Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 23

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Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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8.2 Extension to multi-D

Here we consider the multi-dimensional hyperbolic system

$$u_t + \sum_i \vec{f}^{\,i}(u)_{x_i} = 0$$

hyperbolic system

$$u_t + \sum_i \vec{f}^{\,i}(u)_{x_i} = 0$$

For Euler equs:

$$u = \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ E \end{pmatrix} \begin{matrix} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \rho \vec{v} \\ \\ \end{matrix} , \quad \vec{f}^{\,i} = \begin{pmatrix} \rho v_i \\ \rho v_i v_j + \delta_{ij} p \\ v_i (E + p) \end{pmatrix}$$

$j = 1, 2, 3$

Two basic approaches:

1) Dimensional splitting

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8.2.1 Dimensional splitting

Idea: introduce dimensional splitting in analogy to operator splitting for source terms (Chap. 8.1)

$$\rightsquigarrow \quad \left.\begin{array}{ll} \text{PDE:} & u_t + f^1(u)_{x_1} = 0 \\ \text{IC:} & u^n \end{array}\right\} \xrightarrow[\sim 1D]{\text{evolve } \Delta t} \text{“} u^{n+\frac{1}{3}} \text{”}$$

$$\downarrow$$

$$\left.\begin{array}{ll} \text{PDE:} & u_t + f^2(u)_{x_2} = 0 \\ \text{IC:} & u^{n+\frac{1}{3}} \end{array}\right\} \xrightarrow{\Delta t} \text{“} u^{n+\frac{2}{3}} \text{”}$$

$$\downarrow$$

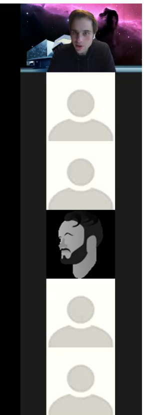
$$\left.\begin{array}{ll} \text{PDE:} & u_t + f^3(u)_{x_3} = 0 \\ \text{IC:} & u^{n+\frac{2}{3}} \end{array}\right\} \xrightarrow{\Delta t} \text{“} u^{n+1} \text{”}$$

For an *explicit conservative method*:

1) $$u^{n+\frac{1}{3}}_{i,j,k} = u^n_{i,j,k} + \frac{\Delta t}{\Delta x_1}\left[(g^1)^n_{i-\frac{1}{2},j,k} - (g^1)^n_{i+\frac{1}{2},j,k}\right] \quad \forall j,k$$

$$(g^1)^n_{i+\frac{1}{2},j,k} = g^1(u^n_{i,j,k}, u^n_{i+1,j,k})$$ numerical flux at cell interface $x_{i+\frac{1}{2}}$

2) $$u^{n+\frac{2}{3}}_{i,j,k} = u^{n+\frac{1}{3}}_{i,j,k} + \frac{\Delta t}{\Delta x_2}\left[(g^2)^{n+\frac{1}{3}}_{i,j-\frac{1}{2},k} - (g^2)^{n+\frac{1}{3}}_{i,j+\frac{1}{2},k}\right] \quad \forall i,k$$

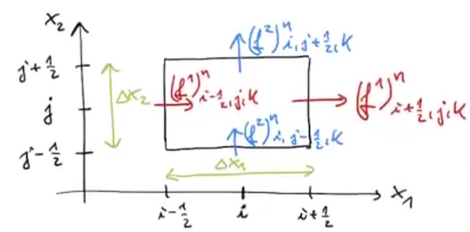


8.2.2 Unsplit finite volume schemes

⤳ account for combined flux contributions in a single step

Consider explicit 3D finite volume scheme:

projection onto x_1-x_2 plane

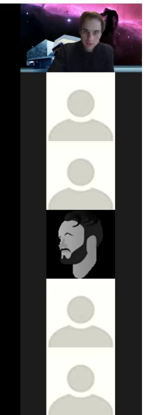


Finite volume discretization of Cartesian domain

$\Delta V_{i,j,k} = (\Delta x_1)_{i,j,k} \times (\Delta x_2)_{i,j,k} \times (\Delta x_3)_{i,j,k}$

$$u^{n+1}_{i,j,k} = u^n_{i,j,k} + \frac{\Delta t}{\Delta x_1}\left[g^1_{i-\frac{1}{2},j,k} - g^1_{i+\frac{1}{2},j,k}\right] + \frac{\Delta t}{\Delta x_2}\left[g^2_{i,j-\frac{1}{2},k} - g^2_{i,j+\frac{1}{2},k}\right] + \frac{\Delta t}{\Delta x_3}\left[g^3_{i,j,k-\frac{1}{2}} - g^3_{i,j,k+\frac{1}{2}}\right]$$

