Title: An overview of Wavelets and MERA

Speakers: Glen Evenbly

Collection: Tensor Networks: from Simulations to Holography III

Date: November 20, 2020 - 1:00 PM

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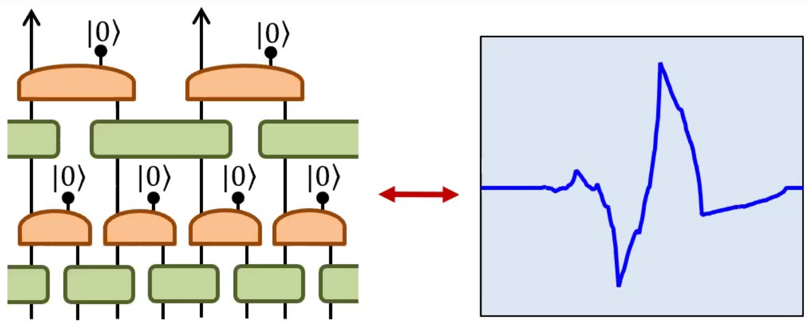
Abstract: The use of wavelet-based constructions has led to significant progress in the analytic understanding of holographic tensor networks, such as the multi-scale entanglement renormalization ansatz (MERA). In this talk I will give an overview of the (past and more recently established) connections between wavelets and MERA, and the discuss the important results that have followed. I will also discuss work currently underway that exploits the wavelet-MERA connection in order to produce new families of wavelets that are optimal for certain tasks, such as image compression.

Tensor Networks: from Simulations to Holography II,
Perimeter Institute 2020

Wavelets and Holographic Tensor Networks

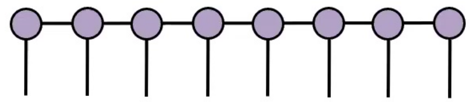


Glen Evenbly

Georgia Tech

Georgia Tech QUANTUM ALLIANCE
Institute for Electronics and Nanotechnology

Introduction



Quantum many-body systems

Tensor Network

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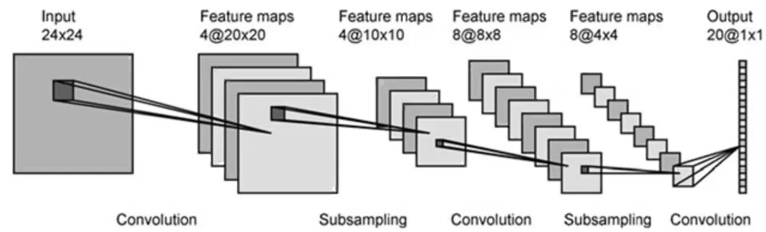
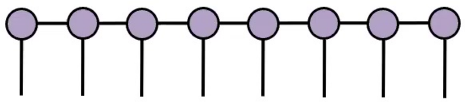
Compressed representation of some correlated data

Data science: data analysis, data compression, machine learning



(e.g. holographic tensor networks are related to convolutional neural networks)

Introduction



Tensor Network

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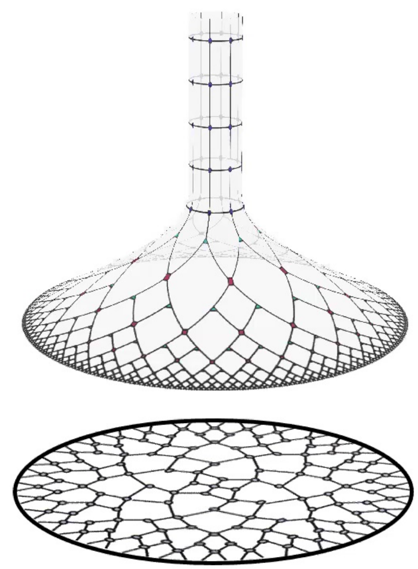
Compressed representation of some correlated data

- Quantum many-body systems
- Data science: data analysis, data compression, machine learning
- Holography: duality between semi-classical gravity and conformal field theories



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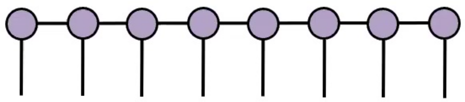

Tensor Network

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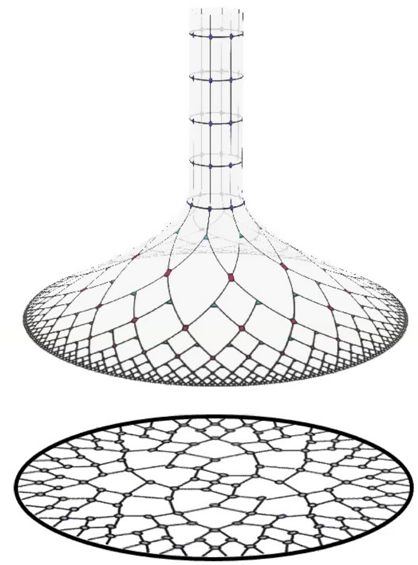
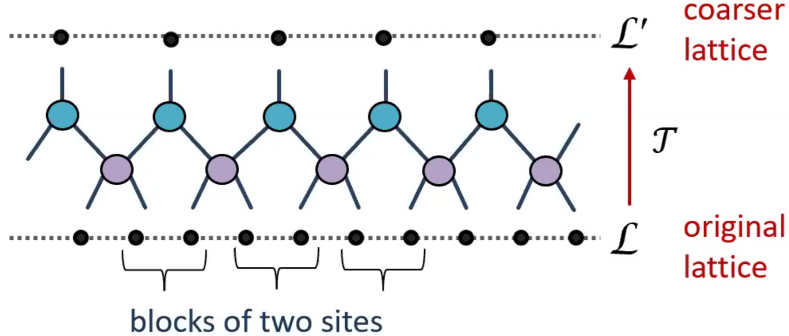


Today: previous and ongoing work on wavelets that ties some of these aspects together

G.E., Steven. R. White, **Phys. Rev. Lett 116.** 140403 (2016)
G.E., Steven. R. White, **Phys. Rev. A 97,** 052314 (2018)
J. C. McCord, **G.E.**, ***in preparation...***

Introduction: MERA

Coarse-graining transformation:
Entanglement Renormalization (ER)



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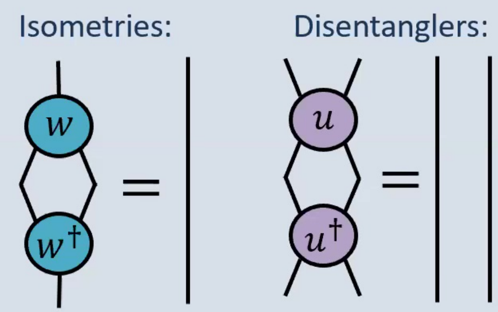
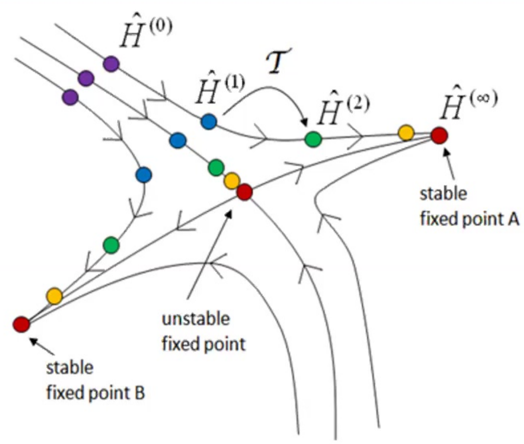


- Starting from a lattice Hamiltonian $H^{(0)}$ we can use ER to generate a sequence of effective (i.e. coarse-grained) Hamiltonians.
- Extract long-range (low-energy) system properties from the effective Hamiltonians.



