Title: q-Opers, QQ-Systems, and Bethe Ansatz

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Abstract: We introduce the notions of (G,q)-opers and Miura (G,q)-opers, where G is a simply-connected complex simple Lie group, and prove some general results about their structure. We then establish a one-to-one correspondence between the set of (G,q)-opers of a certain kind and the set of nondegenerate solutions of a system of XXZ Bethe Ansatz equations. This can be viewed as a generalization of the so-called quantum/classical duality which I studied with D. Gaiotto several years ago. q-Opers generalize classical side, while on the quantum side we have more general XXZ Bethe Ansatz equations. The generalization goes beyond the scope of physics of N=2 supersymmetric gauge theories.

q-Opers, QQ-Systems and Bethe Ansatz
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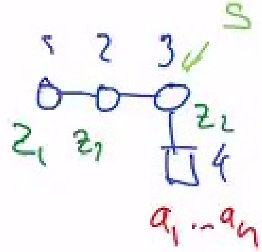
① Motivation

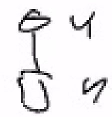
- Enumerative AG [quantum K-theory of Nakajima quiver varieties]
- [Gaiotto, K] Quantum/classical duality
  XXZ / trig Ruijsenaars–Schneider (tRS) model
  Bethe
  q-Opers
- Geometric Langlands

XXZ

quantum K-theory



T^*Fl_n $\hbar$



$T^*Gr_{k,n}$

$$K(T^*Fl_n) = \frac{\mathbb{C}[\underbrace{s^{\pm1}_{1,a}\dots s^{\pm1}_{n-1,a}}_{\frac{n(n-1)}{2}},\ a_1^{\pm1}\dots a_n^{\pm1},\ \hbar^{\pm1},\ z_1^{\pm1}\dots z_n^{\pm1}]}{(\text{Bethe equs})}$$

XXZ $\quad e^{\frac{\partial W}{\partial S}} = 1$

$$V \xleftrightarrow{q\to1} e^{\frac{W}{\log q}}$$ — Yang-Yang

[K, Pushkar, Smirnov, Zeitlin]

$$K(T^*Fl) = \frac{\mathbb{C}[a_1^{\pm1}\dots a_n^{\pm1},\ \hbar^{\pm1},\ z_1^{\pm1},\dots z_n^{\pm1},\ p_1^{\pm1}\dots p_n^{\pm1}]}{(\text{tRS energy equs})}$$

n

$$p_i = \Lambda^i V_i \otimes \Lambda^{i-1} V_{i-1}^* \to \frac{s_{i,1}\cdots s_{i,i}}{s_{i-1,1}\cdots s_{i-1,i-1}}$$

quantum K-theory

$$K(T^*Fl_n) = \frac{\mathbb{C}[S^{\pm1}_{1,a}\dots S^{\pm1}_{n-1,a},\ a_1^{\pm1}\dots a_n^{\pm1},\ \hbar^{\pm1},\ z_1^{\pm1}\dots z_n^{\pm1}]}{(\text{Bethe eqns})}$$

XXZ

$$e^{\frac{\partial W}{\partial S}} = 1$$

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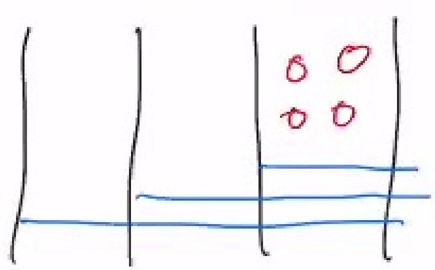
n

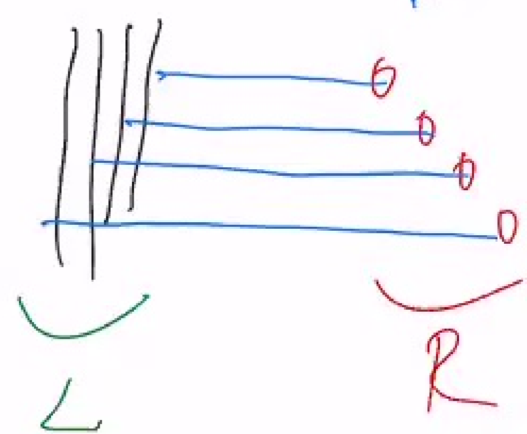
tRS

$$p_i = \Lambda^i V_i \otimes \Lambda^{i-1} V_{i-1}^* \to \frac{S_{i,1}\cdots S_{i,i}}{S_{i-1,1}\cdots S_{i-1,i-1}}$$

