Title: Conformal embedding of random planar maps

Speakers: Nina Holden

Series: Colloquium

Date: January 13, 2021 - 2:00 PM

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Abstract: A planar map is a canonical model for a discrete surface which is studied in probability theory, combinatorics, theoretical physics, and geometry. Liouville quantum gravity provides a natural model for a continuum random surface with roots in string theory and conformal field theory. After introducing these objects, I will present a joint work with Xin Sun where we prove convergence of random planar maps to a Liouville quantum gravity surface under a discrete conformal embedding which we call the Cardy embedding.

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Conformal embedding of random planar maps

Nina Holden

ETH Zürich, Institute for Theoretical Studies

January 13, 2021

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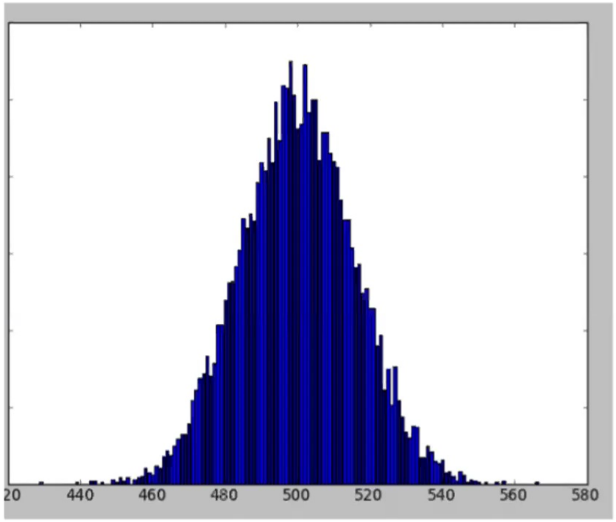
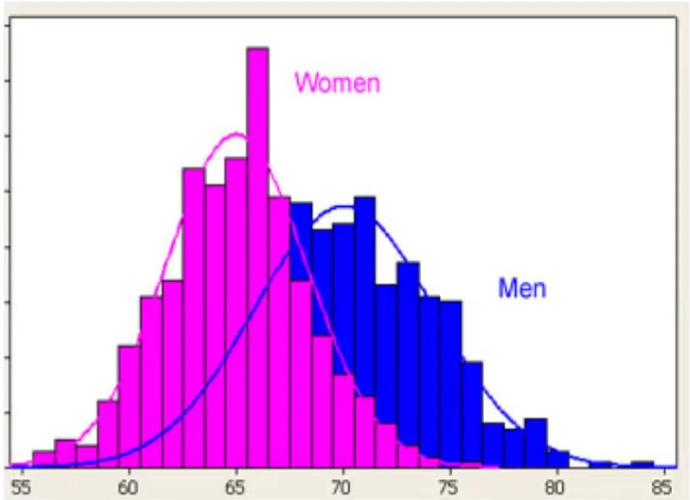
The normal random variable

heads in 1000 coin tosses



Height (inches) in a group of people



Universal: describes the asymptotic behavior of many discrete models, independently of the details of the model

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The normal random variable

Universal: describes the asymptotic behavior of many discrete models, independently of the details of the model

Canonical: uniquely characterized by natural symmetries

Example: Let $X_1, \ldots, X_n \sim \mathcal{N}(0,1)$ be independent. Then

$$\frac{X_1 + \cdots + X_n - n\mathbb{E}[X_1]}{\sqrt{n\,\mathrm{Var}[X_1]}} \text{ has law } \mathcal{N}(0,1).$$

Here $\mathcal{N}(0,1)$ denotes the law of the standard normal random variable.

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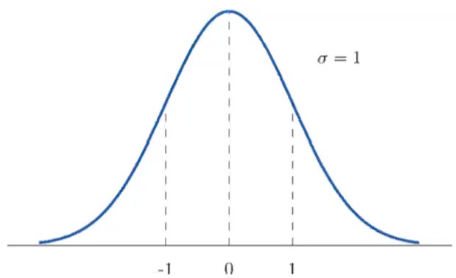
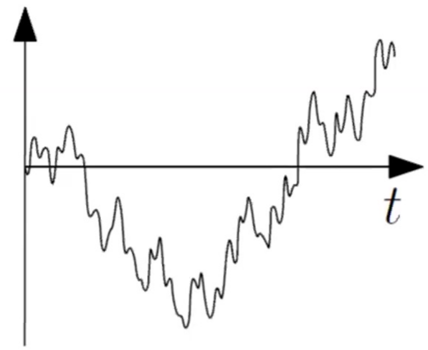
Universal and canonical random objects

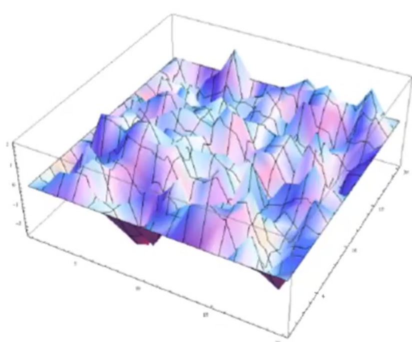


Normal random variable
(number in $\mathbb{R}$)

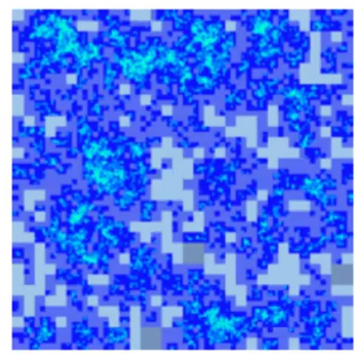


Brownian motion
(function $\mathbb{R}_+ \to \mathbb{R}$)

Schramm-Loewner evolution
(curve $\mathbb{R}_+ \to \mathbb{R}^2$)

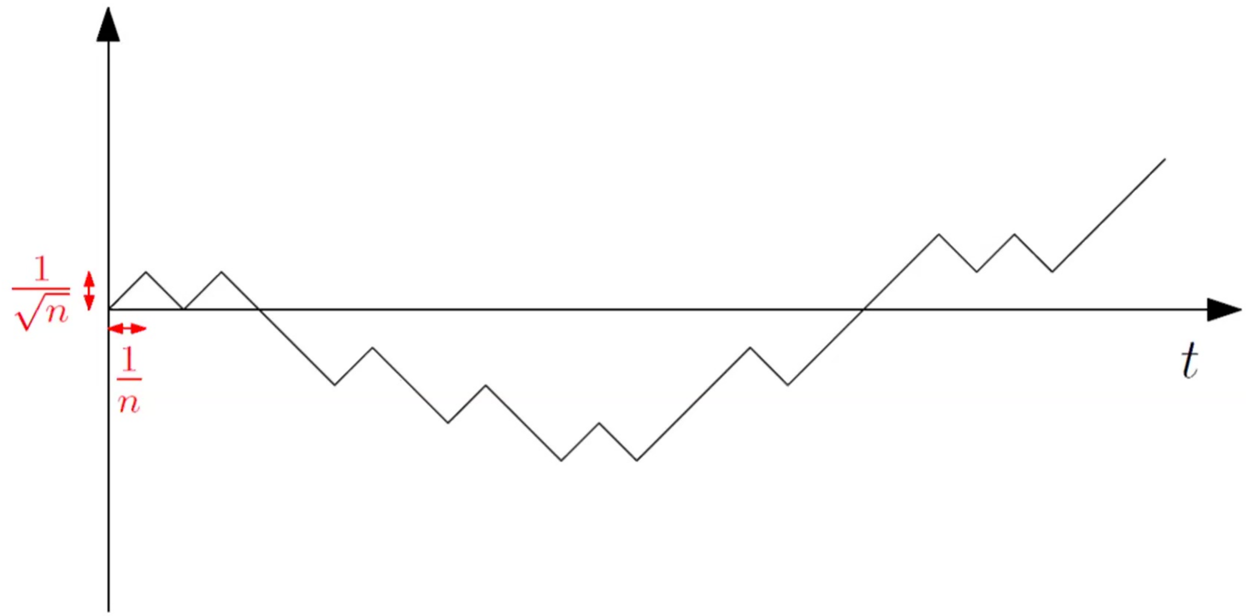


2d Gaussian Free Field
(function $\mathbb{R}^2 \to \mathbb{R}$)



Liouville quantum gravity
(surface)

Simple random walk



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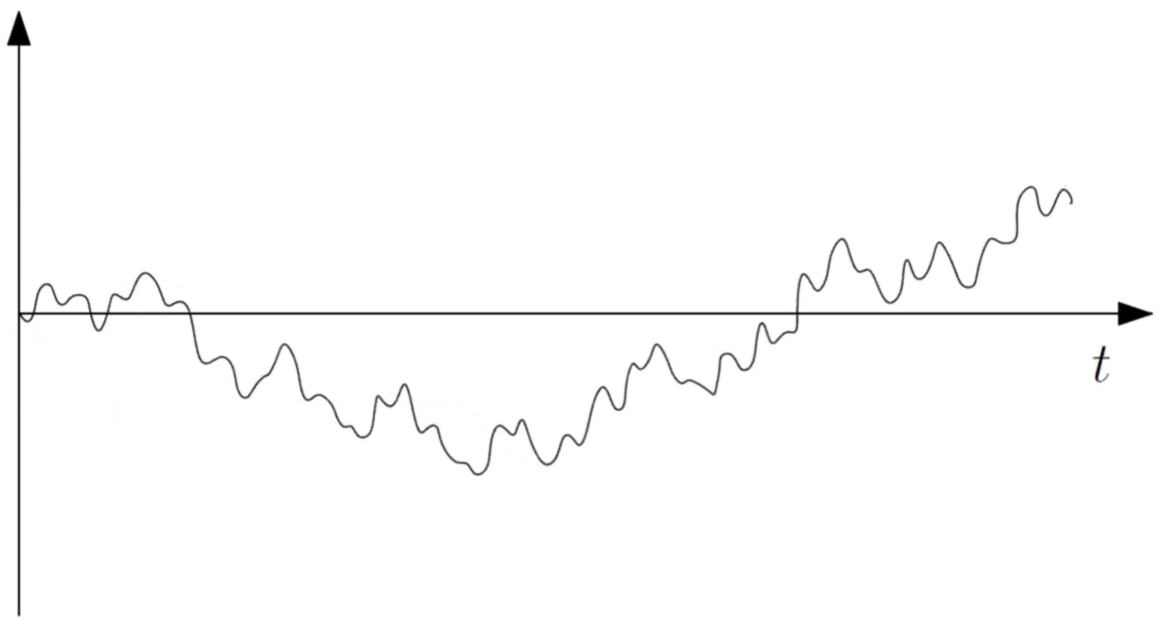
Brownian motion