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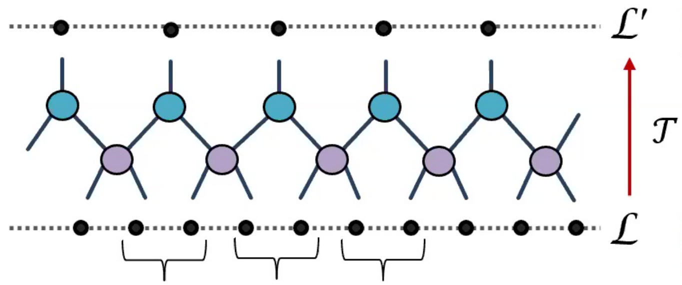

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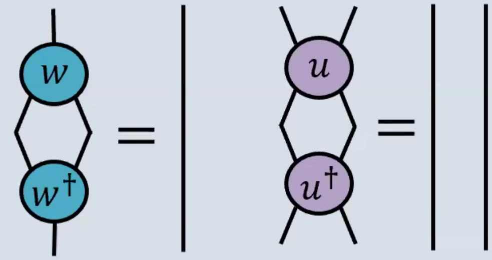
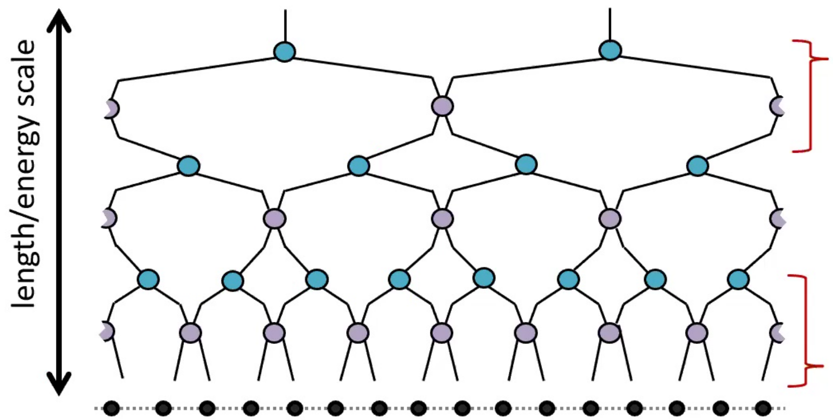


Multi-scale Entanglement Renormalization Ansatz (MERA)



higher layers encode long-ranged properties of the state

MERA = decomposition of a state into information at different length-scales

lower layers encode short-ranged properties

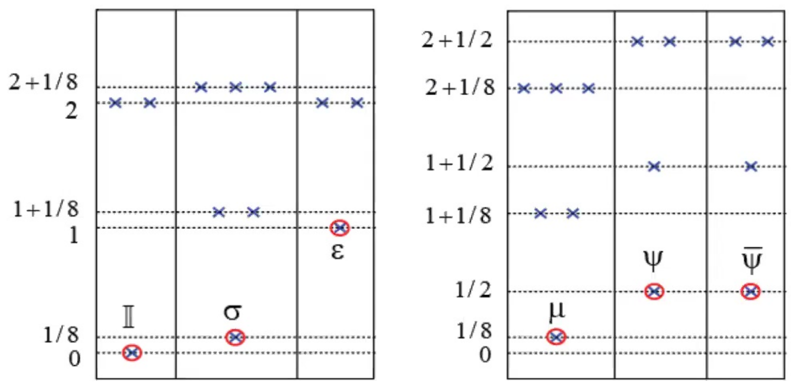
Introduction: MERA

ER and MERA are powerful numeric tools for investigating lattice models (especially critical, scale-invariant systems)

Example: numerical calculation of conformal data from 1D critical Ising model:

$$H = \sum_r \left(-X(r)X(r+1) + Z(r) \right)$$

Scaling Dimensions Δ



primary fields:	exact	MERA
spin:	$\sigma = 0.125$	0.1250003
energy:	$\varepsilon = 1$	1.0000200
disorder:	$\mu = 0.125$	0.1250002
fermion:	$\psi = 0.5$	0.5000004

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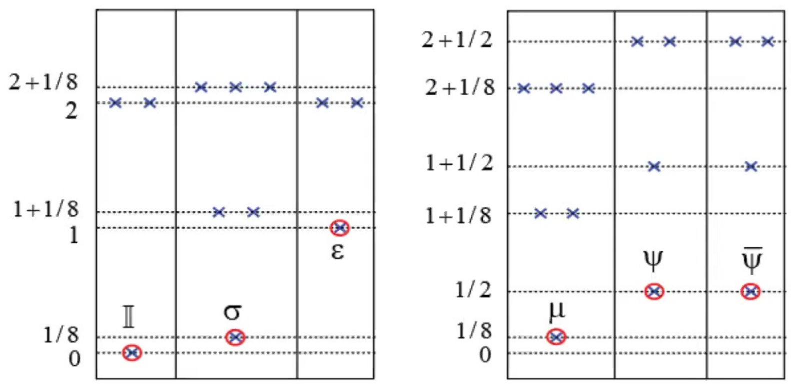


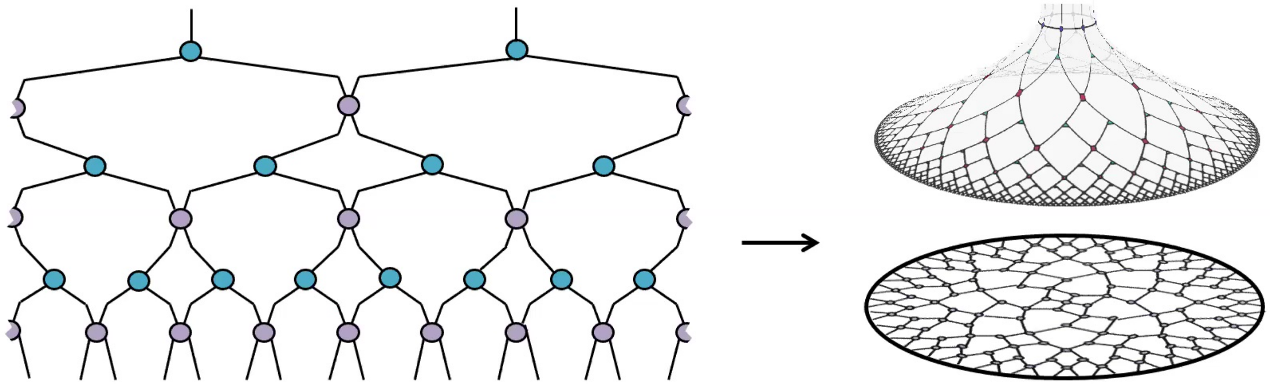
OPE Coefficients

C^{exact}	$C^{\text{MERA}}_{\chi=36}$	error
$C_{\epsilon,\sigma,\sigma} = 1/2$	0.50008	0.016%
$C_{\epsilon,\mu,\mu} = -1/2$	-0.49997	0.006%
$C_{\psi,\mu,\sigma} = \frac{e^{-i\pi/4}}{\sqrt{2}}$	$\frac{1.00068e^{-i\pi/4}}{\sqrt{2}}$	0.068%
$C_{\bar{\psi},\mu,\sigma} = \frac{e^{i\pi/4}}{\sqrt{2}}$	$\frac{1.00068e^{i\pi/4}}{\sqrt{2}}$	0.068%
$C_{\epsilon,\psi,\bar{\psi}} = i$	$1.0001i$	0.010%
$C_{\epsilon,\bar{\psi},\psi} = -i$	$-1.0001i$	0.010%

Pre-2015 status:

- many numeric examples of MERA accurately encoding lattice CFTs
- no analytic examples, no proofs or bounds

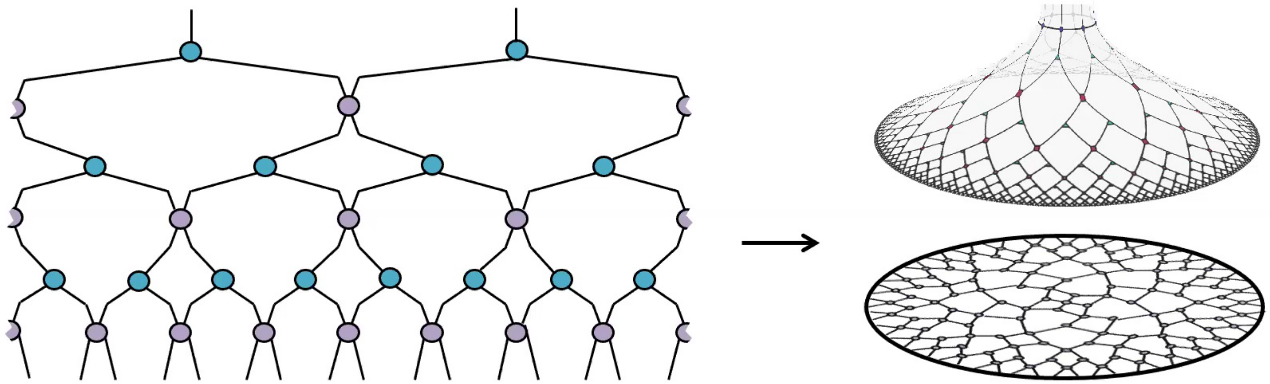
Introduction: MERA



There has been significant interest in exploring the relationship between tensor networks and holography.

Motivation for this proposal (in part): that MERA seems to provide a good representation of CFTs.

