Title: Equivariant Elliptic Cohomology

Speakers: Nora Ganter

Series: Mathematical Physics

Date: November 19, 2020 - 4:00 PM

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Abstract: The subject of equivariant elliptic cohomology finds itself at the interface of topology, string theory, affine representation theory, singularity theory and integrable systems. These connections were already known to the founders of the discipline, Grojnowski, Segal, Hopkins, Devoto, but have come into sharper focus in recent years with a number of remarkable developments happening simultaneously: First, there are the geometric constructions, due to Kitchloo, Rezk and Spong, Berwick-Evans and Tripathy, building on the older program of Stolz-Teichner, and of course Segal. These are now all known to be equivalent to Grojnowski's original definition, thanks to work of Spong. Second, there are new applications, to integrable systems (Aganagic-Okounkov, Felder-Rimanyi-Varchenko-Tarasov) and to representation theory (Yang-Zhao-Zhong, G-Ram). Finally, there is the work of Weber-Rimanyi-Kumar, who build on the techniques of Borisov and Libgober to define Schubert classes in equivariant elliptic cohomology (with an added dynamical parametre), recovering the stable envelopes in type A. This talk will give a gentle introduction to the topic and attempt an overview over these recent developments.

8:02 am Fri 20 Nov

Elliptic_Representation_Theory

Atiyah Symbol With Signs | Hochschild With Action | Elliptic_Representation_Theory | Untitled | Untitled Notebook

THE UNIVERSITY OF MELBOURNE

POSTERA CRESCAM LAUDE

20. 11. 2020

Perimeter Institute

Mathematical Physics Seminar

Equivariant Elliptic Cohomology

An overview talk

by Nora Ganter

The chromatic tower

$\mathbb{CP}^\infty = BU(1)$

e.g. $K(B\mathbb{C}^\times) \leftarrow K_{\mathbb{C}^\times}(pt)$

Atiyah–Segal completion map

$c_1(L_1) +_F c_1(L_2) = c_1(L_1 \otimes L_2)$

Borel-equivariant

Quillen's theorem

$(-)^{\widehat{}}_0$

genuinely equiv

$\mathbb{C}^\times$-(equivariant) Cohomology theories with Chern classes	Complex genera $\phi : MU_* \to R$ complex cobordism ring	affine one dimensional formal group (laws)	one dimensional algebraic groups
H	$[M] \mapsto \begin{cases} \lvert M\rvert & \dim M = 0 \\ 0 & \text{else} \end{cases}$	$x + y$	$H^*(\mathbb{CP}^\infty;\mathbb{C})$ Borel convention $H_{\mathbb{C}^\times}(pt) = \Gamma\mathcal{O}_{\mathbb{G}_a} = \mathbb{C}[x]$
$e_{univ} = x$ $c_1(L_{univ})$	counting genus	additive	$\widetilde{H}(\mathbb{P}^1_1)) = I(0) = x\mathbb{C}[x]$
K	$Td(M)$	$x + y - xy$	$K_{\mathbb{C}^\times}(pt) = \Gamma\mathcal{O}_{\mathbb{G}_m} = \mathbb{C}[z^{\pm 1}]$
$e(L) = 1 - L$ $z = \mathbb{C}_1 = L =$ fund. rep of $\mathbb{C}^\times$	Todd genus complex Euler characteristic	multiplicative	$\widetilde{K}(\mathbb{P}^1_1) = I(1) = (1\text{-}z)\mathbb{C}[z^{\pm 1}]$ $1 - \mathbb{C}_1$
$\mathcal{E}\ell\ell$	Def: $\phi_{ell}(M)$	$E^{\widehat{}}_0$ Exa: $E = \mathbb{C}/\langle 1,\tau\rangle = \mathbb{C}^\times/q^{\mathbb{Z}}$	$\mathcal{E}\ell\ell_{\mathbb{C}^\times}(pt) = \mathcal{O}_E$
$e_{univ} = \theta$	an elliptic genus	elliptic	$\widetilde{\mathcal{E}\ell\ell}_{\mathbb{C}^\times}(\mathbb{P}^1_1) = \mathcal{I}(0)$ not free