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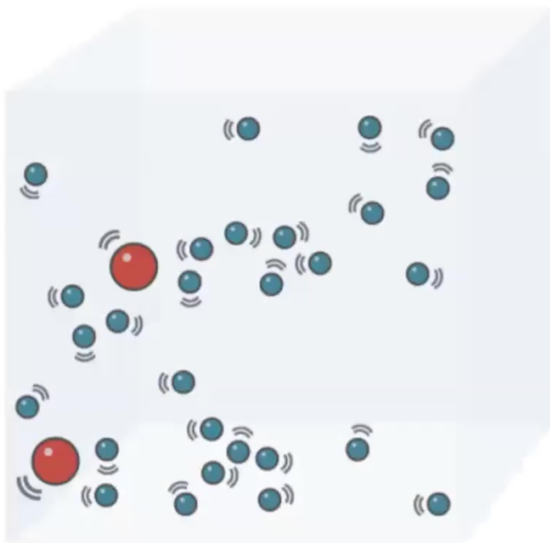
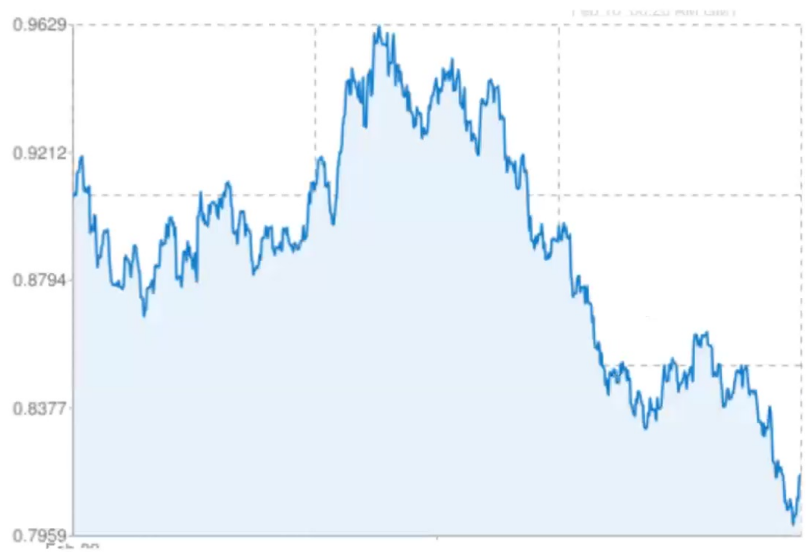
Brownian motion



Pollen grains in water



Historical USD-EUR exchange rate

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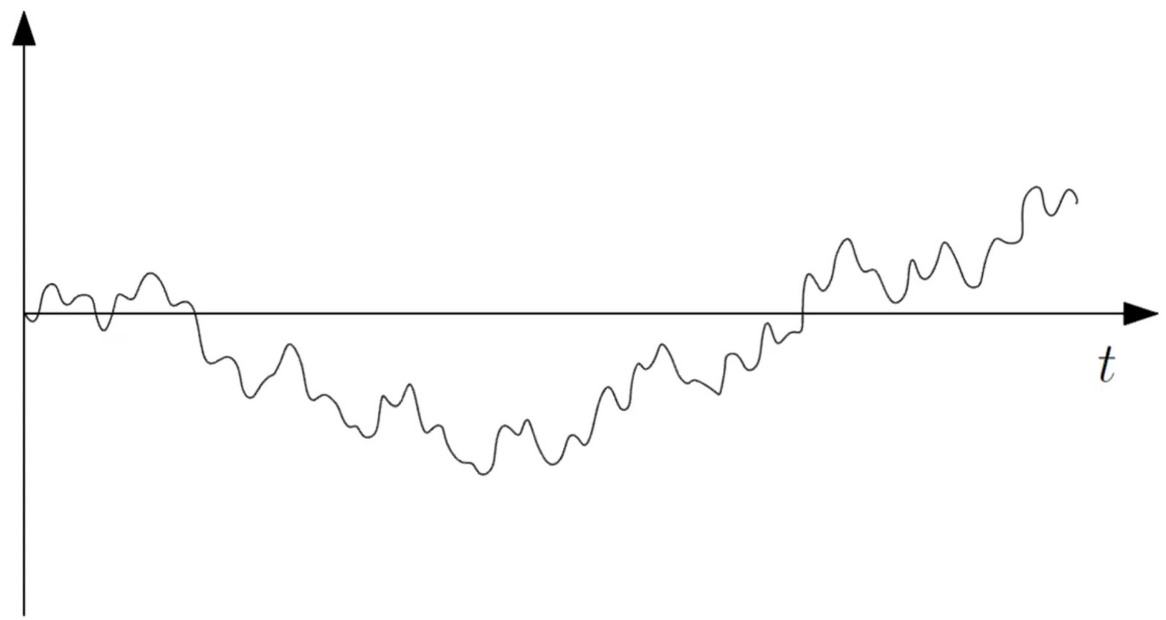
Brownian motion



Brownian motion is the unique mean zero continuous function started from zero with independent and identically distributed increments.

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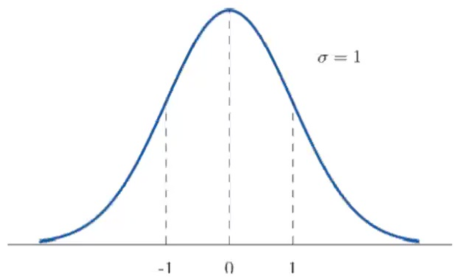
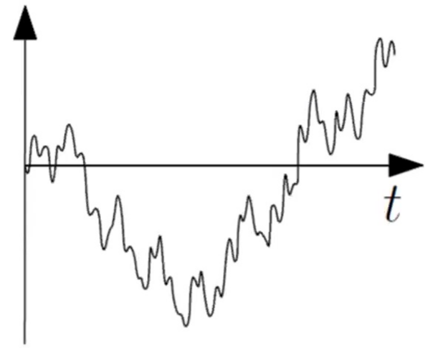
Universal and canonical random objects



Normal random variable
(number in $\mathbb{R}$)



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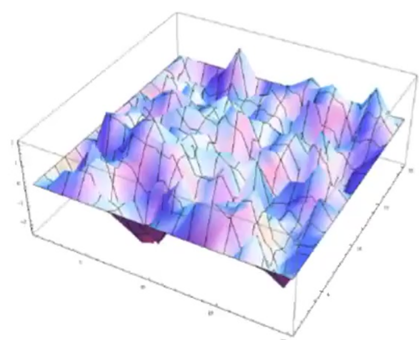

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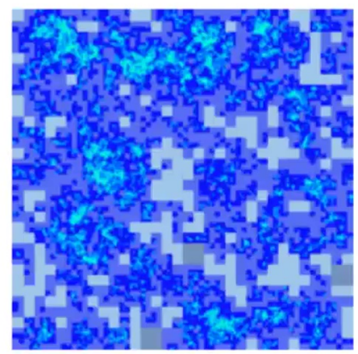

Liouville quantum gravity
(surface)

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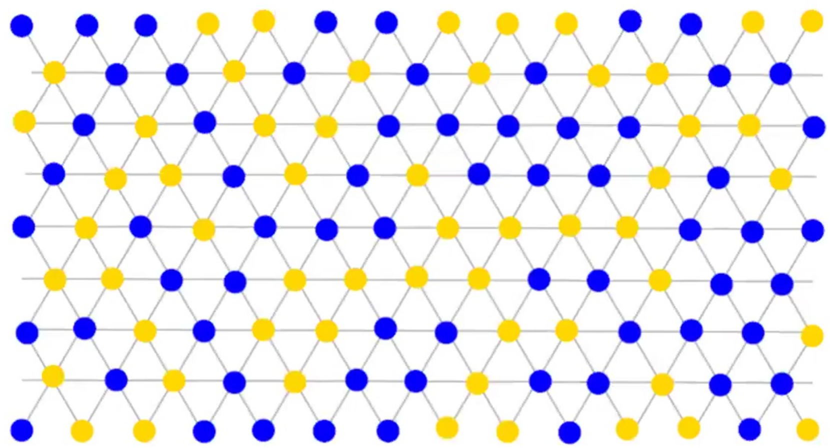
Percolation



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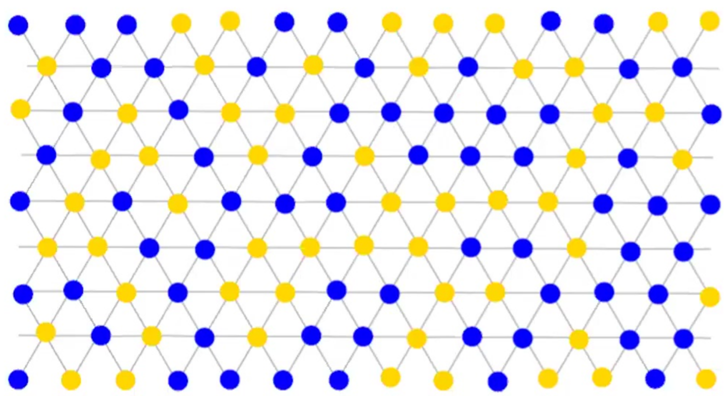
Percolation



Percolation

Porous stone

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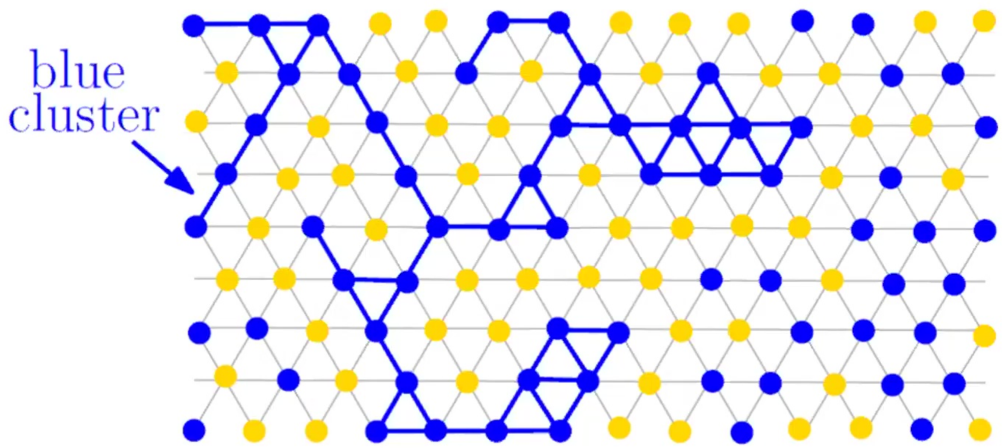
Percolation



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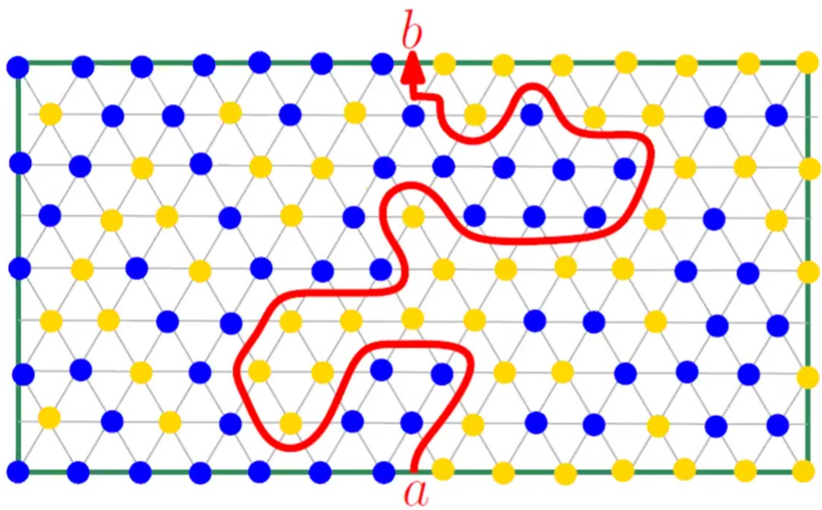
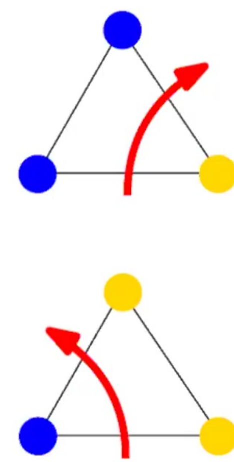
The percolation interface





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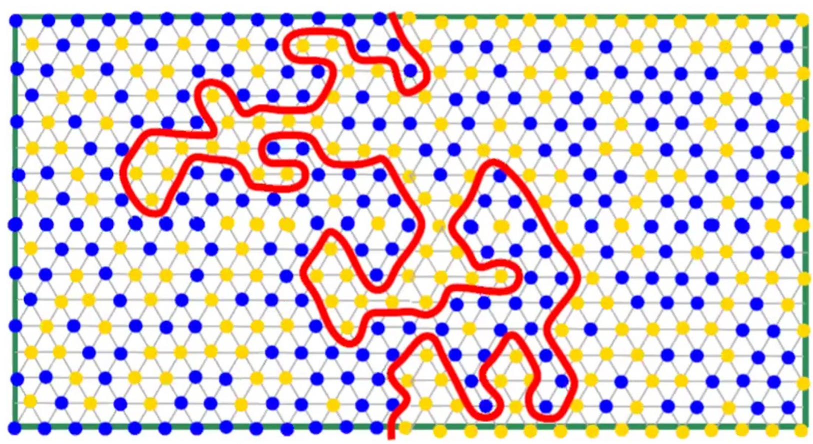
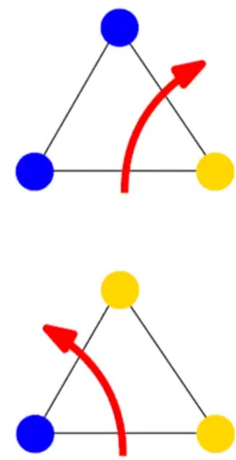
The percolation interface





The percolation interface converges to the Schramm-Loewner evolution.

