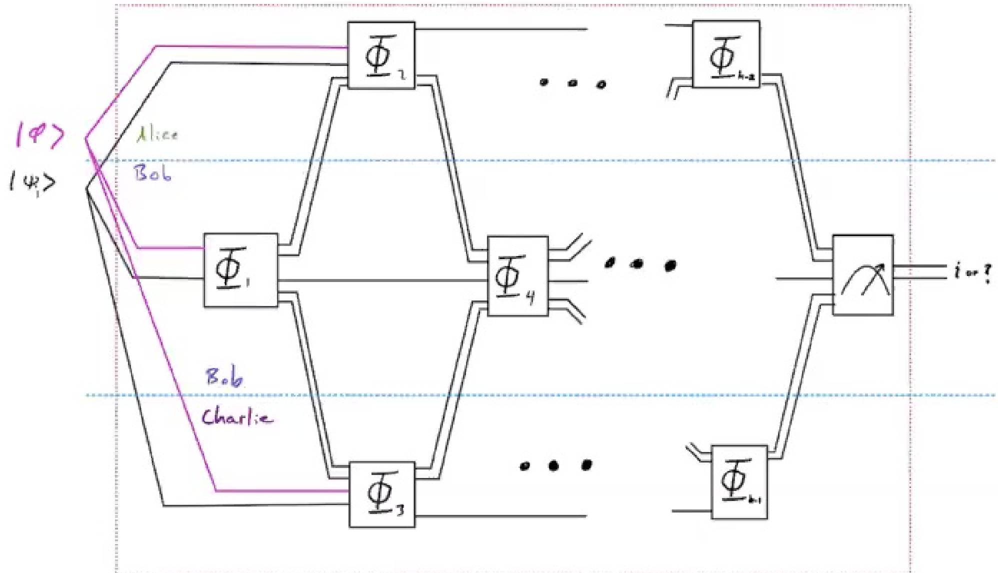
$=$ the Segre variety

LUSD with a resource state

Definition: $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\} \subseteq A \otimes B \otimes C$ is locally (unambiguously) discriminable with resource state $|\varphi\rangle \in \mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$ if

$$\{|\psi_1\rangle \otimes |\varphi\rangle, \ldots, |\psi_n\rangle \otimes |\varphi\rangle\} \subseteq (A \otimes \mathbb{C}^r) \otimes (B \otimes \mathbb{C}^r) \otimes (C \otimes \mathbb{C}^r)$$

is locally discriminable.



LOCC Measurement

Lemma [Bandyopadhyay, Halder, Nathanson 2016]:

Let $S = \{|\psi_1\rangle, \ldots, |\psi_n\rangle\} \subseteq A \otimes B \otimes C$, $\quad |\varphi\rangle \in \mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$.

the following statements are equivalent:

1. S is locally discriminable with $|\varphi\rangle$.
2. $\{|\psi_1\rangle \otimes |\varphi\rangle, \ldots, |\psi_n\rangle \otimes |\varphi\rangle\} \subseteq (A \otimes \mathbb{C}^r) \otimes (B \otimes \mathbb{C}^r) \otimes (C \otimes \mathbb{C}^r)$ is locally discriminable.
3. There exists $\{|\gamma_1\rangle, \ldots, |\gamma_n\rangle\} \subseteq \mathrm{Seg}((A \otimes \mathbb{C}^r) \times (B \otimes \mathbb{C}^r) \times (C \otimes \mathbb{C}^r))$ such that $\langle \gamma_i | (|\psi_j\rangle \otimes |\varphi\rangle) \neq 0 \iff i = j$
4. There exists $\{|\mu_1\rangle, \ldots, |\mu_n\rangle\} \subseteq \mathrm{Im}_{A \otimes B \otimes C}(|\varphi\rangle)$ such that $(|\mu_i\rangle)^T |\psi_j\rangle \neq 0 \iff i = j$.

Proof sketch:

$1 \iff 2$ is clear, $2 \iff 3$ holds by Chefles, so remains to prove $3 \iff 4$.