Introduction: MERA



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Questions that I would often be asked:

Q: Is there a simple (analytic) example of a MERA than encodes a CFT?

A: No, but we have numerically optimized examples (i.e. containing 1000's of optimized parameters).

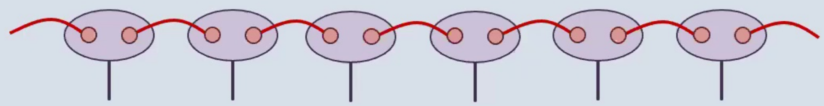
Q: Can one prove that MERA can accurately encode the properties of a CFT?

A: No, but we have numerical evidence that suggests this to be true.

Introduction: MERA

Compare with other networks: Matrix Product States (MPS)

- There exists many useful examples of ground-state MPS for lattice Hamiltonians, e.g. AKLT models.



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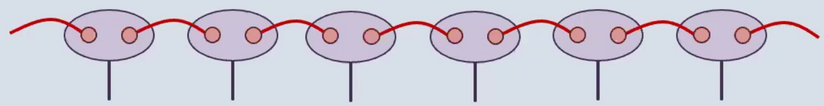

Proposal by Steve White (circa 2014): MERA tensor networks could be related to Wavelets

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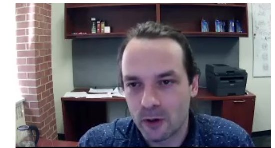

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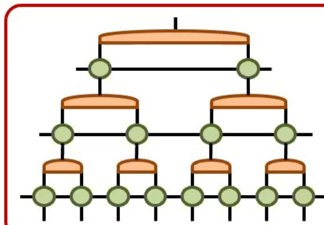
Wavelets are very useful in the theoretical understanding of MERA and CFTs!

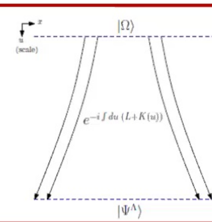
Today:
- recap of wavelet/MERA connection and what we can learn from it
- discuss some ongoing work in using tensor networks to build improved wavelets

Other avenues of progress in building critical MERA without numerics:



Exact holographic networks for the Motzkin chain

R. Alexander, **G.E.**, I. Klich, arXiv:1806.09626 (2018).



Continuous MERA

J. Haegeman, T. J. Osborne, H. Verschelde and F. Verstraete, **Phys. Rev. Lett. 110**, 100402 (2013)

Overview

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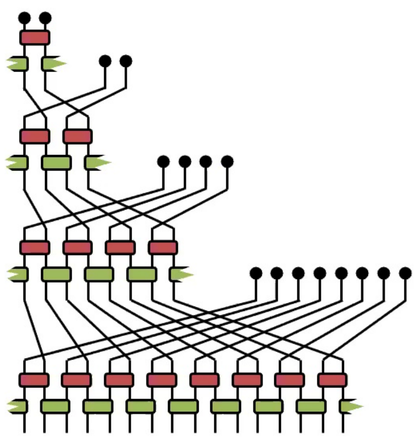
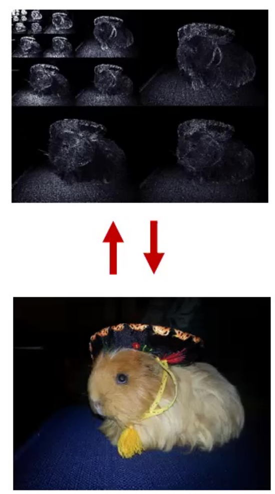
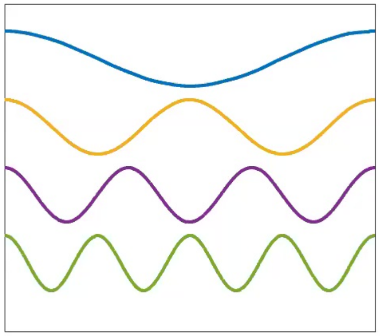
- How to wavelets related to tensor networks (in particular to unitary circuits and MERA)
- What can we learn about MERA from wavelets?
- What can we learn wavelets from MERA?





Introduction to Wavelets

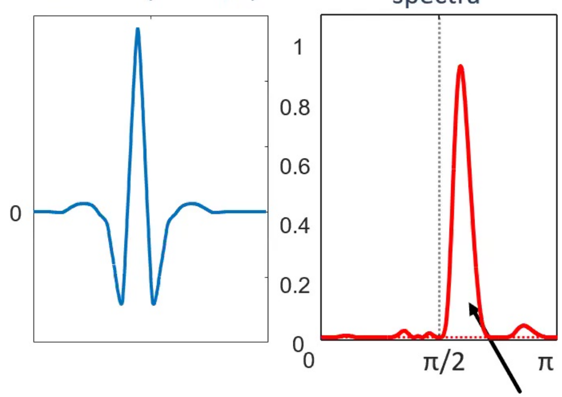


Fourier expansions are ubiquitous in math, science and engineering

- many problems are simplified by expanding in Fourier modes
- smooth functions can be approximated by only a few non-zero Fourier coefficients

wavelet (CDF 9/7)

frequency spectra



narrow band in frequency space

Wavelets are a good compromise between real-space and Fourier-space representations

- compact in real-space and in frequency-space
- developed by math and signal processing communities in late 80's
- applications in signal and image processing, data compression

Daubechies Wavelets

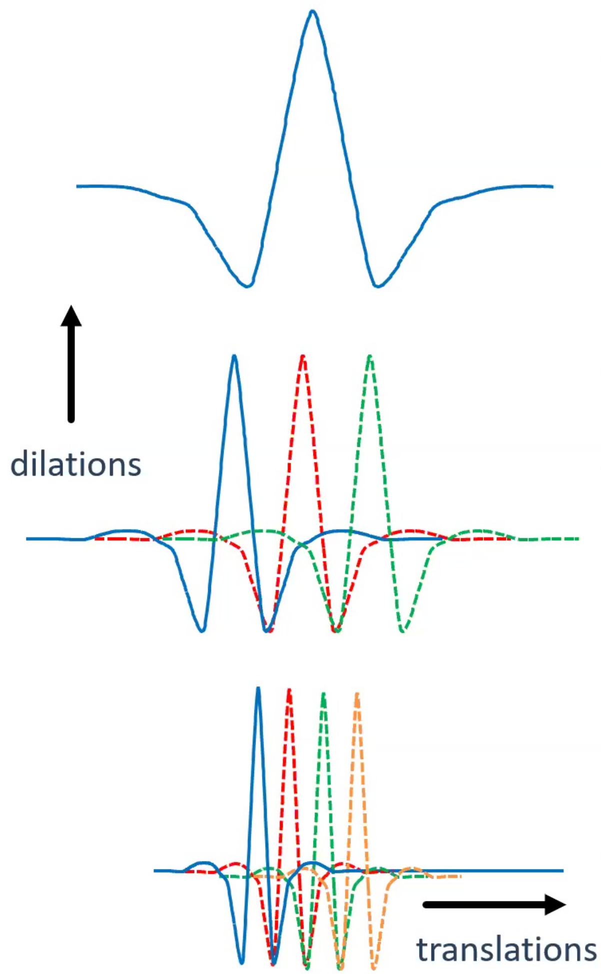
Example:

Daubechies D4 wavelets

- complete, orthonormal basis
- have 2 vanishing moments (orthogonal to constant + linear functions)
- useful for resolving information at different scales