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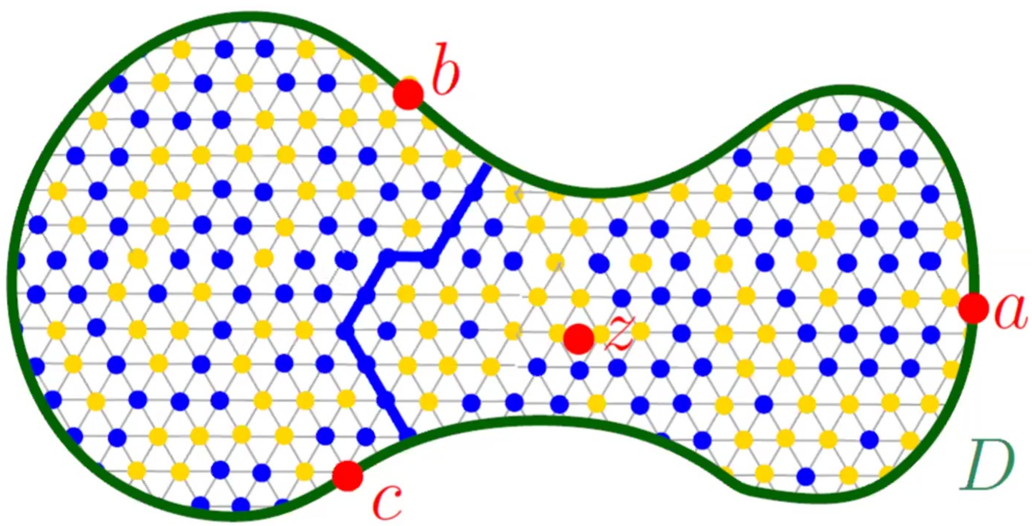
Conformal invariance of percolation (Smirnov'01)



Percolation crossing event

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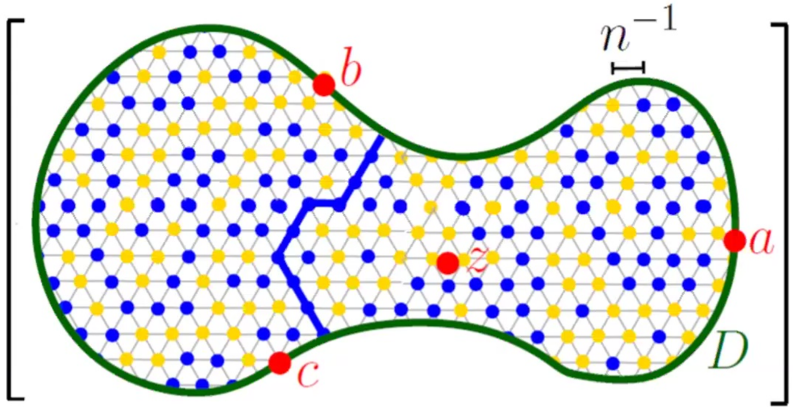
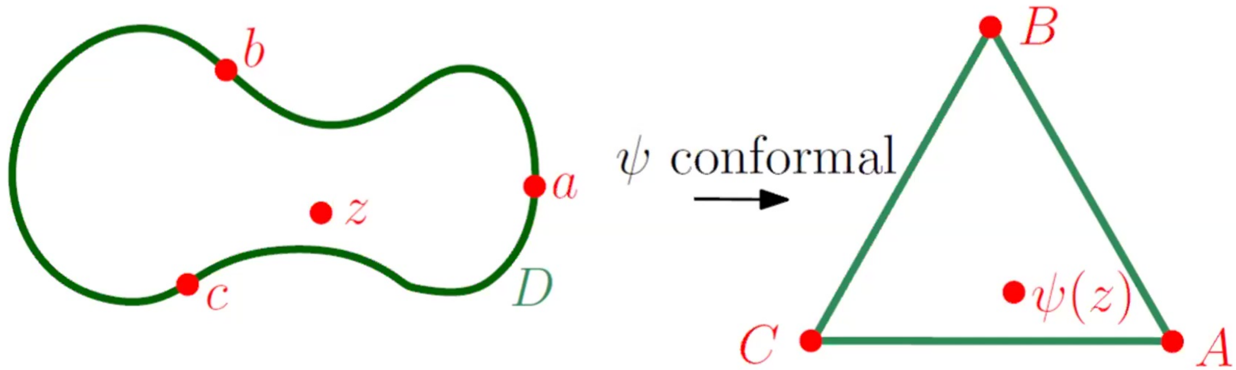
Conformal invariance of percolation (Smirnov'01)

$$\lim_{n\to\infty} \mathbb{P}$$



$$= p_A(\psi(z))$$



p_A is linear such that $p_A(A) = 1$ and $p_A(B) = p_A(C) = 0$

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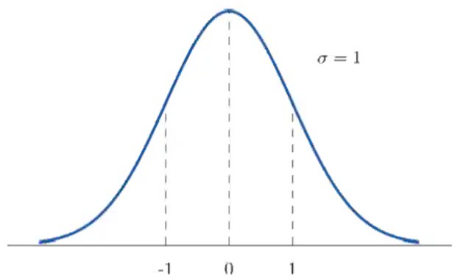
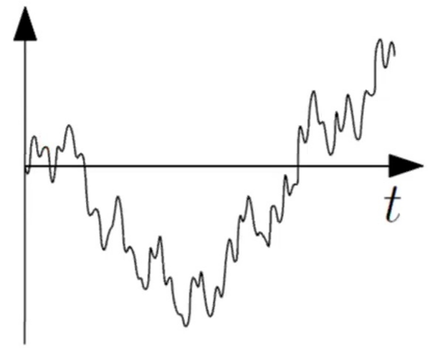
Universal and canonical random objects



Normal random variable
(number in $\mathbb{R}$)



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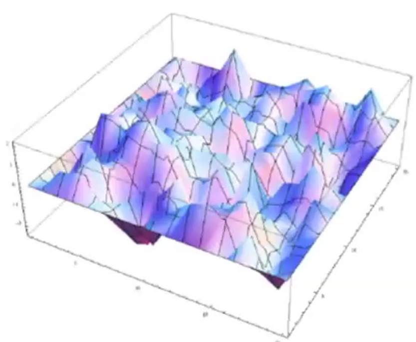

2d Gaussian Free Field
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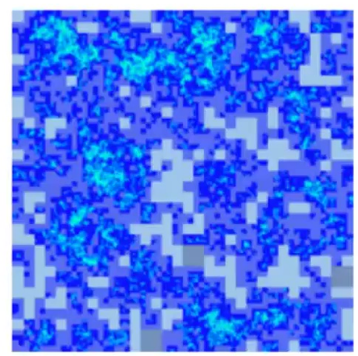

Liouville quantum gravity
(surface)

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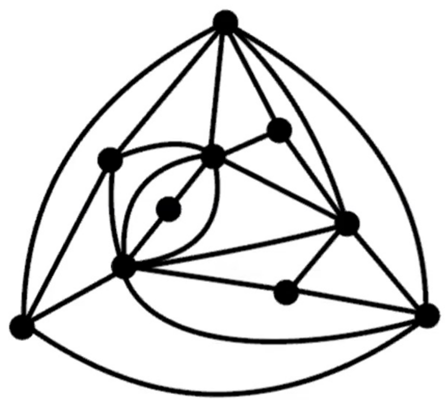
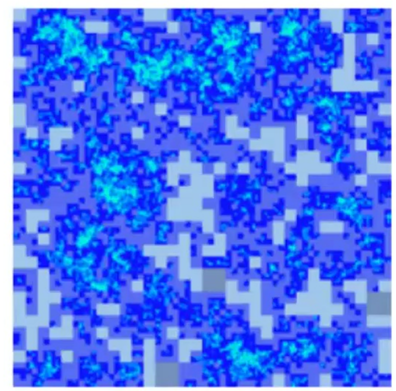
Two random surfaces



random planar map (RPM)



Liouville quantum gravity (LQG)

Main result (informal): RPM converges to LQG as its size goes to infinity.

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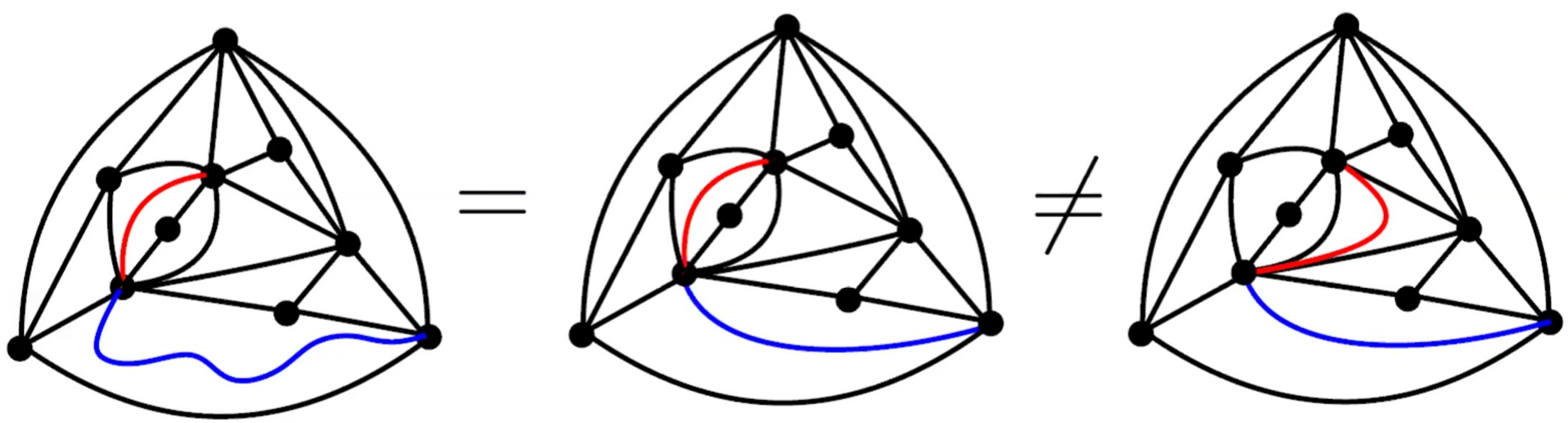
Planar maps

- A **planar map** M is a finite connected graph drawn in the sphere, viewed up to continuous deformations.