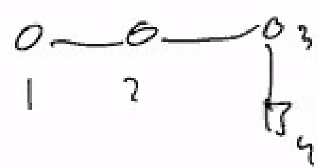
$\mathcal{N}=2^*$ 4d on $\mathbb{R}^2\times S^1\times I_{L,R}$



$\Longrightarrow$



L R

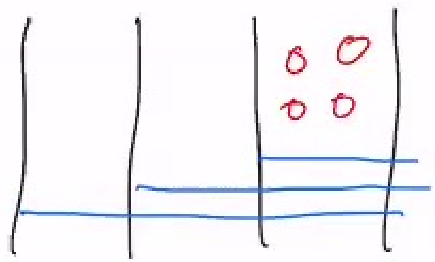


$$\to \frac{S_{i,1}\cdots S_{i,i}}{S_{i-1,1}\cdots S_{i-1,i-1}}$$

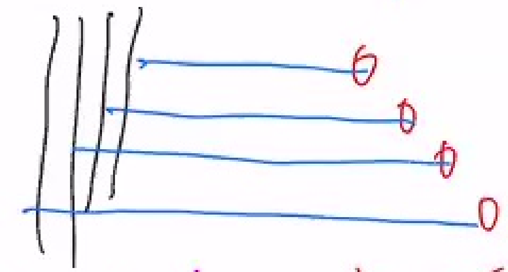
ERS

4d $\mathcal{N}=2^*$ on $\mathbb{R}^2\times S^1\times I_{L,R}$

3d $\mathcal{N}=2^*$ on $\mathbb{R}^2\times S^1$

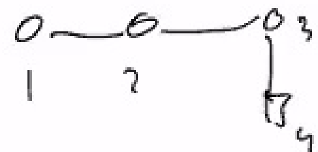


$\Longrightarrow$



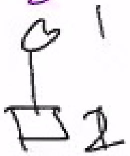
t_1

L $\quad$ t_1 $\quad$ R



$n=2$

$T^*\mathbb{P}^1$



$a_1\ a_2$

$z = \frac{\xi_1}{\xi_2}$

$$H_1 = \frac{\xi_1 - t_1\xi_2}{\xi_1-\xi_2}\, p_1 + \frac{\xi_2 - t_1\xi_1}{\xi_2-\xi_1}\, p_2$$

$$H_1 = a_1 + a_2$$

$$p_1\cdot p_2 = a_1\cdot a_2$$

$$K(T^*\mathbb{P}^1) = \frac{\mathbb{C}[\xi_1,\xi_2,a_1,a_2,t_1,p_1,p_2]}{(\qquad)}$$

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Let G - simple, simply-connected Lie grp.

<u>q-connection on $\mathbb{P}^1$</u> • principal G-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$

$q \in \mathbb{C}^\times$ • $M_q : \mathbb{P}^1 \to \mathbb{P}^1$, $\mathcal{F}_G^q$

$z \mapsto qz$,

Meromorphic (G,q)-connection on $\mathbb{P}^1$ - section A of $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$

U - open subset of $\mathbb{P}^1$

under change of coordinates $A(z) \longmapsto g(qz) A(z) g(z)^{-1}$

<u>(G,q)-oper on $\mathbb{P}^1$</u> : $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$

oper condition $I_U = U \cap M_q^{-1}(U)$

restriction $A : \mathcal{F}_G \to \mathcal{F}_G^q$ to I_U takes values in

$B_-(\mathbb{C}[I_U]) \cdot c \cdot B_-(\mathbb{C}[I$

restriction $A : \mathcal{F}_G \to \mathcal{F}_G^{\mathfrak{g}}$ to D_v takes values in

$$B_-(\mathbb{C}[[u]]) \cdot c \cdot B_-(\mathbb{C}[[u]])$$

$c = \prod_i s_i$

c Coxeter element

$\{e_i, f_i, \check{\alpha}_i\}$
$i = 1, \ldots, r$

$$A(z) = n'(z) \left(\prod_i \phi_i(z)^{\check{\alpha}_i} s_i \right) \cdot n(z)$$

$n', \ n(z) \in N_-(z)$

$N_- = B_- / H$

<u>Miura (G, q)-opers on $\mathbb{P}^1$</u>: $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$

- $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ - (G, q) oper
- $\mathcal{F}_{B_+}$ preserved by A

pick x, $\mathcal{F}_{G,x}$ - G torsor w/ reductions $\mathcal{F}_{B_-,x}$, $\mathcal{F}_{B_+,x}$