N-dim vector space

Introduction to Wavelets



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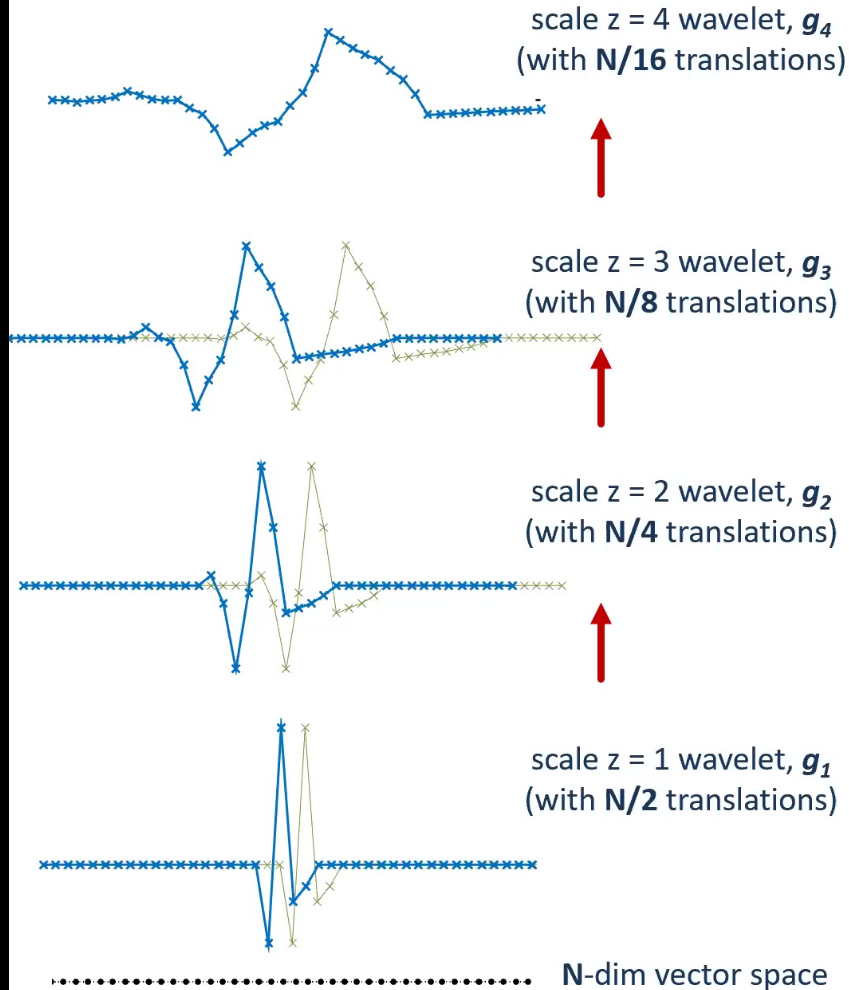

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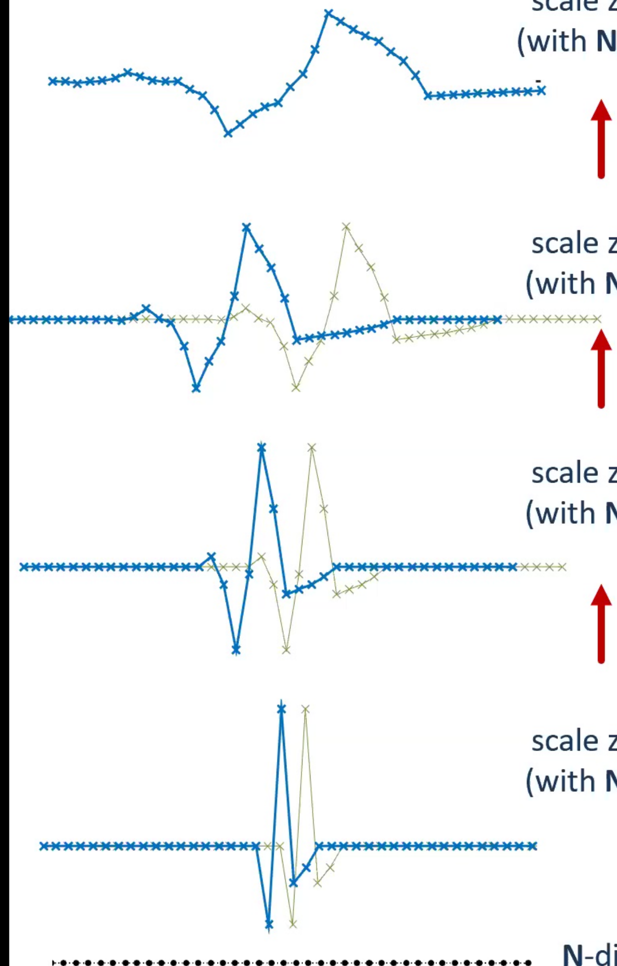

scale z = 4 wavelet, g_4
(with **N/16** translations)

scale z = 3 wavelet, g_3
(with **N/8** translations)

scale z = 2 wavelet, g_2
(with **N/4** translations)

scale z = 1 wavelet, g_1
(with **N/2** translations)

N-dim vector space

Example:

Daubechies D4 wavelets

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large scale wavelets encode long-ranged information

small scale wavelets encode short-ranged information

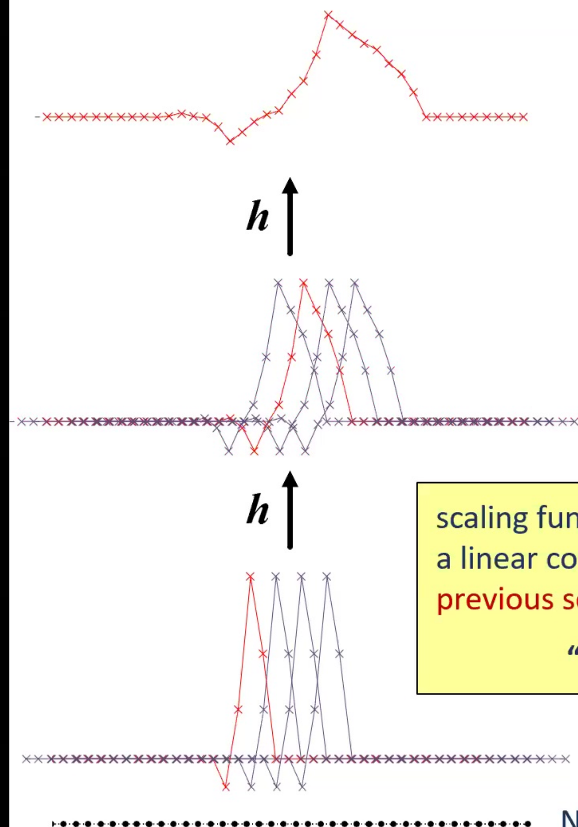
Daubechies Wavelets

How can we construct wavelets?

- first construct scaling function (allows recursive construction of functions at different scales)

D4 scaling sequence

$$\boldsymbol{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} -0.1294 \\ 0.2241 \\ 0.8365 \\ 0.4830 \end{bmatrix}$$

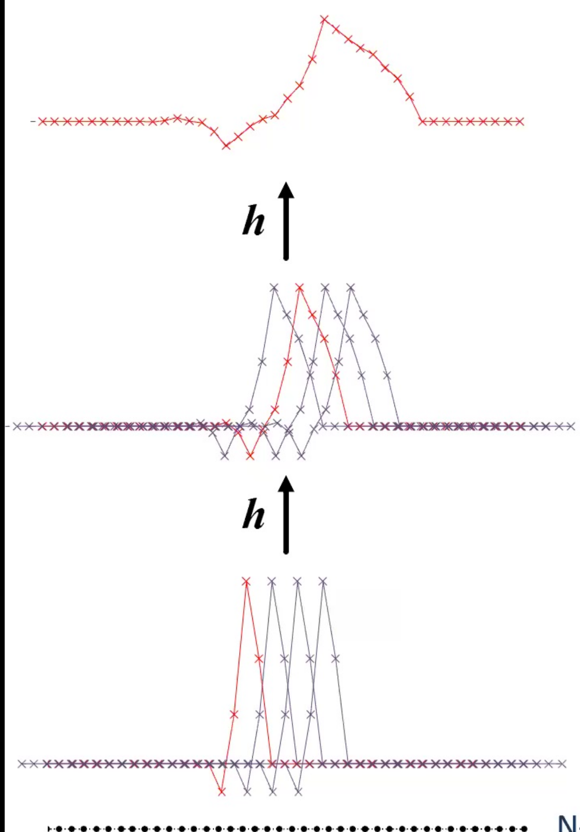


N-dim vector space

scaling function at larger scale defined from a linear combination of scaling functions at previous scale

“Refinement equation”

Daubechies Wavelets



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- wavelets then defined from scaling functions using wavelet sequence