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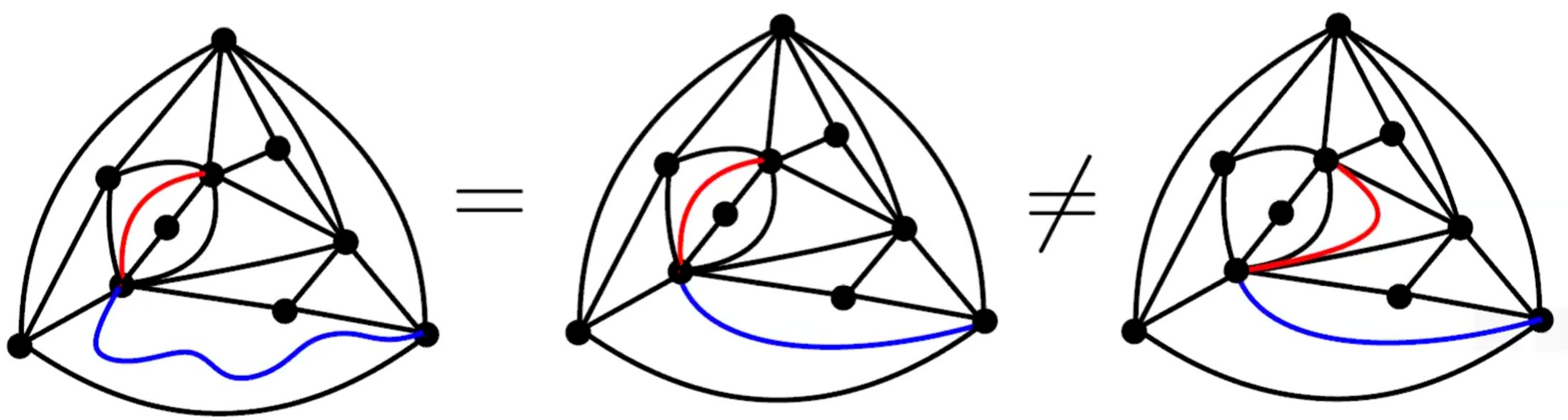
Planar maps

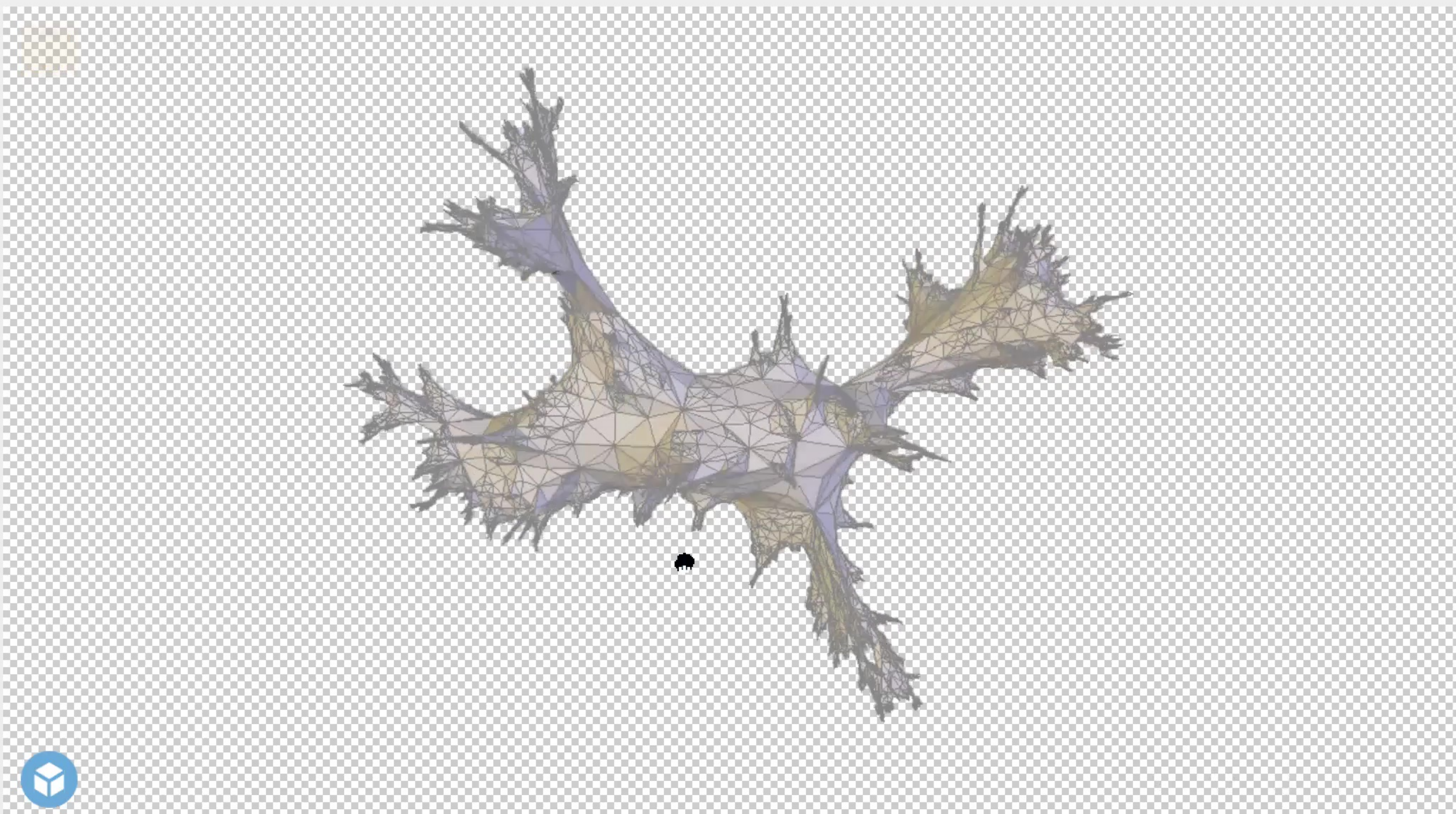
- A **planar map** M is a finite connected graph drawn in the sphere, viewed up to continuous deformations.
- A **triangulation** is a planar map where all faces have three edges.
- Given $n \in \mathbb{N}$ let M be a **uniformly** chosen triangulation with n vertices.



Page personnelle de Nicolas Curi

imo.universite-paris-saclay.fr/~curien/simulation.html

The rest of the zoo can be found in Igor Kortchemski's menagerie.



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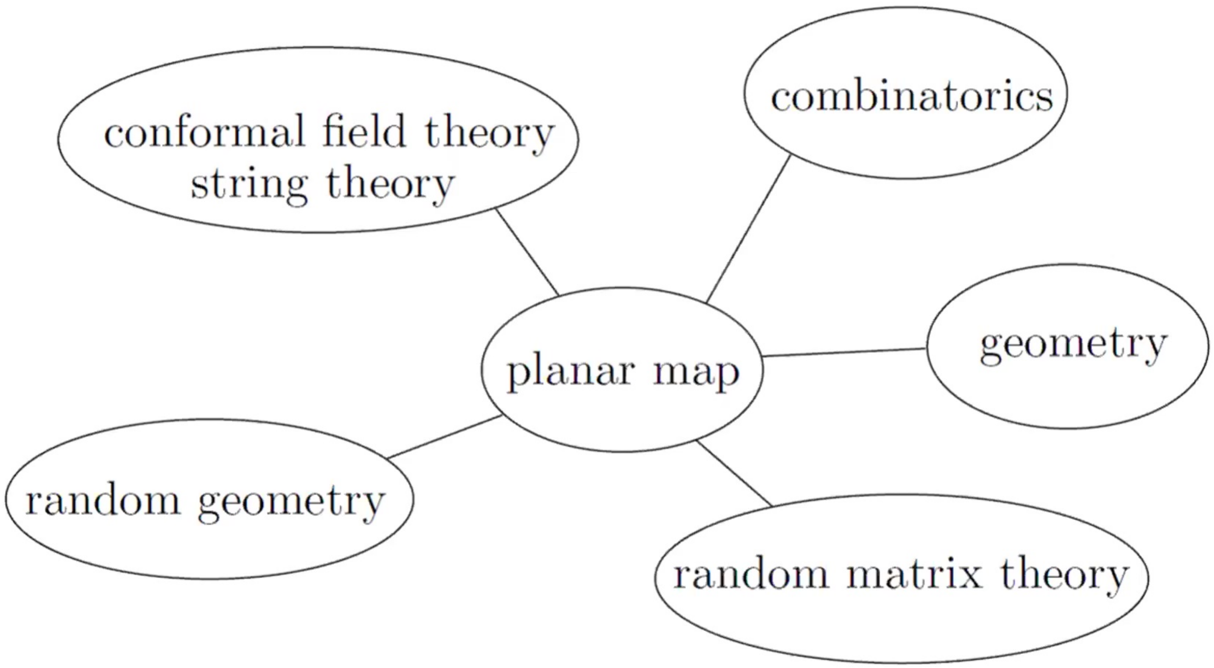
Planar maps



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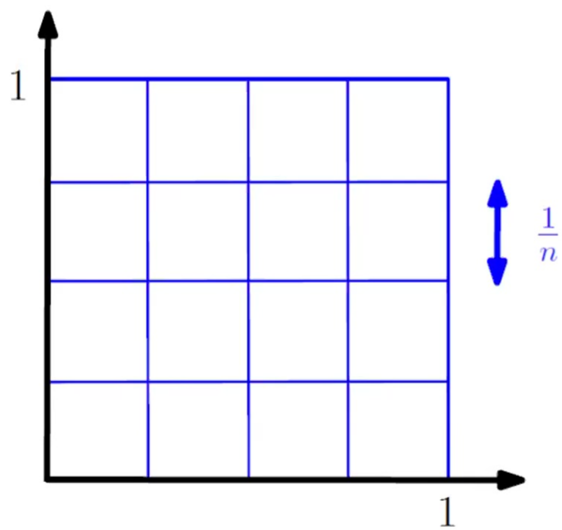
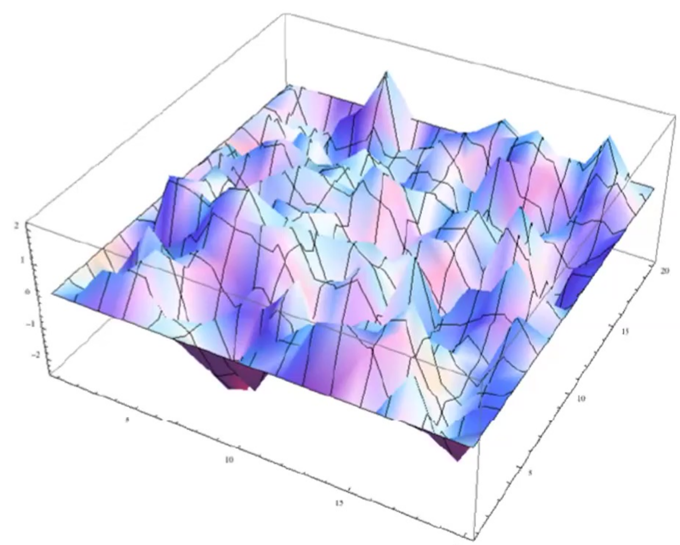
The discrete Gaussian free field

- Hamiltonian $H(f)$ quantifies deviation of f from being harmonic

$$H(f) = \frac{1}{2}\sum_{x\sim y}(f(x)-f(y))^2, \qquad f : \frac{1}{n}\mathbb{Z}^2 \cap [0,1]^2 \to \mathbb{R}.$$