$3 \Rightarrow 4$:

Let $|\mu_i\rangle = [\langle \gamma_i | (\mathbb{1}_{A \otimes B \otimes C} \otimes |\varphi\rangle)]^T \in \mathrm{Im}_{A \otimes B \otimes C}(|\varphi\rangle)$.

$4 \Rightarrow 3$: By statement 4, there exists

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3. There exists $\{|\gamma_1\rangle, \ldots, |\gamma_n\rangle\} \subseteq Seg((A\otimes\mathbb{C}^r)\times(B\otimes\mathbb{C}^r)\times(C\otimes\mathbb{C}^r))$ such that $\langle\gamma_i|(|\psi_j\rangle\otimes|\varphi\rangle) \neq 0 \iff i=j$

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$$(|\mu_i\rangle)^T|\psi_j\rangle \neq 0 \iff i=j.$$

For each $i=1,\ldots,n$, $|\mu_i\rangle = M_A^{(i)}\otimes M_B^{(i)}\otimes M_C^{(i)}|\varphi\rangle \in A\otimes B\otimes C$

for linear maps $M_A^{(i)}: \mathbb{C}^r \to A$, and ditto for B, C.

Let

$$|\gamma_i\rangle = vec(\overline{M_A^{(i)}})\otimes vec(\overline{M_B^{(i)}})\otimes vec(\overline{M_C^{(i)}}) \in Seg((A\otimes\mathbb{C}^r)\times(B\otimes\mathbb{C}^r)\times(C\otimes\mathbb{C}^r)).$$

such that $(|\mu_i\rangle)^T|\psi_j\rangle \neq 0 \iff i=j.$

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$\{|\mu_1\rangle, \ldots, |\mu_n\rangle\} \subseteq \mathrm{Im}_{A\otimes B\otimes C}(|\varphi\rangle)$ such that

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$|\gamma_i\rangle = \mathrm{vec}(\overline{M_A^{(i)}}) \otimes \mathrm{vec}(\overline{M_B^{(i)}}) \otimes \mathrm{vec}(\overline{M_C^{(i)}}) \in \mathrm{Seg}((A\otimes\mathbb{C}^r)\times(B\otimes\mathbb{C}^r)\times(C\otimes\mathbb{C}^r)).$

$\mathrm{vec}(|i\rangle\langle j|) = |i\rangle|j\rangle$

Then $\langle\gamma_i|(|\psi_j\rangle \otimes |\varphi\rangle) = (|\mu_i\rangle)^T|\psi_j\rangle.$ $\blacksquare$

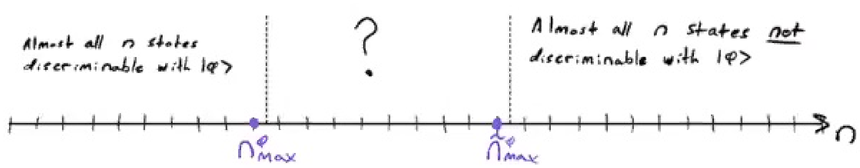
The (generic) usefulness of a state for LUSD

How many states can be locally discriminated with resource state $|\varphi\rangle$?

— Depends on the particular set of states to be discriminated. E.g. n orthogonal product states are always locally discriminable, but n identical states are never locally discriminable.

Let $n^{\varphi}_{max} = \max \{ n \in \mathbb{N} \mid$ Almost all sets of n pure states $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\} \subseteq A \otimes B \otimes C$ are locally discriminable with $|\varphi\rangle \}$

$\tilde{n}^{\varphi}_{max} = \min \{ n \in \mathbb{N} \mid$ Almost all sets of $n+1$ pure states $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\} \subseteq A \otimes B \otimes C$ are NOT locally discriminable with $|\varphi\rangle \}$.