$u^n_{i,j,k}$: cell average, often assigned to centre of cell ⤳ "cell-centred methods"



Two basic approaches:

1) Dimensional splitting

2) unsplit finite volume schemes

8.2.2 Unsplit finite Volume schemes

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8.2.2 Unsplit finite volume schemes

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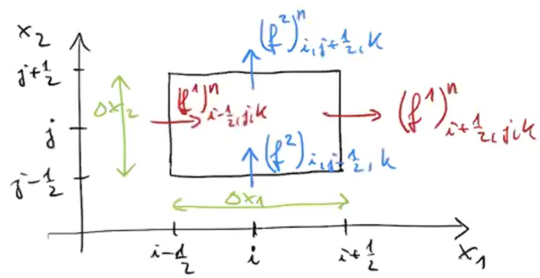

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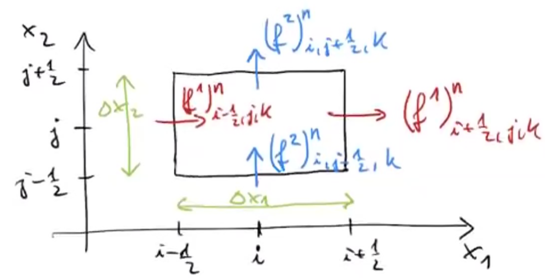
8.2.2 Unsplit finite volume schemes

Consider explicit 3D finite volume scheme:



$$
\begin{aligned}
u^{n+1}_{i,j,k} = u^n_{i,j,k} &+ \frac{\Delta t}{\Delta x_1}\left[g^1_{i-\frac{1}{2},j,k} - g^1_{i+\frac{1}{2},j,k}\right] \\
&+ \frac{\Delta t}{\Delta x_2}\left[g^2_{i,j-\frac{1}{2},k} - g^2_{i,j+\frac{1}{2},k}\right] \\
&+ \frac{\Delta t}{\Delta x_3}\left[g^3_{i,j,k-\frac{1}{2}} - g^3_{i,j,k+\frac{1}{2}}\right]
\end{aligned}
$$

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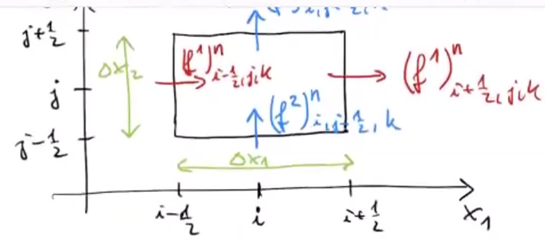

$$u^{n+1}_{i,j,k} = u^n_{i,j,k} + \frac{\Delta t}{\Delta x_1}\left[g^1_{i-\frac{1}{2},j,k} - g^1_{i+\frac{1}{2},j,k}\right]$$

$$+ \frac{\Delta t}{\Delta x_2}\left[g^2_{i,j-\frac{1}{2},k} - g^2_{i,j+\frac{1}{2},k}\right]$$

$$+ \frac{\Delta t}{\Delta x_3}\left[g^3_{i,j,k-\frac{1}{2}} - g^3_{i,j,k+\frac{1}{2}}\right]$$

$u^n_{i,j,k}$: cell average, often assigned to centre of cell

Gener

$$+ \frac{\Delta t}{\Delta x_3}\left[g_{i,j,k-\frac{1}{2}} - g_{i,j,k+\frac{1}{2}} \right]$$

$u^n_{i,j,k}$: cell average, often assigned to centre of cell

Generalization of Godunov's scheme:

$\rightsquigarrow$ apply fluxes across each intercell boundary in exact same way as for 1D problems

$$u^n_{i,j,k} \simeq \frac{1}{\Delta V_{i,j,k}} \int_{\Delta V_{i,j,k}} u(x_1, x_2, x_3, t)\, dx_1 dx_2 dx_3$$

$$g^1_{i+\frac{1}{2},j,k} = f^1\left(u_{i+\frac{1}{2},j,k}(0)\right)$$

$\uparrow$ solution to the local Riemann problem along cell interface

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ΔV_{ijk}

$$g^1_{i+\frac{1}{2},j,k} = f^1(u_{i+\frac{1}{2},j,k}(0))$$

↑ solution to the local Riemann problem along cell interface

$$g^2_{i,j+\frac{1}{2},k} = f^2(u_{i,j+\frac{1}{2},k}(0))$$

$$g^3_{i,j,k+\frac{1}{2}} = f^3(u_{i,j,k+\frac{1}{2}}(0))$$

where

- $u_{i+\frac{1}{2},j,k}\left(\frac{x_1}{t}\right)$ is the solution to the Riemann problem