Grojnowski's equivariant elliptic cohomology

$$E_\tau = \mathbb{C}/\langle 1, \tau\rangle = \mathbb{C}^\times/q^{\mathbb{Z}} \text{ and } E_T = \Lambda^\vee \otimes E_\tau.$$

$$\mathcal{E}\ell\ell_T : T\text{-spaces} \longrightarrow q\mathcal{C}oh(E_T)$$

The theory is reverse engineered from the stalks. → Grojnowski

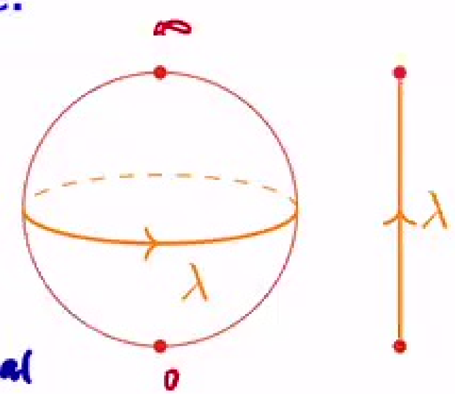
$\mathcal{E}\ell\ell_T(T/K_\lambda) = \mathcal{E}\ell\ell_{K_\lambda}(pt)$

Example:

$pt^{\mathbb{C}_\lambda} = \mathbb{P}^1_\lambda$

$T \circlearrowright$

(complex) one dimensional orbit



$$K_\lambda = \ker(e^{2\pi i\lambda})$$

$$\mathcal{E}\ell\ell_T(T/K_\lambda) = \mathcal{E}\ell\ell_{K_\lambda}(pt)$$

By Mayer Vietoris: $\mathbb{P}^1_\lambda \setminus \{0, \infty\} = \mathbb{C}^\times_\lambda$

$K_\lambda \hookrightarrow T \twoheadrightarrow \mathbb{C}^\times_\lambda$

$$\mathcal{E}\ell\ell_T(\mathbb{P}^1_\lambda) = \textit{equalizer}\left(\mathcal{O}_{E_T} \oplus \mathcal{O}_{E_T} \rightrightarrows \mathcal{O}_{E_{K_\lambda}}\right)$$
$$= \{(a, b) \in \mathcal{E}\ell\ell_T(pt)^2 \mid a - b \in \mathcal{I}(E_{K_\lambda})\}.$$

Scheme valued theory: $E_{\mathbb{P}^1_\lambda} = E_T \sqcup_{E_{K_\lambda}} E_T.$

Moment graph toolkit: can study flag varieties, quiver varieties

Key new feature:

The ideal sheaves $\mathcal{I}(0)$ and $\mathcal{I}(E_{K_\lambda})$ are no longer trivial, as they were in the additive and multiplicative case. Instead, they are locally free of rank one.

Instead of Thom isomorphism, the theory acquires a twist by the Thom sheaf. line bundle on E_τ = base of sheaf valued theory Ell_τ.

The role of Euler class (in H and K a trivializing section) is played by the theta function

$$\theta(z,q) = (z^{\frac{1}{2}} - z^{-\frac{1}{2}}) \prod_{n\geq 1} (1-q^n z)(1-q^n z^{-1})$$

circle $\mathbb{C}^\times$
$z = \mathbb{C}_1$

or, in our example,

$$\theta(z,q) = (z^{\frac{\lambda}{2}} - z^{-\frac{\lambda}{2}}) \prod_{n\geq 1} \left(1-q^n z^{\lambda}\right)\left(1-q^n z^{-\lambda}\right)$$

torus T
$\lambda \in \Lambda$
weight

Substantial body of unwritten work by Grojnowski; further work by Rosu, Ando, Morava, C