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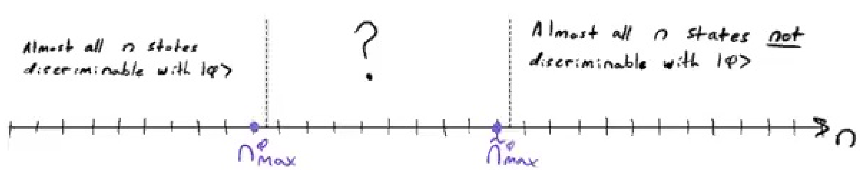
$\tilde{n}^{\varphi}_{max} = \min \left\{ n \in \mathbb{N} \;\middle|\; \text{Almost all sets of } n+1 \text{ pure states } \{|\psi_1\rangle, \ldots, |\psi_n\rangle\} \subseteq A \otimes B \otimes C \text{ are NOT locally discriminable with } |\varphi\rangle. \right\}$



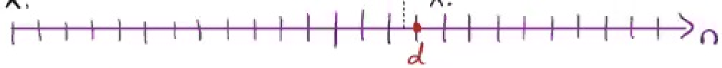
Theorem [Lovitz, Johnston 2020]: For any resource state $|\varphi\rangle$, $n^{\varphi}_{max} = \tilde{n}^{\varphi}_{max} = \dim\left(\overline{\mathrm{Im}_{A \otimes B \otimes C}(|\varphi\rangle)}\right)$.

Proving the main result on generic LUSD

Theorem [Lovitz, Johnston 2020]: For any resource state $|\varphi\rangle$, $n^{\varphi}_{\max} = \tilde{n}^{\varphi}_{\max} = \dim\left(\overline{\mathrm{Im}_{A\otimes B\otimes C}(|\varphi\rangle)}\right)$.

Let $X = \mathrm{Im}_{A\otimes B\otimes C}(|\varphi\rangle)$ and $d = \dim(\overline{X})$.

Lemma 1: Almost all linear subspaces of dimension $abc - n$ non-trivially intersect X. | Almost all linear subspaces of dimension $abc - n$ trivially intersect $\overline{X}$.



Lemma 2 [BHN16]: $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\} \subseteq A\otimes B\otimes C$ are locally discriminable with $|\varphi\rangle$ if and only if there exists $\{|\mu_1\rangle, \ldots, |\mu_n\rangle\} \subseteq X$ such that $(|\mu_i\rangle)^T|\psi_j\rangle \neq 0 \iff i=j$.

Proof that $\tilde{n}^{\varphi}_{\max} \leq d$: By Lemma 1, almost all linear subspaces of dimension $abc - d$ trivially intersect X. Hence, for almost all sets of states $\{|\psi_1\rangle, \ldots, |\psi_d\rangle\} \subseteq A\otimes B\otimes C$,

$$\operatorname{span}\{|\psi_1\rangle, \ldots, |\psi_d\rangle\}^{\perp} \cap X = \{0\}.$$

Orthogonal complement with respect to bilinear pairing

Theorem [Lovitz, Johnston 2020]: For any resource state $|\varphi\rangle$, $n^{\varphi}_{max} = \tilde{n}^{\varphi}_{max} = \dim\left(\overline{Im_{A\otimes B\otimes C}(|\varphi\rangle)}\right)$.

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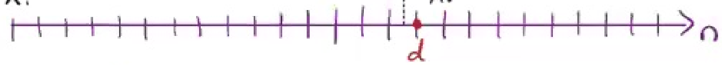

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Orthogonal complement with respect to bilinear pairing

$\Longrightarrow \{|\psi_1\rangle, \ldots, |\psi_d\rangle, |\psi_{d+1}\rangle\}$ is not locally discriminable with $|\varphi\rangle$ for an $|\psi_{d+1}\rangle \in A\otimes B\otimes C$, so $\tilde{n}^{\varphi}_{max} \leq d$. ■

Examples (Usefulness of particular $|\varphi\rangle$)

Theorem [Lovitz, Johnston 2020]: For any resource state $|\varphi\rangle$, $n^{\varphi}_{max} = \tilde{n}^{\varphi}_{max} = \dim\left(\overline{\mathrm{Im}_{A\otimes B\otimes C}(|\varphi\rangle)}\right)$.

Example: In the bipartite setting, for any Schmidt-rank-r state $|\varphi\rangle$,

$$n^{\varphi}_{max} = \tilde{n}^{\varphi}_{max} = \dim(\{\text{Schmidt rank} \le r \text{ states}\}) = ab - (a - \min\{a,r\})(b - \min\{b,r\}),$$

where $a = \dim(A)$, $b = \dim(B)$.