$\mathcal{F}_{G,x} \simeq G$, $\mathcal{F}_{B_-,x} \simeq a$

$n', n(z) \in \underline{N}(z)$ $\qquad N_- = B_-/H$

Miura (G,q)-opers on $\mathbb{P}^1$: $\quad (\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$

- $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ - (G,q) oper
- $\mathcal{F}_{B_+}$ preserved by A

pick x, $\quad \mathcal{F}_{G,x}$ ~ G torsor w/ reductions $\mathcal{F}_{B_-,x}$, $\mathcal{F}_{B_+,x}$

$$\mathcal{F}_{G,x} = G, \qquad \mathcal{F}_{B_-,x} \simeq a \cdot B_-, \qquad \mathcal{F}_{B_+,x} \simeq b \cdot B_+$$

$$a^{-1} \cdot b \in B_- \backslash G / B_+ \simeq W_G$$

generic relative condition at x $\quad$ if $a^{-1} \cdot b \in J \subset B_- \cdot B_+$

Bruhat cell

Structure theorems

Th1: $\forall$ Miura (G,q)-oper s.t. $\mathcal{F}_{B_+}, \mathcal{F}_{B_-}$ are in generic relative position $\forall\, x \in V \subset \mathbb{P}^1$: if $g(qz)A(z)g(z)^{-1} \in B_+(z)$

then $g(z) \in B_+(z)\cdot N_-(z)$

Th2:

- $A(z) \in N_-(z)\cdot \prod_i \left(\varphi_i(z)^{\check{\alpha}_i}\cdot s_i\right)\cdot N_-(z) \cap B_+(z)$
- Any element from $\prod_i g_i(z)^{\check{\alpha}_i}\cdot \exp\left(\frac{\varphi_i(z)}{g_i(z)} t_i \cdot e_i\right)$

  $g_i(z)$ – rational

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(G, q)-opers w/ regular singularities

Let $\Lambda_i(z) \in \mathbb{C}[z]$,

$$A(z) = n'(z) \cdot \prod_i \left(\Lambda_i^{\check{\alpha}_i}(z) \cdot s_i\right) \cdot n(z)$$

roots of Λ – sing. of oper

Miura oper:

$$A(z) = \prod_i g_i^{\check{\alpha}_i}(z) \exp\left(\frac{\Lambda_i(z)}{g_i(z)} e_i\right)$$

Z-twisted (G, q)-oper:

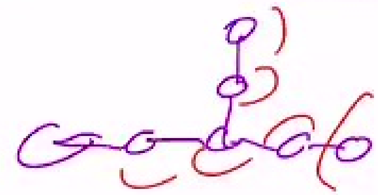
Z – reg. semisimple

$$A(z) = g(qz)\, Z\, g^{-1}(z)$$

$$Z = \prod_i z_i^{\check{\alpha}_i}$$

Miura-Plücker (G,q)-oper

$SL(2) \subset G$



V_i - irrep of G, ω_i - weight, highest weight vector v_i

W_i: v_i, $f_i v_i$

$i = 1, \dots, r$

$$A_i(z) = v(qz) Z_i \, v(z)^{-1} \Big|_{W_i} = \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ji}} \\ 0 & g_i^{-1}(z) \cdot \prod_{j \neq i} g_j(z)^{-a_{ji}} \end{pmatrix}$$

Th: $MP_qOp \longleftrightarrow Q\tilde{Q}$-System

Th: $MP_qOp \subset M_qOp$

Cartan connection

$$A^H(z) = \prod_i g_i(z)^{\check{\alpha}_i} \qquad A^H \rightsquigarrow Z$$

$$A^H(z) = \prod_i g_i(qz)^{\check{\alpha}_i} \cdot Z.$$

$$g_i(z) = z_i \frac{y_i(qz)}{y_i(z)}$$

Theorem:

$$\left\{ Z\text{-twisted Miura-Plücker } (G,q)\text{-opers} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Set of nondegenerate} \\ \text{polynomial solutions} \\ \text{of } QQ_G\text{-system} \end{array} \right\}$$

$$\tilde{\xi}_i Q_i^-(z) Q_i^+(qz) - \xi_i Q_i^-(qz) Q_i^+(z) = \Lambda_i(z) \cdot \prod_{j>i} \left(Q_j^+(qz)\right)^{-a_{ji}} \prod_{j<i} \left(Q_j^+(z)\right)^{-a_{ji}}$$

$$\tilde{\xi}_i = z_i \prod_{j>i} z_i^{a_{ji}}$$

$$\xi_i = z_i^{-1} \prod_{j<i} z_j^{-a_{ji}}$$

$$Q_i^+(z) = y_i(z)$$

$$v(z) = \prod_{i=1}^{r} y_i(z)^{\check{\alpha}_i} \prod_{i=1}^{r} e^{-\frac{Q_i^-(z)}{Q_i^+(z)} e_i} \times \ldots$$