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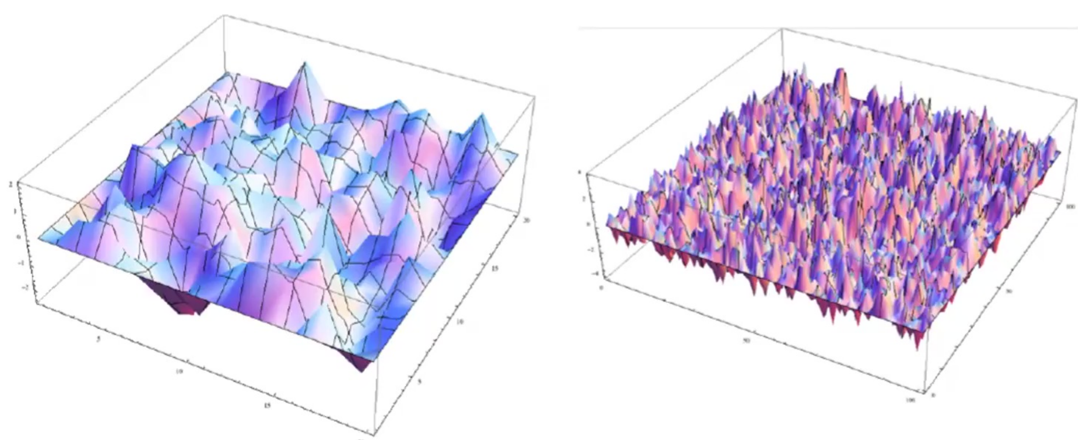
- Discrete zero-boundary Gaussian free field h_n:
  - $h_n|_{\partial[0,1]^2} = 0$,
  - probability density relative to the product of Lebesgue measure proportional to

$$\exp(-H(h_n)).$$



$n = 20, \; n = 100$

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The discrete Gaussian free field

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- For fixed $z, w \in (0,1)^2$ and $c = \frac{1}{2\pi}$,

$$h_n(z) \sim \mathcal{N}(0, c\log n + O(1)),$$
$$\mathrm{Cov}(h_n(z), h_n(w)) = c\log|z-w|^{-1} + O(1).$$

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The Gaussian free field (GFF)

- The **Gaussian free field** (GFF) h is the limit of h_n when $n \to \infty$.
- The GFF is a **random distribution (generalized function)**.
  - $h(z)$ is **not** well-defined.
  - The average $h_\epsilon(z)$ of $h(z)$ on $\partial B(z, \epsilon)$ is well-defined for $\epsilon > 0$.
- Natural analogue of Brownian motion with two time dimensions but still one space dimension.
- Formal path integral definition: With $D\phi$ "Lebesgue measure on the space of functions" and $S(\phi) := \frac{1}{2} \int |\nabla \phi|^2 \, d^2 z$ the action,

$$h \sim \exp(-S(\phi)) \, D\phi.$$

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The discrete Gaussian free field

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Liouville quantum gravity (LQG)

- If $h : [0,1]^2 \to \mathbb{R}$ is smooth and $\gamma \in (0,2)$, then the following defines a **measure** μ and a **distance function (metric)** D on $[0,1]^2$:

$$\mu(U) = \int_U e^{\gamma h(z)} d^2 z, \qquad D(z_1, z_2) = \inf_{P: z_1 \to z_2} \int_P e^{\frac{\gamma h(z)}{2}} \, dz,$$

where $U \subset [0,1]^2$ and $z_1, z_2 \in [0,1]^2$.

- γ-Liouville quantum gravity (LQG): h is the **Gaussian free field**.
- The definition of an LQG surface does not make literal sense since h is a distribution and not a function.
- **Measure** μ and **distance function (metric)** D defined by considering regularization h_ϵ of h.[1]

$$\mu(U) = \lim_{\epsilon \to 0} \epsilon^{\frac{\gamma^2}{2}} \int_U e^{\gamma h_\epsilon(z)} d^2 z, \qquad D(z_1, z_2) = \lim_{\epsilon \to 0} c_\epsilon \inf_{P: z_1 \to z_2} \int_P e^{\frac{\gamma h_\epsilon(z)}{d(\gamma)}} \, dz.$$

[1] Metric construction: Gwynne-Miller'19, Ding-Dubedat-Dunlap-Falconet'19, Dubedat-Falconet-Gwynne-Pfeffer-Sun'19. Hausdorff dim. $([0,1]^2, D)$ denoted by $d(\gamma)$.

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- LQG for $\gamma = \sqrt{8/3}$ is called **pure LQG**.

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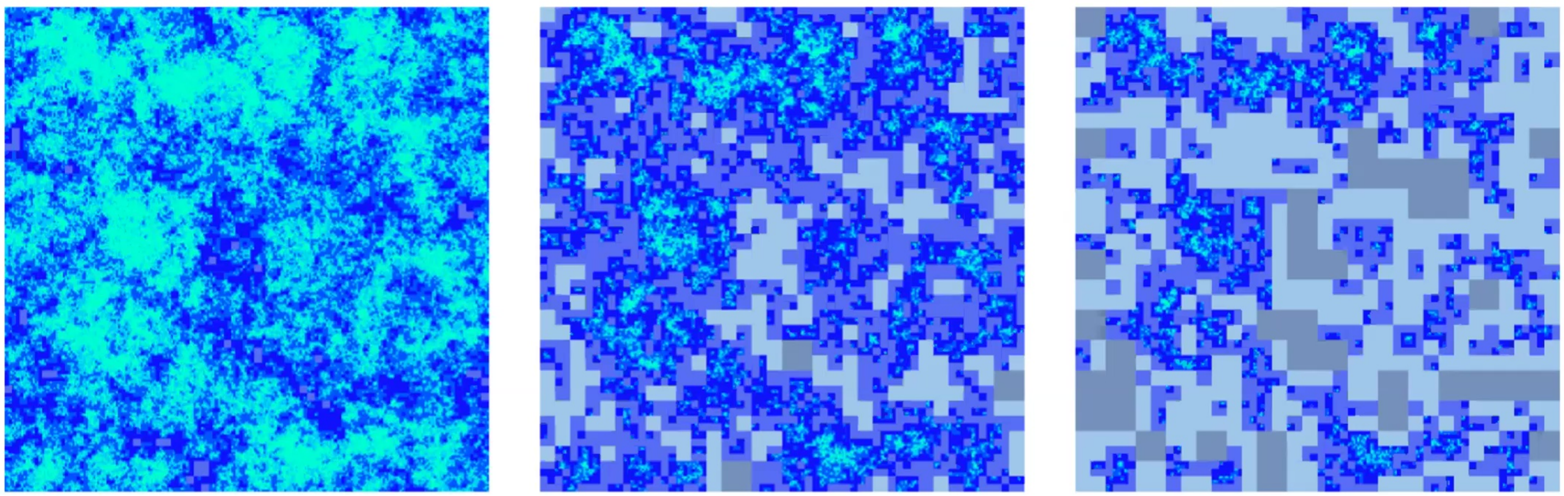
Illustration of LQG area measure



$\gamma = 1$ $\gamma = 1.5$ $\gamma = 1.75$

Random area measure $\mu = e^{\gamma h} \, d^2 z$ defined by

$$\mu(U) = \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} \int_U e^{\gamma h_\epsilon(z)} d^2 z, \quad U \subset [0,1]^2.$$

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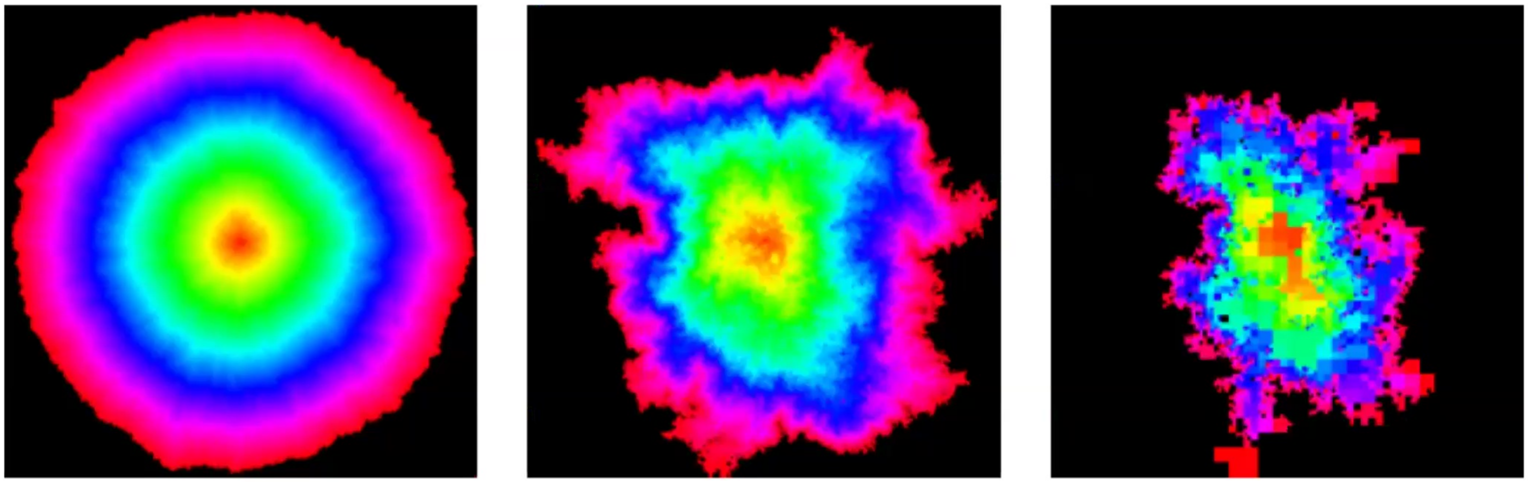
Illustration of LQG metric



$\gamma = 0.25$ $\gamma = 1$ $\gamma = 1.5$

Random metric defined by

$$D(z_1, z_2) = \lim_{\epsilon \to 0} c_\epsilon \inf_{P: z_1 \to z_2} \int_P e^{\gamma h_\epsilon(z)/d(\gamma)}\, dz, \quad z_1, z_2 \in [0,1]^2.$$

Simulations by J. Miller

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Random planar maps converge to LQG

Two models for random surfaces:

- Random planar maps (RPM)
- Liouville quantum gravity (LQG)

Conjectural relationship used by physicists to predict/calculate the dimension of random fractals and exponents of statistical physics models via the Knizhnik-Polyakov-Zamolodchikov (KPZ) formula.

What does it mean for a RPM to converge?

- Metric structure (Le Gall'13, Miermont'13, ...)
- Statistical physics observables (Duplantier-Miller-Sheffield'14, ...)
- Conformal structure (H.-Sun'19)

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Conformal embedding of random planar maps

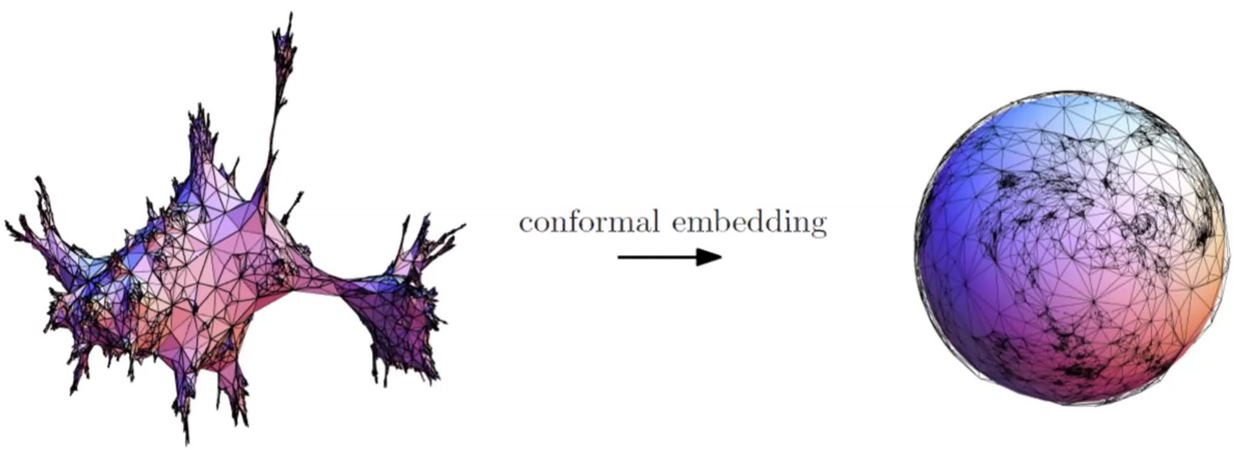

Figure by Nicolas Curien.