Example [Walgate, Scott 2008]: For a trivial resource state,

$$n^{\varphi}_{max} = \dim(\mathrm{Seg}(A\times B\times C)) = (a-1)+(b-1)+(c-1)+1.$$

Example: For a GHZ resource state

$$|\tau_r\rangle = \frac{1}{\sqrt{r}}\sum_{i=1}^{r} |i\rangle|i\rangle|i\rangle, \quad n^{\varphi}_{max} = \tilde{n}^{\varphi}_{max} \ge r\big((a-1)+(b-1)+(c-1)+1\big),$$

with equality in many cases (see [Abo, Ottaviani, Peterson 2009] for details).

What are the most useful states for generic LUSD? (And how useful are they?)

For $A \otimes B \otimes C = \mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$,

$$\overline{\mathrm{Im}_{\mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r}(|\varphi\rangle)} = \overline{(GL_r \times GL_r \times GL_r) \cdot |\varphi\rangle}$$

$=$ The SLOCC orbit of $|\varphi\rangle$

Proposition: For an algebraic group G acting on an algebraic variety X, and any $k \in \mathbb{N}$, the set $\{x \in X \mid \dim(\overline{G \cdot x}) \leq k\}$ is Zariski closed ($\Rightarrow$ Measure zero).

Corollary: Almost all states $|\varphi\rangle \in \mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$ maximize $\dim(\overline{(GL_r \times GL_r \times GL_r) \cdot |\varphi\rangle})$, and hence are maximally useful for generic LUSD in $\mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$.

Example: $(\mathbb{C}^2)^{\otimes m}$ [Gour, Kraus, Wallach 2017] [Gour, Wallach 2011] (and references therein)

n_{max} for generic $|\varphi\rangle$

m		Example of a maximally useful state $|\varphi\rangle$ for LUSD
2	4	$|00\rangle + |11\rangle$

Benjamin Lovitz

$$\overline{\mathrm{Im}_{\mathbb{C}^r\otimes\mathbb{C}^r\otimes\mathbb{C}^r}(|\varphi\rangle)} = \overline{(GL_r \times GL_r \times GL_r)\cdot|\varphi\rangle}$$

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n^*_{max} for generic $|\varphi\rangle$

m		Example of a maximally useful state $\|\varphi\rangle$ for LUSD
2	4	$\|00\rangle + \|11\rangle$
3	8	$\|000\rangle + \|111\rangle$
4	13	$(\|00\rangle + \|11\rangle)^{\otimes 2} + 2(\|00\rangle - \|11\rangle)^{\otimes 2} + 3(\|01\rangle + \|10\rangle)^{\otimes 2}$
≥ 5	$4m-(m-1)$	$\sqrt{m-2}\,\|11\cdots1\rangle + \|10\cdots0\rangle + \|01\cdots0\rangle + \cdots + \|0\cdots01\rangle$

Entangled Subspaces

— Linear subspaces of $A \otimes B \otimes C$ for which every element is "entangled."

Theorem: [Parthasarathy 2004, Bhat 2006, Cubitt, Montanaro, Winter 2008, Walgate, Scott 2008]

- Max dimension of a subspace of $A \otimes B \otimes C$ without product states is $abc - \dim(\mathrm{Seg}(A \times B \times C))$
- Max dimension of a subspace of $A \otimes B$ without states of Schmidt rank $\leq r$ is $(a - \min\{a, r\})(b - \min\{b, r\})$
- Explicit constructions

Definitions:

- The tensor rank of a state $|\psi\rangle \in A \otimes B \otimes C$ is the minimum number of product states which can be put in superposition to create $|\psi\rangle$.
- $\sigma_r = \overline{\{|\psi\rangle \in A \otimes B \otimes C \text{ of tensor rank} \leq r\}}$
  $= \overline{\mathcal{I}m_{A \otimes B \otimes C}(|\tau_r\rangle)}$, where $|\tau_r\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} |i\rangle|i\rangle|i\rangle$
- A linear subspace $W \subseteq A \otimes B \otimes C$ is r-entangled if $W \cap \sigma_r = \{0\}$.