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$$g^2_{i,j+\frac{1}{2},k} = f^2(u_{i,j+\frac{1}{2},k}(0))$$

$$g^3_{i,j,k+\frac{1}{2}} = f^3(u_{i,j,k+\frac{1}{2}}(0))$$

where

- $u_{i+\frac{1}{2},j,k}\left(\frac{x_1}{t}\right)$ is the solution to the Riemann problem

$$u_t + f^1(u)_{x_1} = 0$$

$$u(x_1,0) = \begin{cases} u^n_{i,j,k}, & x_1 < 0 \\ u^n_{i+1,j,k}, & x_1 > 0 \end{cases}$$

and similarly for the other spatial directions

Remark: As in 1D, appr

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$$g^2_{i,j+\frac{1}{2},k} = f^2\left(u_{i,j+\frac{1}{2},k}(0)\right)$$

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and similarly for the other spatial directions

Remark: As in 1D, approximate Riemann solvers can be used for the above Riemann problems

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8.2.3 Higher-order unsplit schemes: MUSCL-Hancock methods

Van Leer / Hancock 1977: J. Comp. Phys

23 263 (1977)
23 276 (1977)
32 101 (1979)

modification of Godunov's method to achieve higher accuracy

Idea:

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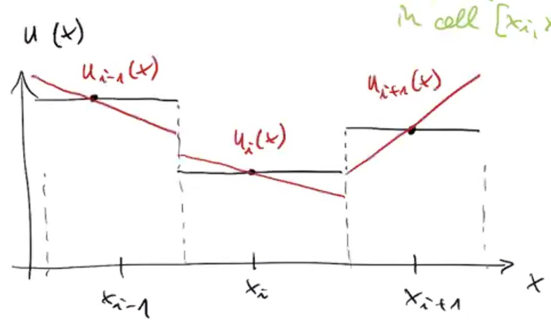
32 101 (1979)

modification of Godunov's method to achieve higher accuracy

Idea: piece-wise linear (or higher-order) local reconstruction

$$u_i(x) = u_i^n + \frac{(x - x_i)}{\Delta x} \Delta_i, \quad x \in [0, \Delta x]$$

suitably chosen slope of $u_i(x)$ in cell $[x_i, x_i + \Delta x]$



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slope of $u_i(x)$
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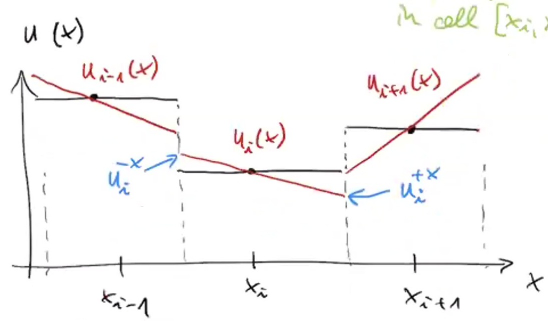

REA algorithm:

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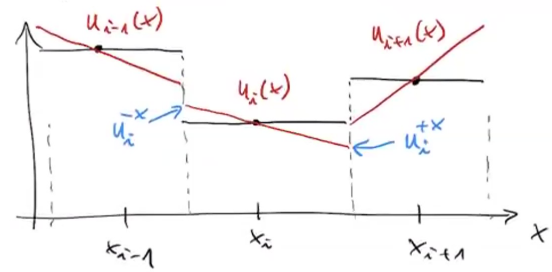

REA algorithm:

(I) Reconstruction

reconstruct cell averages $u^n_{i,j,k}$ independently in x, y, z, select appropriate slopes Δ_{ijk} and obtain <u>boundary extrapolated values</u> (BEV)

$$u^{-x}_{ijk} = u^n_{ijk} - \frac{1}{2}\Delta_i \; ; \quad u^{+x}_{ijk} = u^n_{ijk} + \frac{1}{2}\Delta_i$$

$$u^{-y}_{ijk} = \dots \; ; \quad u^{-y}_{ijk} = \dots$$

$$u^{-z}_{ijk} = \dots \qquad u^{-z}_{ijk} = \dots$$

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$u_{i,j,k} = \dots \qquad u_{i,j,k} = \dots$