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$y_i(z)$

Theorem:

$$\left\{ Z\text{-twisted Miura-Plücker } (G,q)\text{-opers} \right\} \longleftrightarrow \left\{ \text{Set of nondegenerate polynomial solutions of } QQ_G\text{-system} \right\}$$

$$\tilde{\xi}_i Q_i^-(z) Q_i^+(qz) - \xi_i Q_i^-(qz) Q_i^+(z) = \Lambda_i(z) \cdot \prod_{j>i} \left(Q_j^+(qz)\right)^{-a_{ji}} \prod_{j<i} \left(Q_j^+(z)\right)^{-a_{ji}}$$

$$\tilde{\xi}_i = z_i \prod_{j>i} z_i^{a_{ji}}$$

$$\xi_i = z_i^{-1} \prod_{j<i} z_j^{-a_{ji}}$$

$$\left[\begin{array}{l} Q_i^+(z) = y_i(z) \\ v(z) = \prod_{i=1} y_i(z)^{\check{\alpha}_i} \prod_{i=1} e^{-\frac{Q_i^-(z)}{Q_i^+(z)} e_i} \cdots \end{array}\right]$$

Theorem: $MP(G,q)O_p \subset M(G,q)O_p$

Bäcklund transformations:

$$A(z) \longmapsto A^{(i)}(z) = e^{\mu_i(qz) f_i} A(z) e^{-\mu_i(z) f_i}$$

$$\mu_i(z) = \frac{\prod_{j \neq i} (Q_j^+)^{-a_{ji}}}{Q_i^+(z) Q_i^-(z)}$$

results in

$$Q_j^+ \longmapsto Q_j^+ , \quad j \neq i$$

$$Q_i^+ \longmapsto Q_i^- , \quad Z \longmapsto s_i(Z)$$

$$\left\{ \begin{array}{c} \text{Nondegenerate} \\ \text{QQ system} \end{array} \right\} \longleftrightarrow \left\{ \text{Bethe eqns} \right\}^*$$

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$$\upsilon(z) = \prod_{i=1}^{r} y_i(z)^{\alpha_i} \prod_{i=1} e^{Q_i^+(z)} \ldots$$

Theorem: $MP(G,q)O_p \subset M(G,q)O_p$

Bäcklund transformations:

$$A(z) \longmapsto A^{(i)}(z) = e^{\mu_i(qz) f_i} A(z) e^{-\mu_i(z) f_i}$$

$$\mu_i(z) = \frac{\prod_{j \neq i} (Q_j^+)^{-a_{ji}}}{Q_i^+(z) Q_i^-(z)}$$

results in

$$Q_j^+ \longmapsto Q_j^+ , \quad j \neq i$$

$$Q_i^+ \longmapsto Q_i^- , \quad Z \longmapsto s_i(Z)$$

$$\left\{ \begin{matrix} \text{nondegenerate} \\ QQ \text{ system} \end{matrix} \right\} \longleftrightarrow \{\text{Bethe eqns}\}^*$$

$4(\hat{g})$

Theorem: $w_0 = s_{i_1} \ldots s_{i_\ell}$ - maximal element in W_G

$$\{ w_0\text{-generic } QQ \text{ system} \} \longleftrightarrow \left\{ \begin{matrix} \text{nondegenerate } Z\text{-twisted} \\ \text{Miura } (G,q) \text{ op} \end{matrix} \right\}$$

$$\{ W_0\text{-generic } Q\tilde{Q} \text{ system} \} \longleftrightarrow \left\{ \begin{matrix} \text{nondegenerate } Z\text{-twisted} \\ \text{Miura } (G,q) \text{ op} \end{matrix} \right\}$$

$(SL(2),q)$

$$\mathcal{L} \subset W \qquad \mathcal{L} \to W \to W/\mathcal{L}$$

$$A: \mathcal{L} \xrightarrow{\sim} (W/\mathcal{L})^q$$

$$s = \begin{pmatrix} Q^-(z) \\ Q^+(z) \end{pmatrix}$$

$$s(qz) \wedge Z\, s(z) = \Lambda(z)$$

$$\Downarrow$$

$$\zeta\, Q^+(qz)\, Q^-(z) - \zeta^{-1} Q^+(z)\, Q^-(qz) = \Lambda(z)$$

$$\Downarrow$$

tRS

$$Q^+ = z - p_1$$

$$Q^- = z - p_2$$