Results: (Lovitz, Johnston)

- The maximum dimension of an r-entangled subspace

element is entangled.

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Results: (Lovitz, Johnston)

- The maximum dimension of an r-entangled subspace is $abc - \dim(\sigma_r)$.
- Explicit constructions
- Analogous results for Bosonic, Fermionic systems.
- Application to entanglement witnesses

Theorem: [Parthasarathy 2004, Bhat 2006, Cubitt, Montanaro, Winter 2008, Walgate, Scott 2008]

- Max dimension of a subspace of $A \otimes B \otimes C$ without product states is $abc - \dim(\mathrm{Seg}(A \times B \times C))$
- Max dimension of a subspace of $A \otimes B$ without states of Schmidt rank $\leq r$ is $(a - \min\{a, r\})(b - \min\{b, r\})$
- Explicit constructions

Definitions:

- The tensor rank of a state $|\psi\rangle \in A \otimes B \otimes C$ is the minimum number of product states which can be put in superposition to create $|\psi\rangle$.
- $\sigma_r = \overline{\{|\psi\rangle \in A \otimes B \otimes C \text{ of tensor rank} \leq r\}}$
  $= \overline{I_{m_{A\otimes B\otimes C}}(|\tau_r\rangle)}$, where $|\tau_r\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} |i\rangle|i\rangle|i\rangle$
- A linear subspace $W \subseteq A \otimes B \otimes C$ is r-entangled if $W \cap \sigma_r = \{0\}$.

Results: (Lovitz, Johnston)

- The maximum dimension of an r-entangled subspace is $abc - \dim(\sigma_r)$.
- Explicit constructions
- Analogous results for Bosonic, Fermionic systems.
- Application to entanglement witnesses.

• Application to entanglement witnesses

[1] Marcus Huber and Julio I. de Vicente.
Structure of multidimensional entanglement in multipartite systems.
Physical Review Letters, 110:030501, 2013.

[2] Benjamin Lovitz and Nathaniel Johnston.
Entangled subspaces and generic local state discrimination with pre-shared entanglement.
arXiv preprint, quant-ph/2010.02876, 2020.

[3] W. Dür, G. Vidal, and J. I. Cirac.
Three qubits can be entangled in two inequivalent ways.
Physical Review A, 62:062314, 2000.

[4] Somshubhro Bandyopadhyay, Saronath Halder, and Michael Nathanson.
Entanglement as a resource for local state discrimination in multipartite systems.
Physical Review A, 94:022311, 2016.

[5] Toby Cubitt, Ashley Montanaro, and Andreas Winter.
On the dimension of subspaces with bounded schmidt rank.
Journal of Mathematical Physics, 49(2):022107, 2008.

[6] B. V. Rajarama Bhat.
A completely entangled subspace of maximal dimension.
International Journal of Quantum Information, 04(02):325–330, 2006.

[7] K. R. Parthasarathy.
On the maximal dimension of a completely entangled subspace for finite level quantum systems.
Proceedings Mathematical Sciences, 114(4):365–374, 2004.

[8] Hirotachi Abo, Giorgio Ottaviani, and Chris Peterson.
Induction for secant varieties of segre varieties.
Transactions of the American Mathematical Society, 361(2):767–792, 2009.

[9] Anthony Chefles.
Condition for unambiguous state discrimination using local operations and classical communication.
Physical Review A, 69:050307, 2004.

[10] Gilad Gour, Barbara Kraus, and Nolan R Wallach.
Almost all multipartite qubit quantum states have trivial stabilizer.
Journal of Mathematical Physics, 58(9):092204, 2017.

[11] Gilad Gour and Nolan R Wallach.
Necessary and sufficient conditions for local manipulation of multipartite pure quantum states.
New Journal of Physics, 13(7):073013, 2011.

[12] Jonathan Walgate and A J Scott.
Generic local distinguishability and completely entangled subspaces.
Journal of Physics A: Mathematical and Theoretical, 41(37):375305, 2008.

Thank you