$u^{-z}_{i,j,k} = \dots \qquad u^{-z}_{i,j,k} = \dots$

(II) (a) Evolution of BEV by half a timestep $\frac{\Delta t}{2}$:

$$\hat{u}^{l}_{i,j,k} = u^{l}_{i,j,k} + \frac{1}{2}\frac{\Delta t}{\Delta x}\left[f^1(u^{-x}_{i,j,k}) - f^1(u^{+x}_{i,j,k})\right] + \frac{1}{2}\frac{\Delta t}{\Delta y}\left[f^2(u^{-y}_{i,j,k}) - f^2(u^{+y}_{i,j,k})\right] + \frac{1}{2}\frac{\Delta t}{\Delta z}\left[f^3(u^{-z}_{i,j,k}) - f^3(u^{+z}_{i,j,k})\right]$$

for $l = -x, x, -y, y, -z, z$

(II) (b) Solutio

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(II) (b) Solution of piece-wise constant data Riemann problem:

at each cell interface $(i+\frac{1}{2}, j, k)$ solve the x/y/z-split 1D Riemann problem

$$u_t + f(u)_x = 0$$

$$u(x,0) = \begin{cases} \hat{u}^{+x}_{i,j,k} & x < 0 \\ \hat{u}^{-x}_{i,j,k} & x > 0 \end{cases}$$

to find $u_{i+\frac{1}{2},j,k}\left(\frac{x}{t}\right)$ (analogous for y, z)

The corresponding intercell fluxes are found as in the 1D

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Riemann problem

$$u_t + f(u)_x = 0$$

$$u(x,0) = \begin{cases} \hat{u}^{+x}_{i,j,k} & x < 0 \\ \hat{u}^{-x}_{i,j,k} & x > 0 \end{cases}$$

to find $u_{i+\frac{1}{2},j,k}\left(\frac{x}{t}\right)$ (analogous for y,z)

The corresponding intercell fluxes are found as in the 1D Godunov method:

$$g^1_{i+\frac{1}{2},j,k} = f^1\left(u_{i+\frac{1}{2},j,k}(0)\right)$$

$$g^2_{i,j+\frac{1}{2},k} = f^2\left(u_{i,j+\frac{1}{2},k}(0)\right)$$

$$g^3_{i,j,k+\frac{1}{2}} = f^3\left(u_{i,j,k+\frac{1}{2}}(0)\right)$$

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are found as in the 1D Godunov method:

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$$g^3_{i,j,k+\frac{1}{2}} = f^3(u_{i,j,k+\frac{1}{2}}(0))$$

(III) Average $u^{n+1}_{i,j,k} = \frac{1}{\Delta x \Delta y \Delta z} \int u^n(x,t^{n+1})\, d^3x$

$u^n_{i,j,k}(\frac{x}{t})$

Remarks: 1) This scheme is 2nd order accurate, numerical fluxes can be found using approximate Riemann solvers

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fluxes can be found using approximate Riemann solvers

2) Note that we did not have to consider the so-called generalized Riemann problem

$$u_t + f(u)_x = 0$$

$$u(x,0) = \begin{cases} u_{i-1,jik}(x), & x<0 \\ u_{i,jik}(x), & x>0 \end{cases}$$

The BEVs are evolved by $\frac{\Delta t}{2}$, which cancels the error introduced to linear order and provides stabilit

Riemann problem

$$u_t + f(u)_x = 0$$

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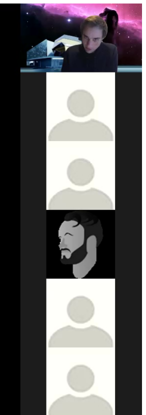
The BEVs are evolved by $\frac{\Delta t}{2}$, which cancels the error introduced to linear order and provides stability to the scheme.

" the reconstruction method gives rise to higher accuracy "

4)



4) Higher-order schemes can be devised using higher-order reconstruction.

For example:

$$u_{i,j,k}(x) = u_{i,j,k} + \frac{(x-x_i)}{\Delta x}\Delta_i^{(1)} + \frac{1}{2(\Delta x)^2}\left[(x-x_i)^2 - \frac{(\Delta x)^2}{12}\right]$$

$$\Delta_i^{(2)}$$

$\Delta_i^{(1)}$, $\Delta_i^{(2)}$ estimates for 1st & 2nd derivative

$\rightsquigarrow$ gives rise to 3rd-order accuracy.

→ <u>piece-wise parabolic method</u> (PPM

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For example;

$$u_{i,j,k}(x) = u_{i,j,k} + \frac{(x-x_i)}{\Delta x}\Delta_i^{(1)} + \frac{1}{2(\Delta x)^2}\left[(x-x_i)^2 - \frac{(\Delta x)^2}{12}\right]$$

$$\Delta_i^{(2)}$$

$\Delta_i^{(1)}$, $\Delta_i^{(2)}$ estimates for 1st & 2nd derivative

$\leadsto$ gives rise to 3rd-order accuracy.

$\longrightarrow$ piece-wise parabolic method (PPM)
(Colella & Woodward 1984)