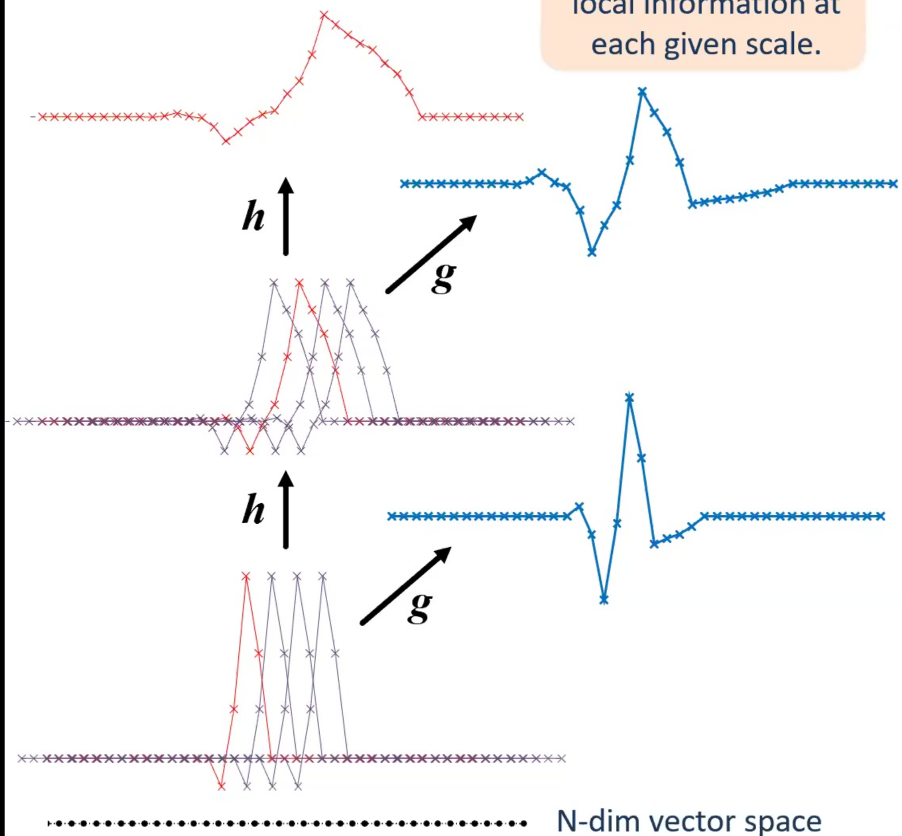
D4 wavelet sequence

$$\boldsymbol{g} = \begin{bmatrix} -h_4 \\ h_3 \\ -h_2 \\ h_1 \end{bmatrix} = \begin{bmatrix} -0.4830 \\ 0.8365 \\ -0.2241 \\ -0.1294 \end{bmatrix}$$

Daubechies Wavelets

Scaling functions: transform data between different length scales

Wavelets: encode the local information at each given scale.



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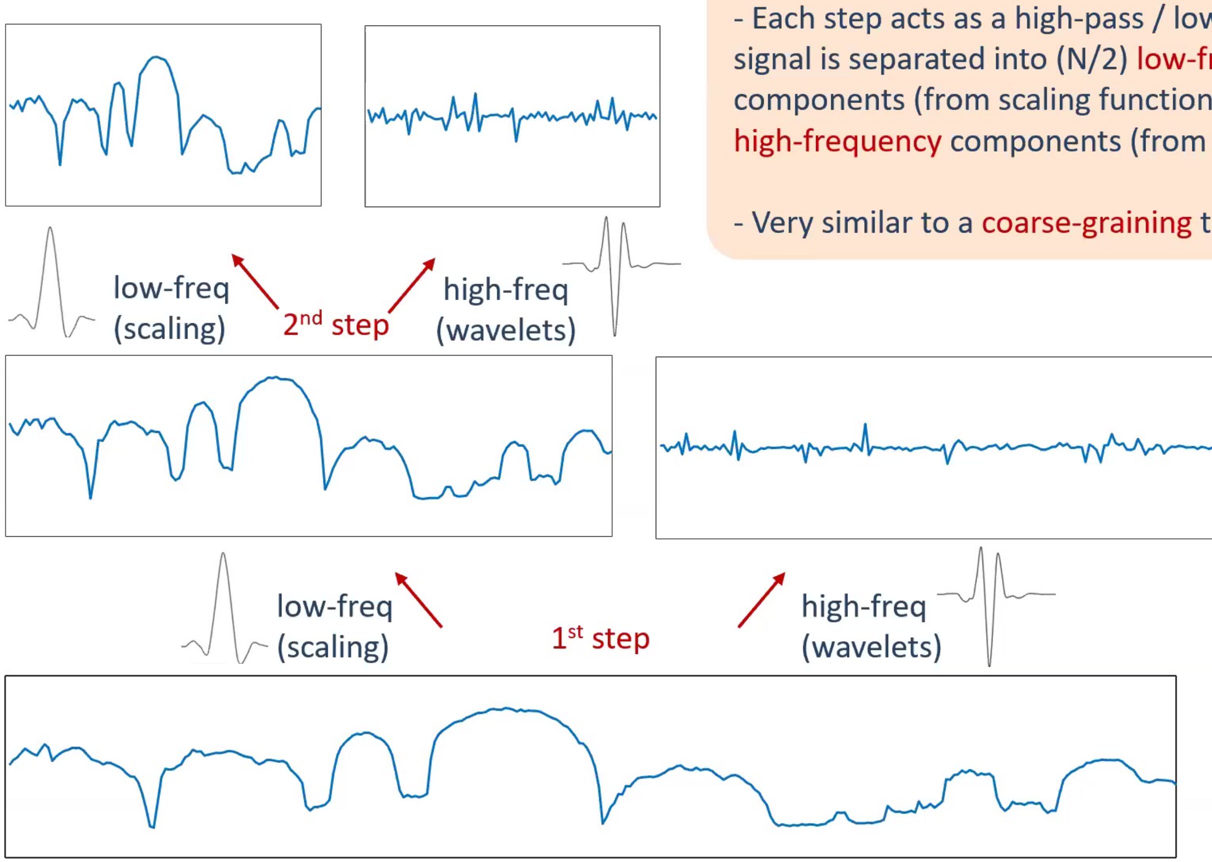
Introduction to Wavelets

- A discrete wavelet transform (DWT) is a type of multi-resolution analysis (MRA)

- Each step acts as a high-pass / low-pass filter: the signal is separated into (N/2) low-frequency components (from scaling functions) and (N/2) high-frequency components (from wavelets)

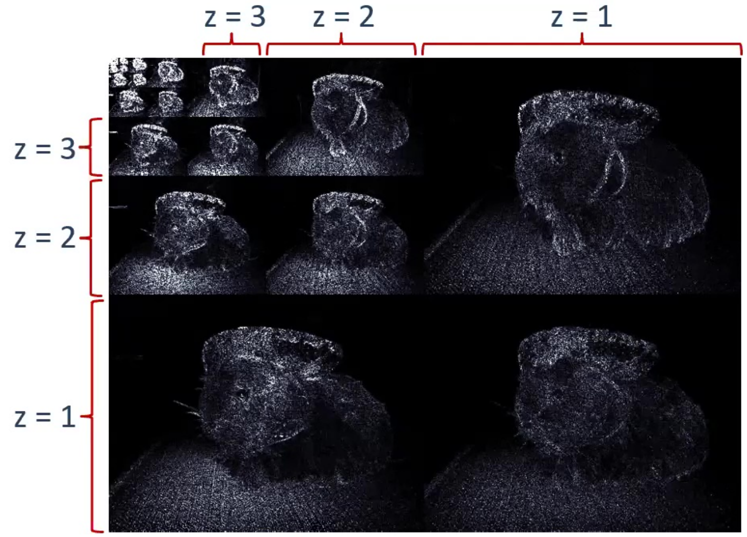
- Very similar to a coarse-graining transformation!



original signal on N-lattice points

Introduction to Wavelets

- the discrete wavelet transform decomposes the image into the information at different scales \`z\`
- bright pixels in transformed image represent large high-freq components (i.e. sharp changes in the image)
- transformed image still contains all of the information of the original image (we have just made a change of basis!)