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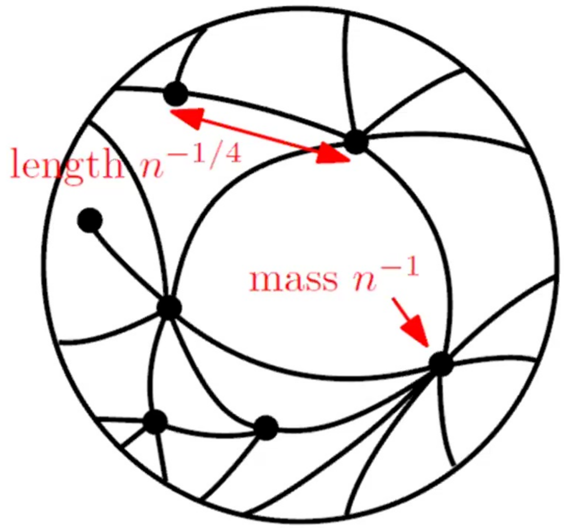
Conformal embedding of random planar maps



conformally embedded planar map

The embedded planar map induces the following on $\mathbb{S}^2$:

a **measure** μ_n and a **metric (distance function)** D_n.

Conjecture: For a large class of conformal embeddings, (D_n, μ_n) converge as $n \to \infty$.

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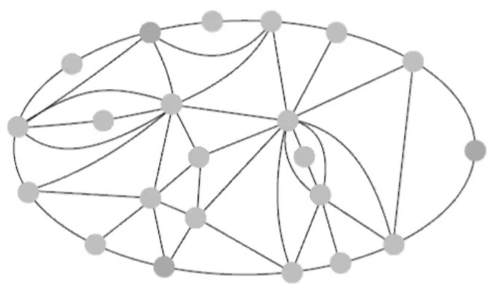
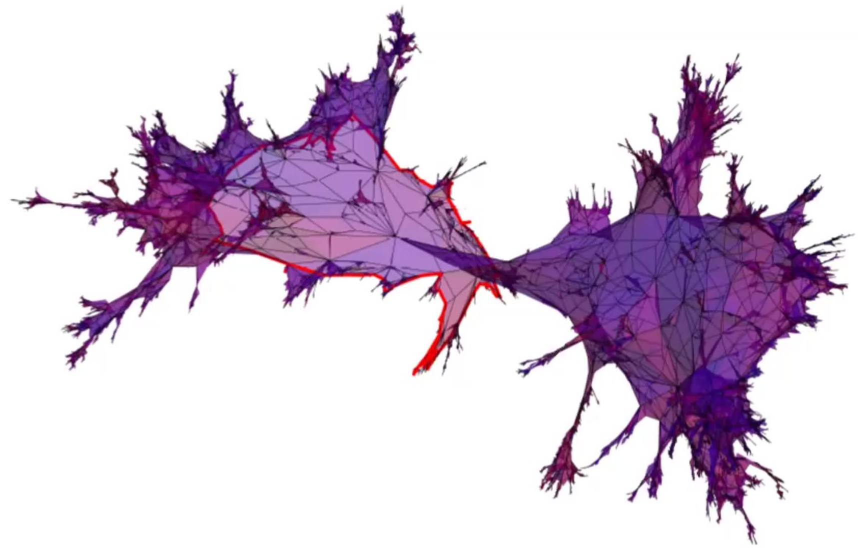
Triangulations with a boundary

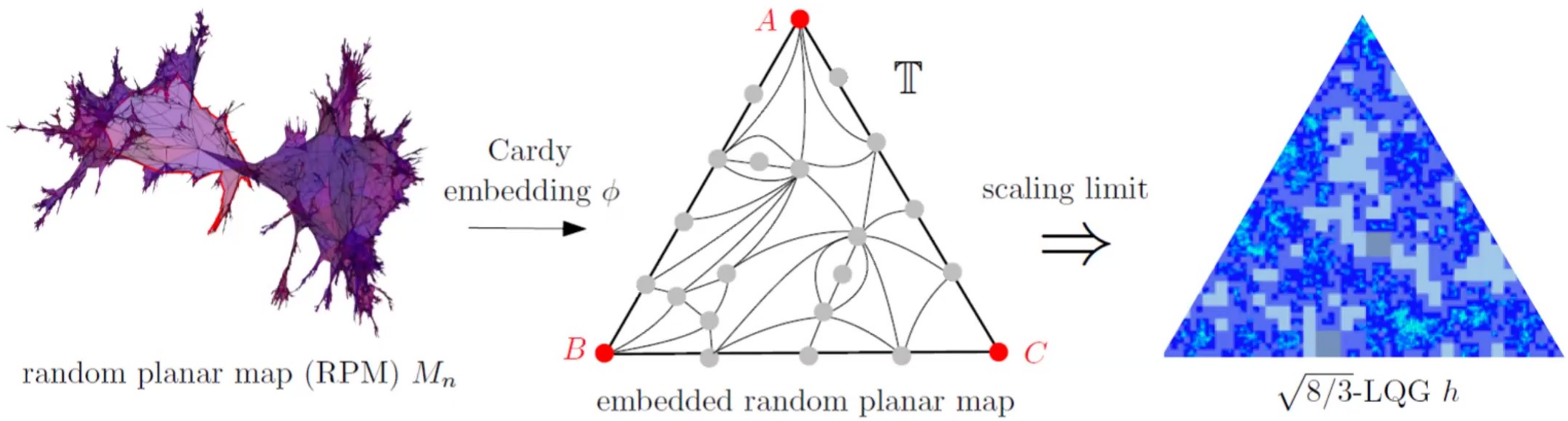




A **triangulation with boundary** is a planar map where all faces have three edges, except for the **exterior face**, which has arbitrary degree and simple boundary.

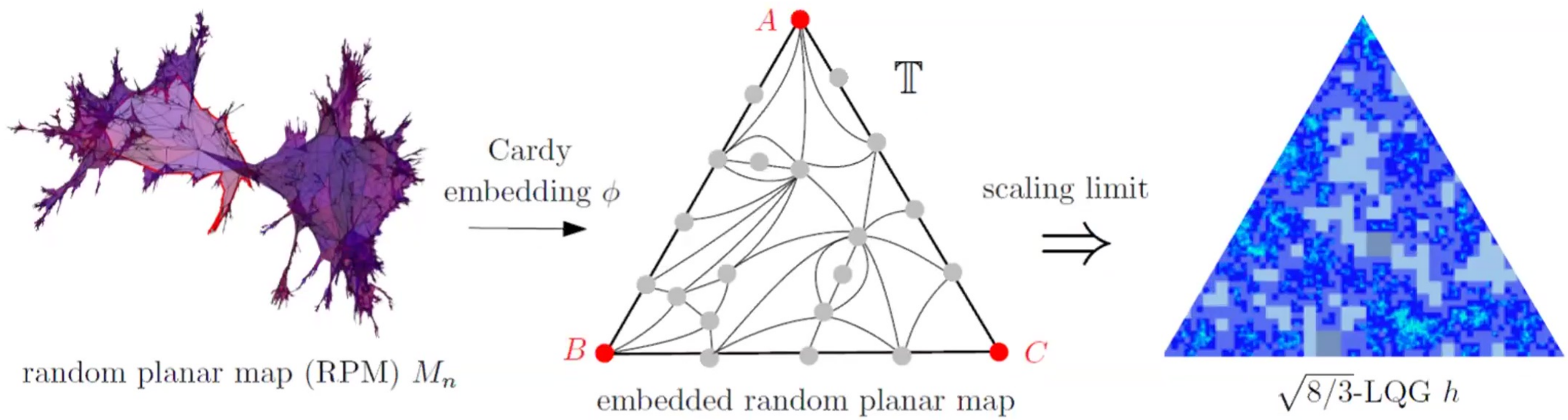
Simulation by Bettinelli

Conformally embedded RPM converge to $\sqrt{8/3}$-LQG



- Uniform triangulation M_n with n vertices, boundary length $\lceil n^{1/2} \rceil$.
- **Cardy embedding:** uses properties of percolation on the RPM.
- Let μ_n be renormalized counting measure on the vertices in $\mathbb{T}$.
- Let D_n be a metric (distance function) on $\mathbb{T}$ prop. to graph distances.
- Let μ be $\sqrt{8/3}$-LQG area measure in $\mathbb{T}$, and D the associated metric.

Conformally embedded RPM converge to $\sqrt{8/3}$-LQG



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Theorem 1 (H.-Sun'19)

In the above setting, $(\mu_n, D_n) \Rightarrow (\mu, D)$.

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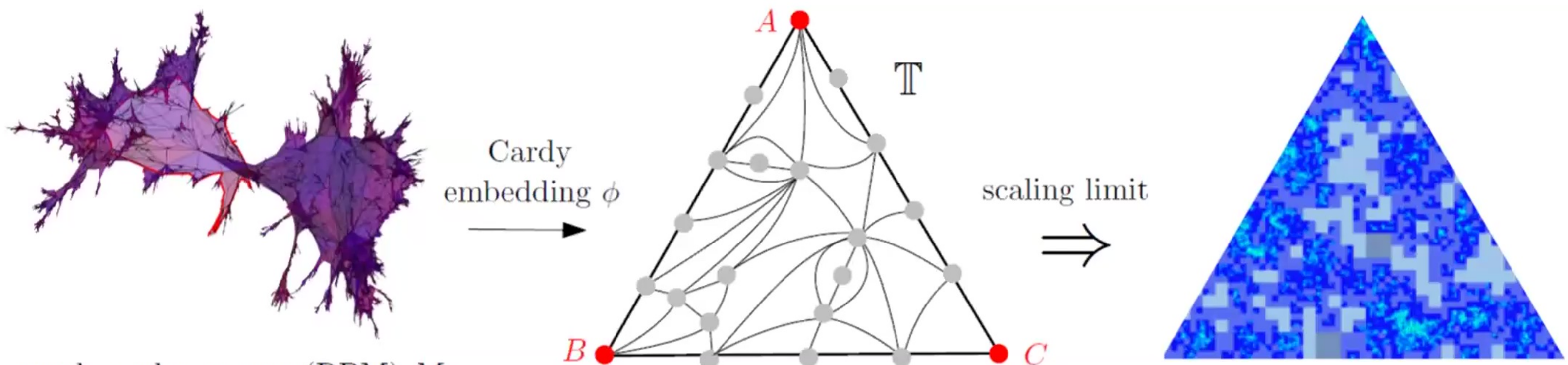
Conformally embedded RPM converge to $\sqrt{8/3}$-LQG



random planar map (RPM) M_n

embedded random planar map

$\sqrt{8/3}$-LQG h

Theorem 1 (H.-Sun'19)

In the above setting, $(\mu_n, D_n) \Rightarrow (\mu, D)$.