Transchromatic phenomena

	G/B	$\mathcal{L}(G/B)$	$\mathcal{L}^2(G/B)$
H	$\mathcal{H}_T(G/B)$	$\mathcal{H}(\mathcal{L}(G/B))^{\Lambda^\vee}$	$\mathcal{H}_{\widetilde{\mathcal{L}^2 G}}(\mathcal{L}^2(G/B))^{\Lambda^\vee \oplus \Lambda^\vee}$
K	$\mathcal{K}_T(G/B)$	$\mathcal{K}_{\widetilde{\mathcal{L}G}}(\mathcal{L}(G/B))^{\Lambda^\vee}$	
$\mathcal{E}\ell\ell$	$\mathcal{E}\ell\ell_T(G/B)$		

New geometric constructions:

Kitchloo Loops and dominant K-theory

Rezk and Spong Double loops and cohomology

Spong Comparison Theorem

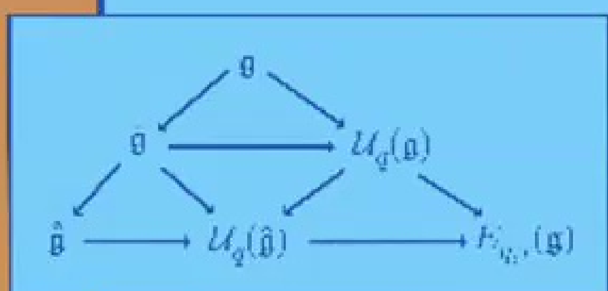
Berwick-Evans and Tripathy strongly influenced by but different to Stolz and Teichner

Huan and Rezk building on Devoto, G. Orbifold loop spaces

Mathematical Surveys and Monographs
Volume 58

Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations

Pavel I. Etingof
Igor B. Frenkel
Alexander A. Kirillov, Jr.



American Mathematical Society

Push-pull formalism

(non singular)

In $\mathcal{E}\ell\ell_T$ what should be the Weyl character formula (push-forward along $G/B \longrightarrow pt$) becomes the Weyl-Kac formula.

Ginzburg-Kapranov-Vasserot:
1995: Elliptic algebras and equivariant elliptic cohomology (technical report)
1997: Residue construction of Hecke algebras

Zhao-Zhong:
2017: Representations of the elliptic affine Hecke algebras carry out Ginzburg-Kapranov-Vasserot program in the Hopkins-Lurie setting. Obtain a push-pull formalism involving theta functions.

Schubert classes via resulutions of singularities

2019 Rimanyi-Weber: Elliptic classes of Schubert cells via Bott-Samelson resolution

2019 Kumar-Rimanyi-Weber: Elliptic classes of Schubert varieties

Definition in the spirit of Borisov and Libgober, i.e., choose a resolution of singularities and define an elliptic class with a correction factor depending on the exceptional divisor, and prove that this is independent of choices.

To remedy the fact that the singularities are not log Kawamata, Rimanyi and Weber find the need to introduce an additional parameter – the same dynamical parameter as in Aganagic and Okounkov's work.

Where both are defined, the elliptic stable envelope basis and the Borisov-Libgober elliptic Schubert classes of Kumar-Rimanyi-Weber coincide.

Elliptic stable envelopes

Aganagic-Okounkov:
2016 Stable envelope basis for $\mathcal{E}\ell\ell_{\mathbf{T}}(T^*G/B)$ in type A.

Application to q-difference equations.

Greatly clarify quantum Knizhnik-Zamolodchikov equations.

$\mathcal{E}\ell\ell_{\mathbf{T}}(X)$ needs to be varied to tell this story, the base acquires additional parameters:

$h = y^{-1}$ and the dynamical parameters parametrized by $Pic(X) \otimes E_\tau$, allowing to build in shifts and twists of line bundles needed to make sense of the q-difference equation.

Further work by Felder-Tarasov-Varchenko, by Konno by Smirnov and others.

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$$\mathcal{E}_T = \Lambda^\vee \otimes E_\tau$$

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lattice

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