2D discrete wavelet transform

Original Image (“bubbles” the guinea pig)

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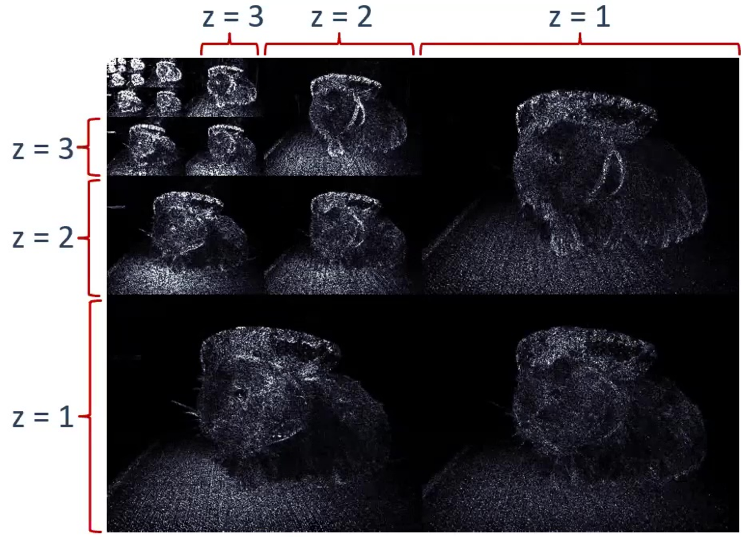

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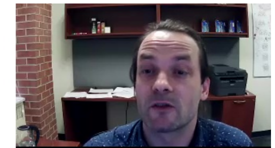
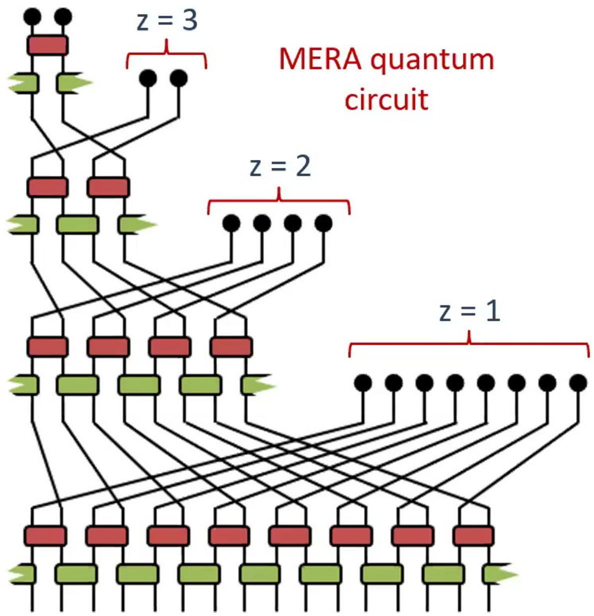
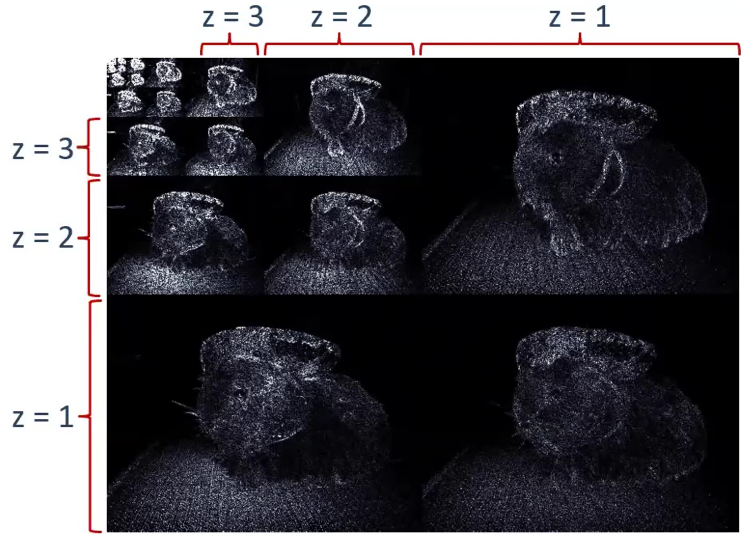


Introduction to Wavelets



The discrete wavelet transform displays many similarities to coarse-graining and the MERA. Are these similarities superficial or something deeper?



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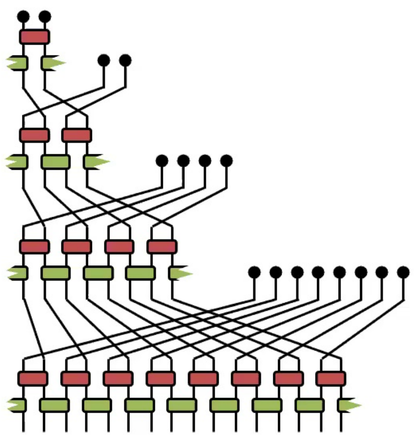
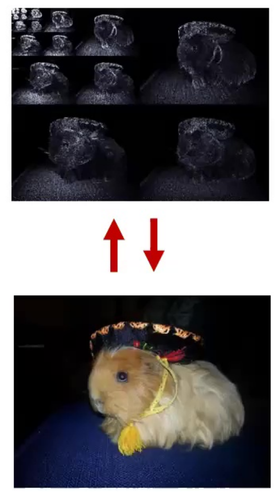
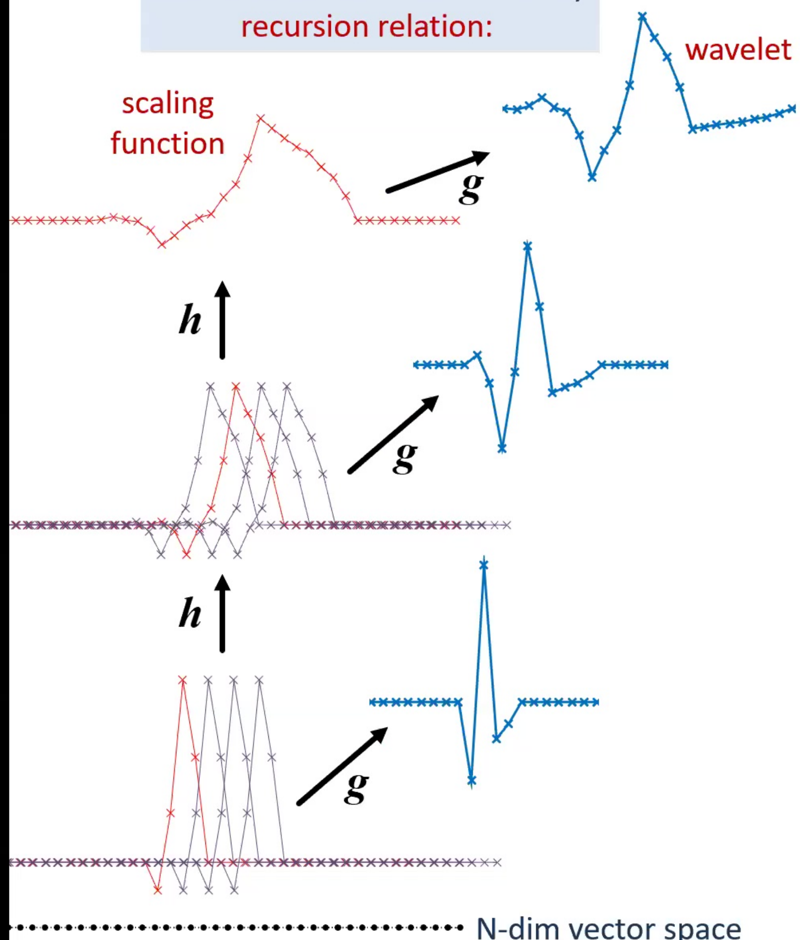




Circuit representation of wavelets

Wavelet transform described by recursion relation:



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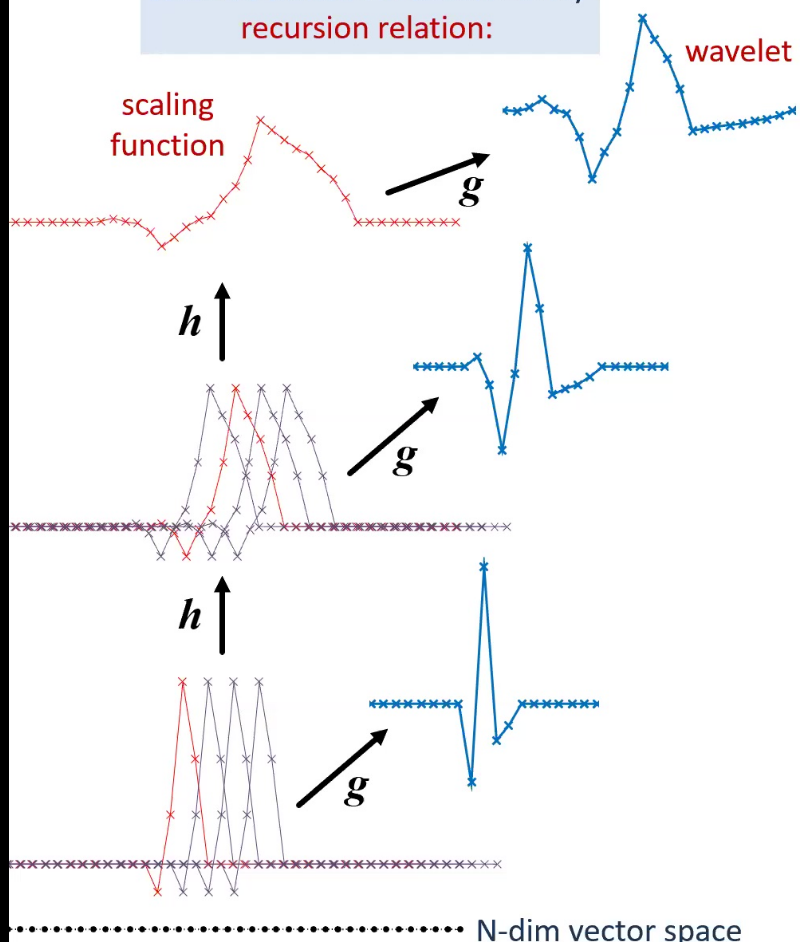
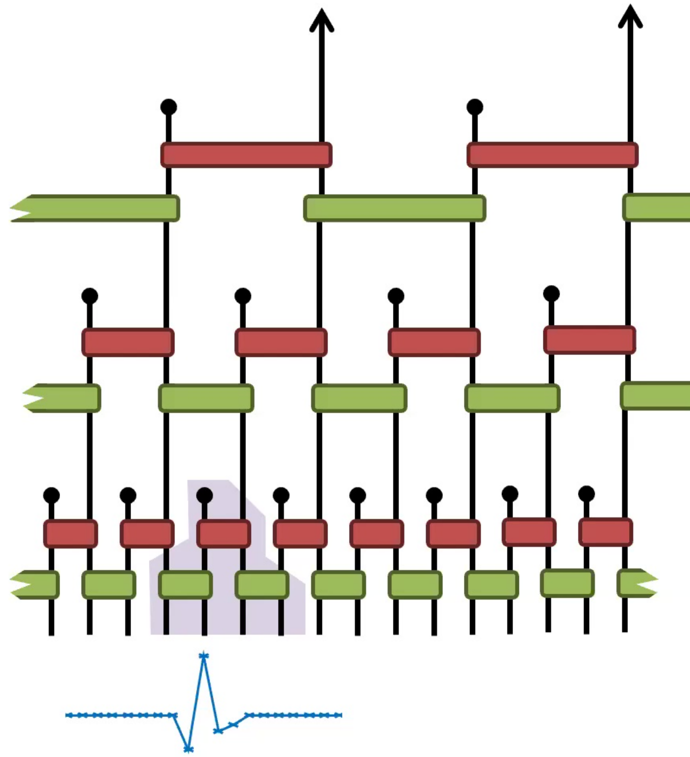
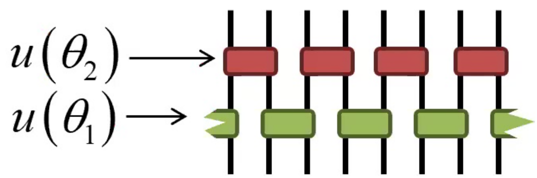


Recursion relation can be encoded as a (classical) unitary circuit:



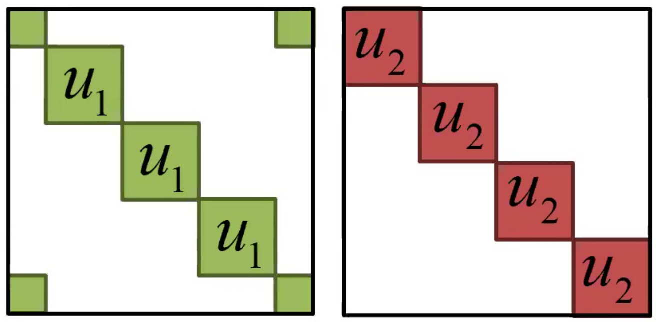
Circuit representation of wavelets

Diagrammatic notation:



2x2 unitary rotation matrix

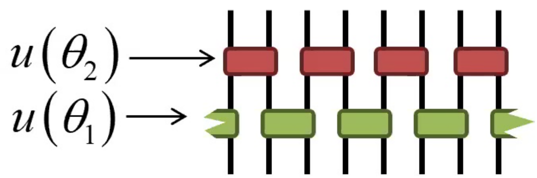
$$u(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$



Classical circuit here represents **direct sum** of matrices (not **tensor product!**)

Circuit representation of wavelets

Diagrammatic notation:



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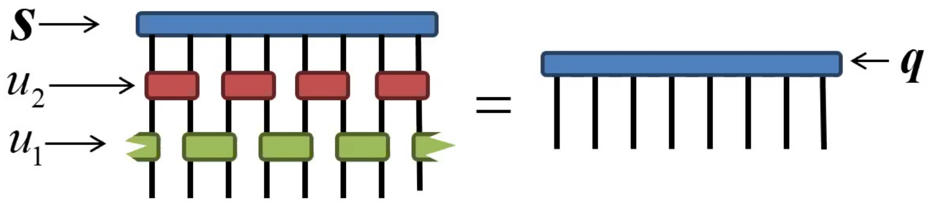
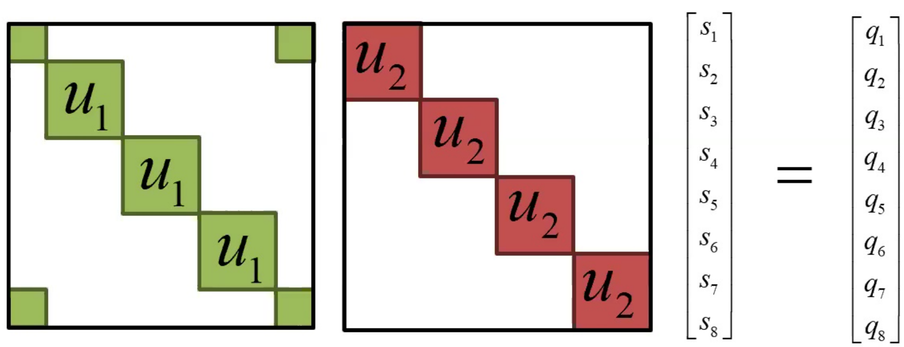
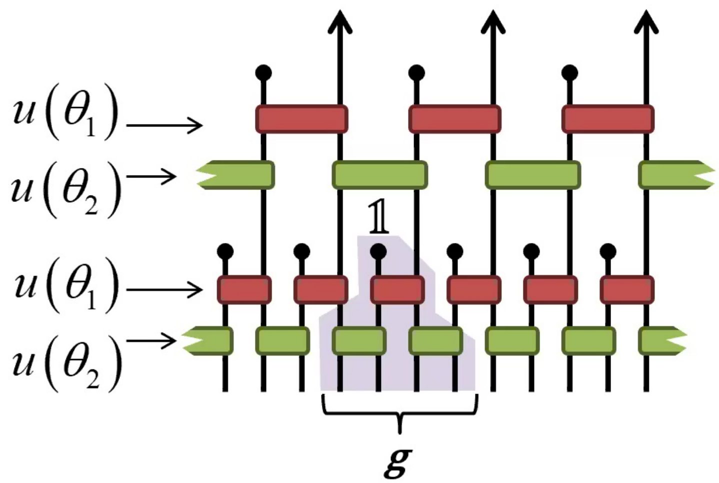


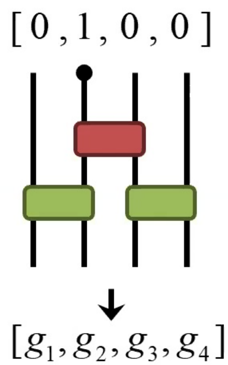
Wavelet transform maps from vector of **N scalars** to vector of **N scalars**!

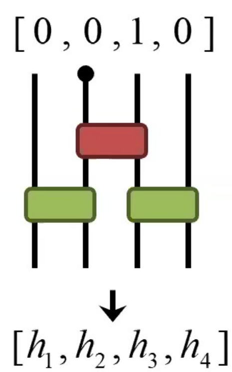


Circuit representation of wavelets



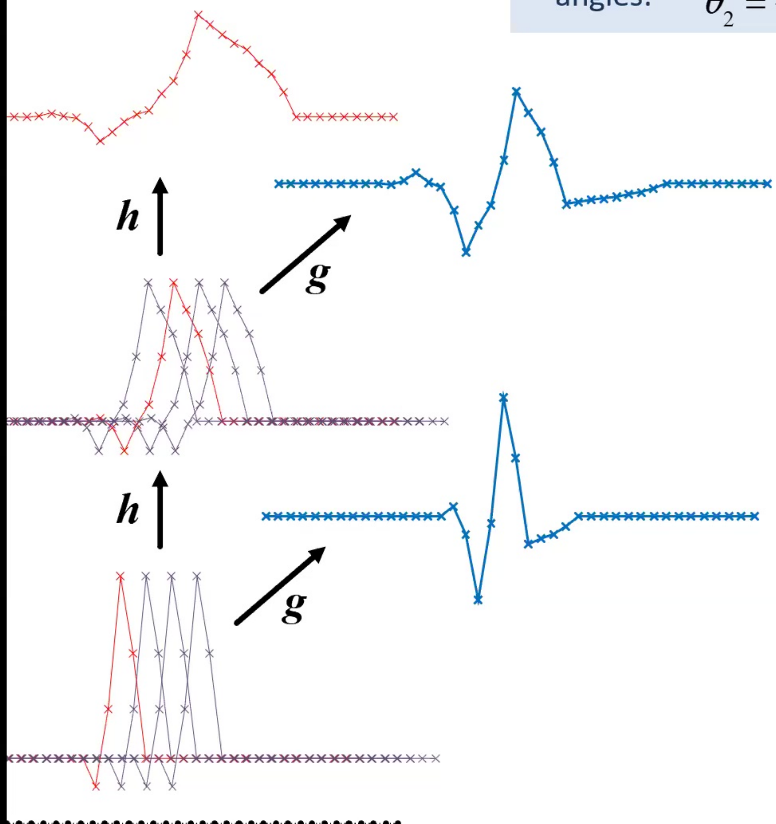
wavelet sequence
associated to inverse
transforming unit vector
(odd sub-lattice)