More precisely, $\exists$ coupling of M_n and h s.t. with probability 1, as $n \to \infty$,

- $\mu_n(A) \to \mu(A)$, $\forall$ balls $A \subset \mathbb{T}$ (measure convergence)
- $D_n(z, w) \to D(z, w)$, uniformly in $z, w \in \mathbb{T}$ (metric convergence)

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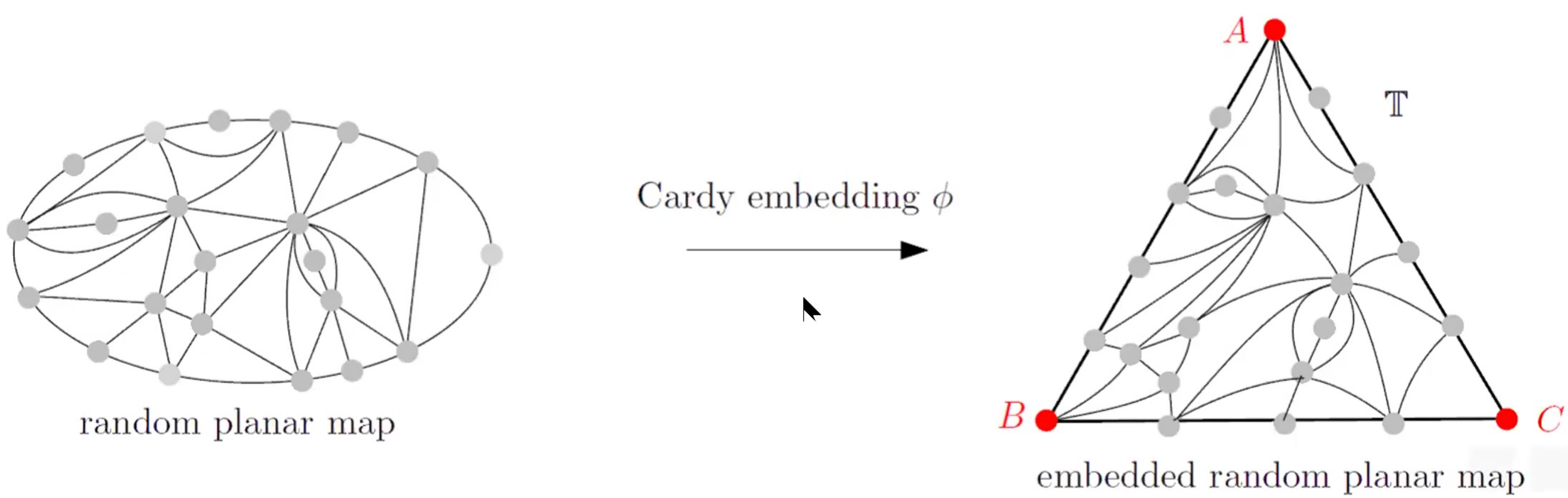
Cardy embedding: percolation-based embedding



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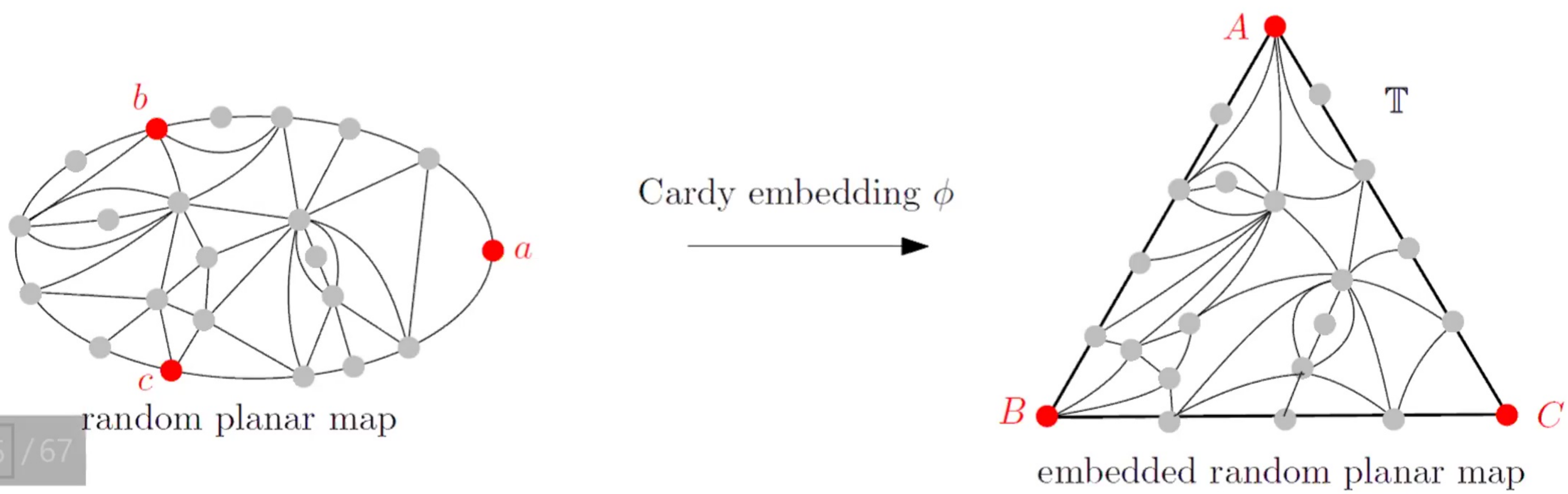
Cardy embedding: percolation-based embedding



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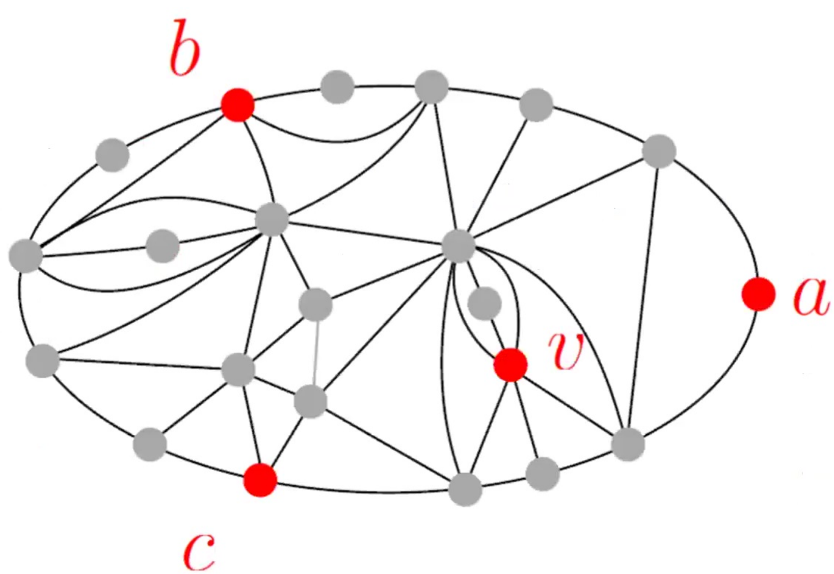
Cardy embedding: percolation-based embedding

- What is the "correct" position of v in $\mathbb{T}$?



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Cardy embedding: percolation-based embedding

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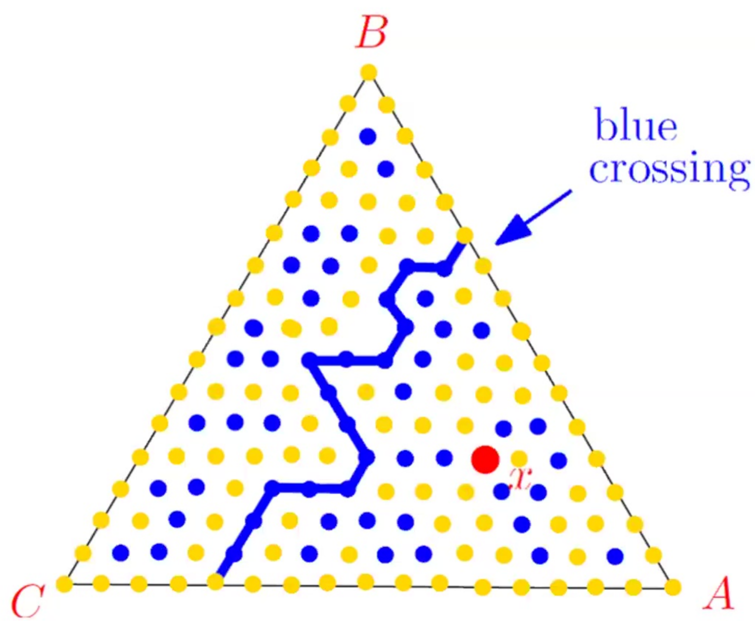

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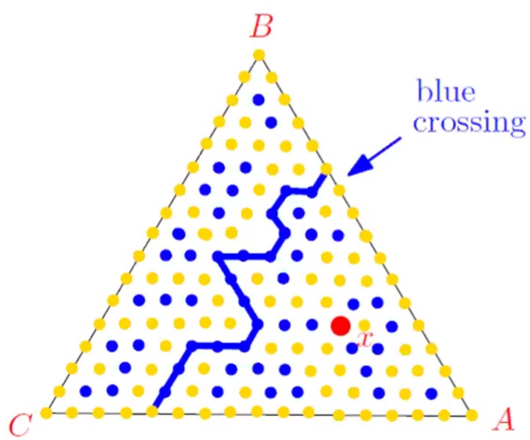
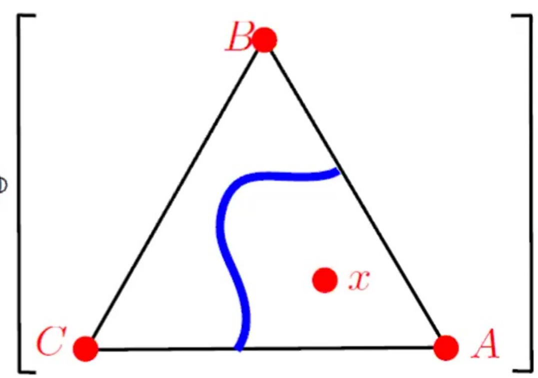
Cardy embedding: percolation-based embedding

- What is the "correct" position of v in $\mathbb{T}$?



$p_A(x) = \mathbb{P}$



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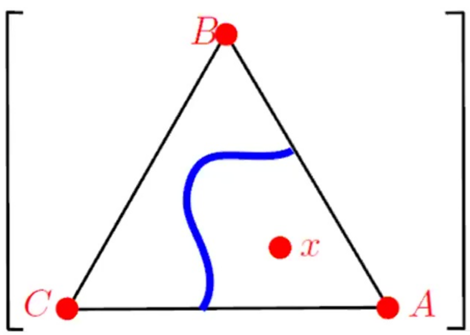
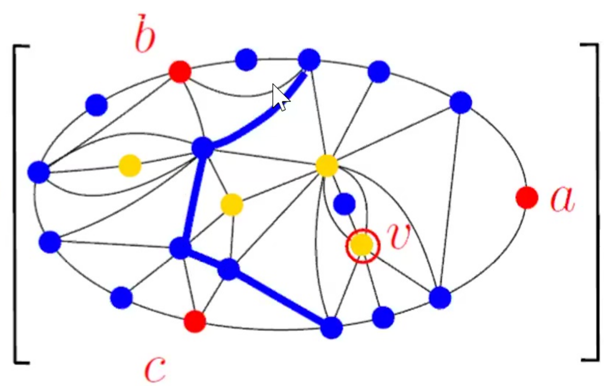
Cardy embedding: percolation-based embedding

- What is the "correct" position of v in $\mathbb{T}$?
- Map $v \in V(M)$ to $x \in \mathbb{T}$ such that

$$(p_A(x), p_B(x), p_C(x)) = (\widehat{p}_a(v), \widehat{p}_b(v), \widehat{p}_c(v)).$$

$p_A(x) = \mathbb{P}$ 

$\widehat{p}_a(v) = \mathbb{P}$ 

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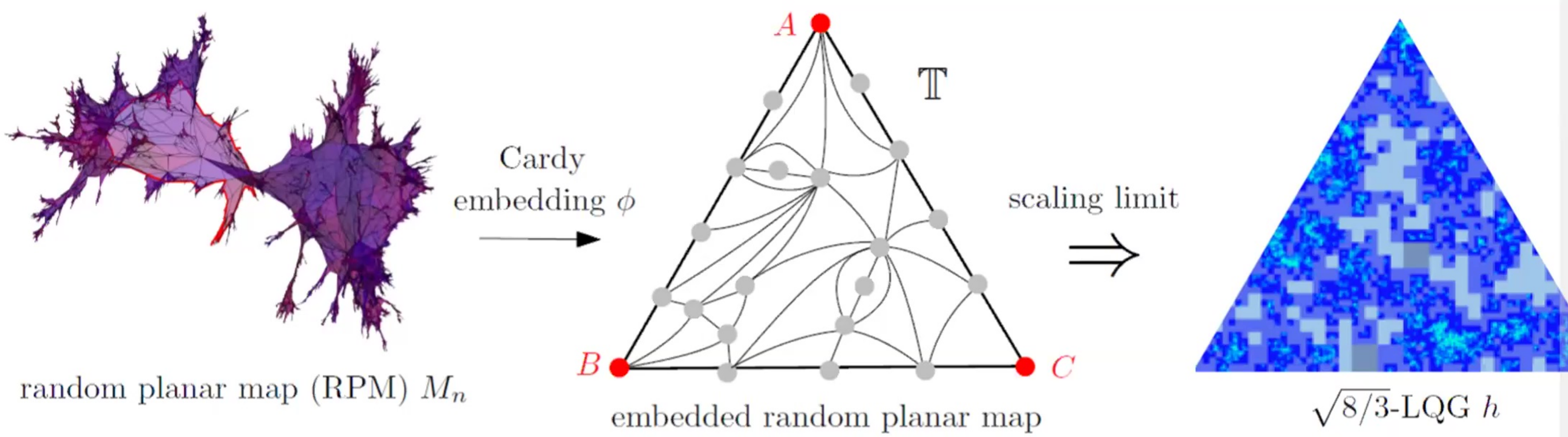
RPM ⇒ LQG under conformal embedding



Our result is for **uniform triangulations** and the **Cardy embedding**, but is also believed to hold for other

1. conformal embeddings,
2. local map constraints, and
3. universality classes of random planar maps.

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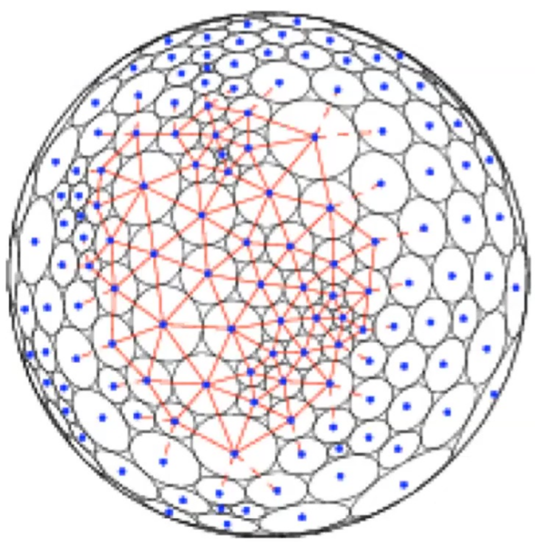
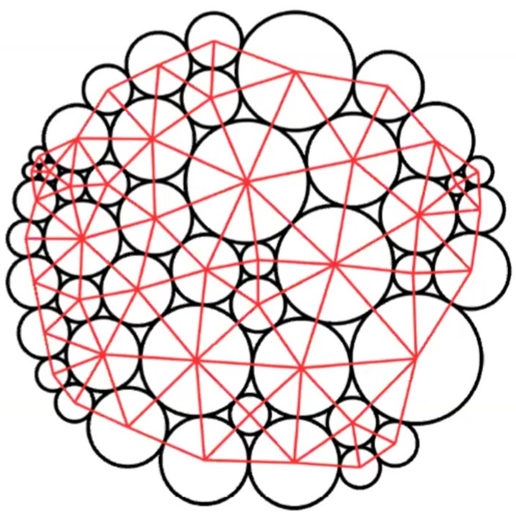
Discrete conformal embeddings

- Circle packing
- Riemann uniformization
- Tutte embedding
- Cardy embedding



circle packing (sphere topology)



circle packing (disk topology)

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Discrete conformal embeddings

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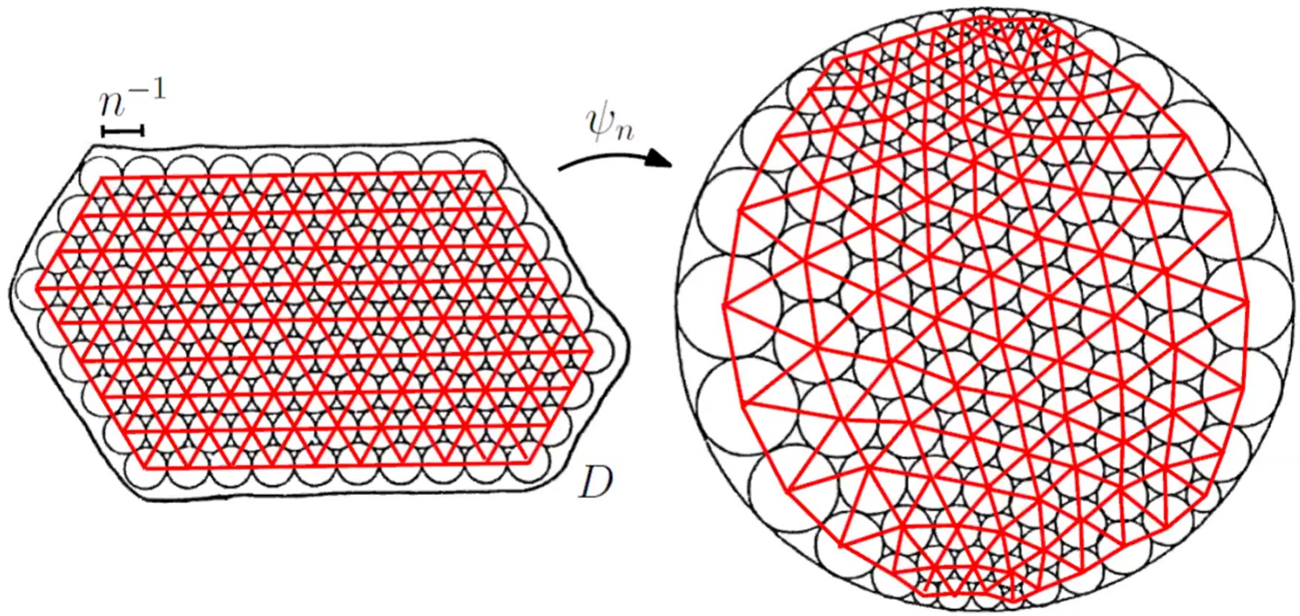

Rodin-Sullivan'87, confirming Thurston's conjecture

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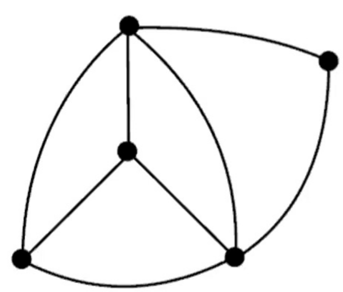
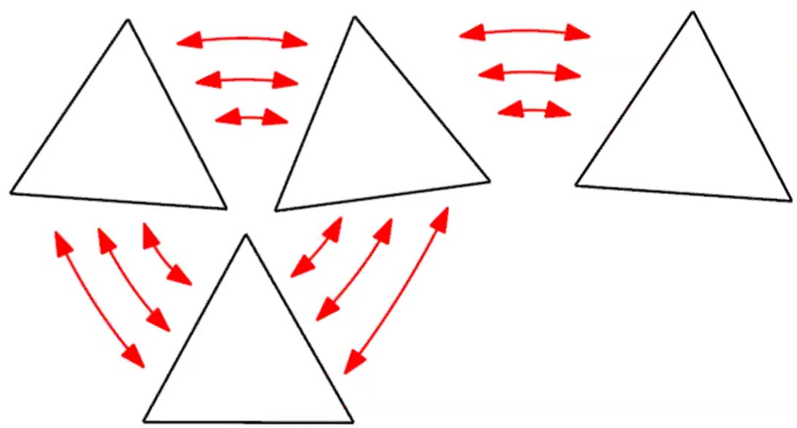


Random planar map



Riemannian manifold

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Discrete conformal embeddings

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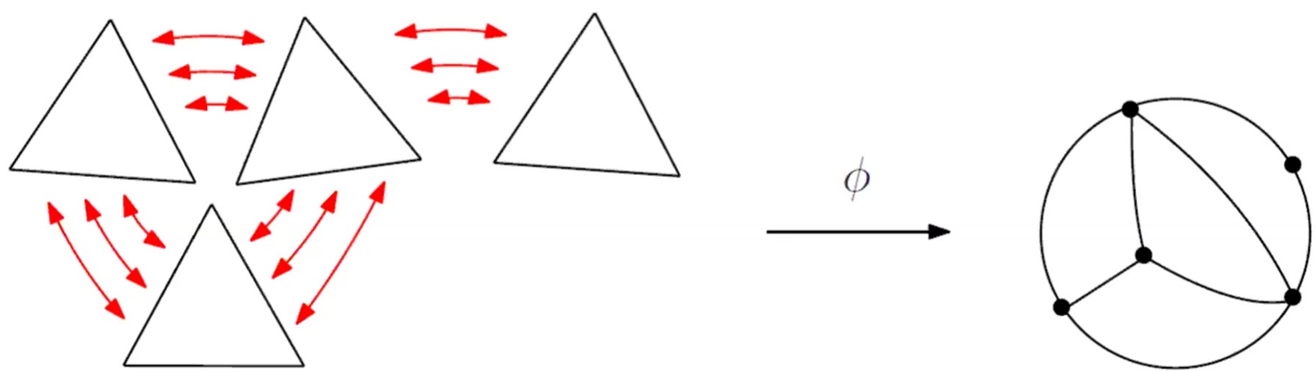
Uniformization theorem: For any simply connected Riemann surface M there is a conformal map ϕ from M to either $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{S}^2$.



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Discrete conformal embeddings

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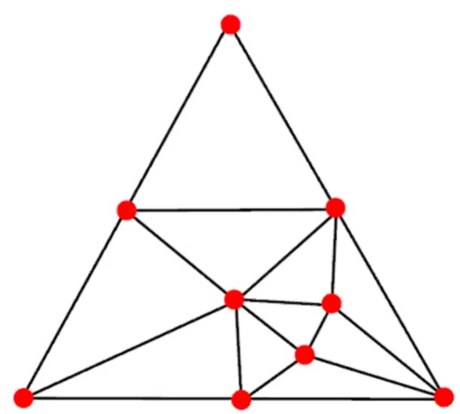

Tutte embedding

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Discrete conformal embeddings

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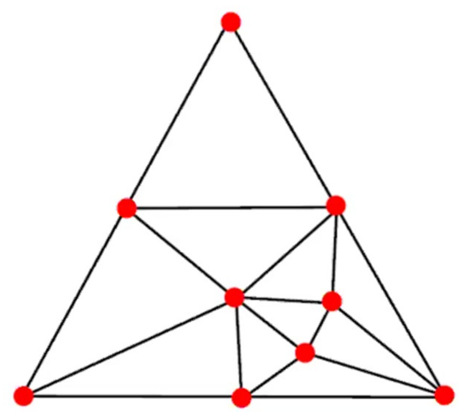

Tutte embedding

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Summary

- Universal and canonical random 2d objects
- Cardy embedded uniform triangulations $\Rightarrow \sqrt{8/3}$-LQG
- Future direction: universality (other embeddings and planar maps)

Thanks for your attention!