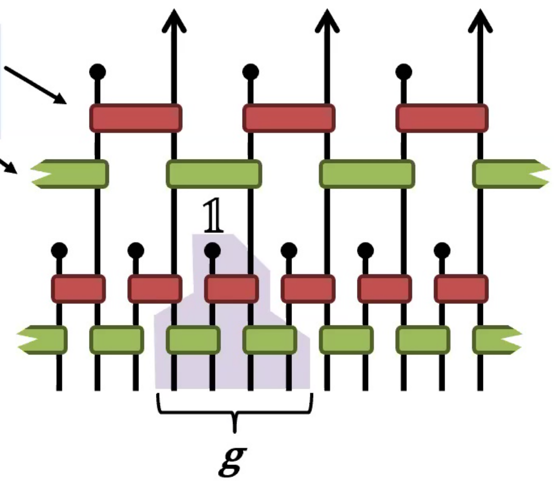


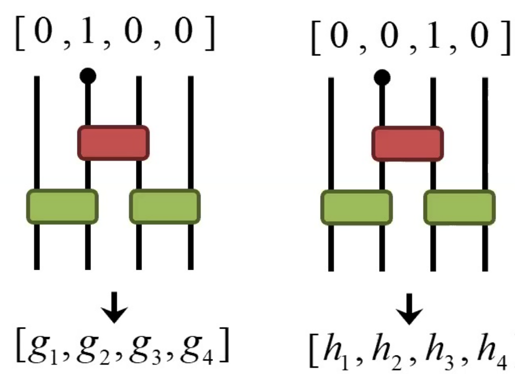
scaling sequence
associated to inverse
transforming unit vector
(even sub-lattice)

Circuit representation of wavelets

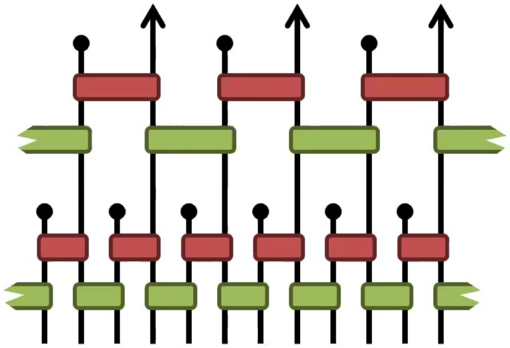


Set rotation angles: $\theta_1 = \pi / 12$, $\theta_2 = -\pi / 6$

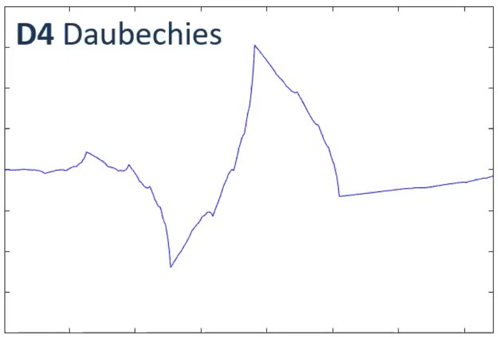




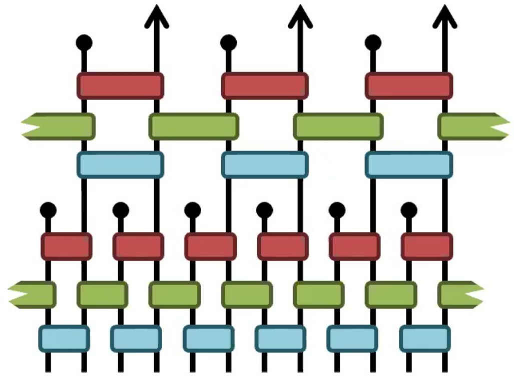
Circuit representation of wavelets



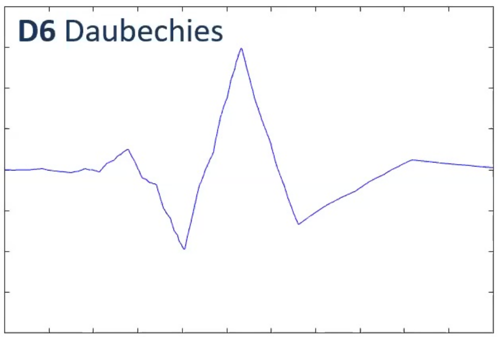
Daubechies D4 wavelets can be represented as unitary circuits with exactly the same structure as MERA (but direct-sum rather than tensor-product).



orthogonal to **constant** + **linear** functions



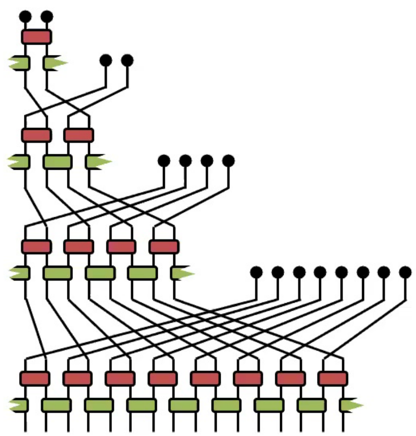
Higher-order Daubechies wavelets (or other wavelet types, such as symlets or coiflets) can also be represented as MERA-like circuits.

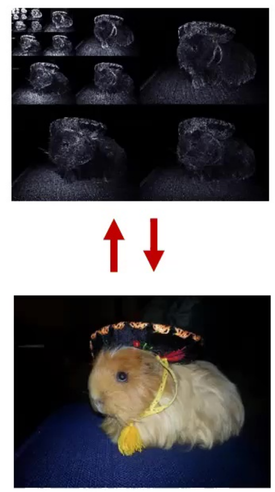


orthogonal to **constant** + **linear** + **quadratic** functions

Overview

- What are wavelets? What are they useful for?
- How to wavelets related to tensor networks (in particular to unitary circuits and MERA)
- What can we learn about MERA from wavelets?
- What can we learn wavelets from MERA?





Free spinless fermions

consider 1D free spin-less fermions:

$$H_{\mathrm{FF}} = \frac{1}{2}\sum_r \big(\underbrace{\hat{a}^\dagger_{r+1}\hat{a}_r + h.c.}_{\text{hopping term}}\big) - \mu \sum_r \underbrace{\hat{a}^\dagger_r \hat{a}_r}_{\text{chemical potential}}$$

diagonalize via Fourier transform:

$$H_{\mathrm{FF}} = \int_{-\pi}^{\pi} \Lambda_k \hat{c}^\dagger_k \hat{c}_k \, dk$$

dispersion relation:

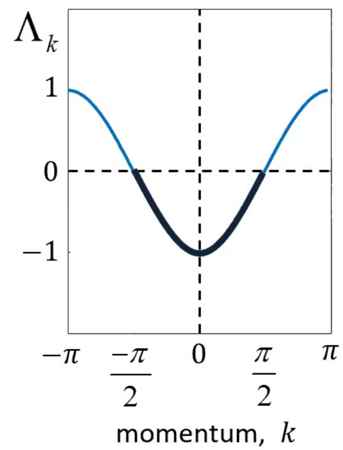
$$\Lambda_k = \mu - \cos\frac{2\pi k}{N}$$

Fourier modes:

$$\hat{c}_k = \frac{1}{\sqrt{N}} \sum_r \hat{a}_r e^{-i2\pi k r/N}$$

Ground state is given by filling in negative energy states (fermi-sea):

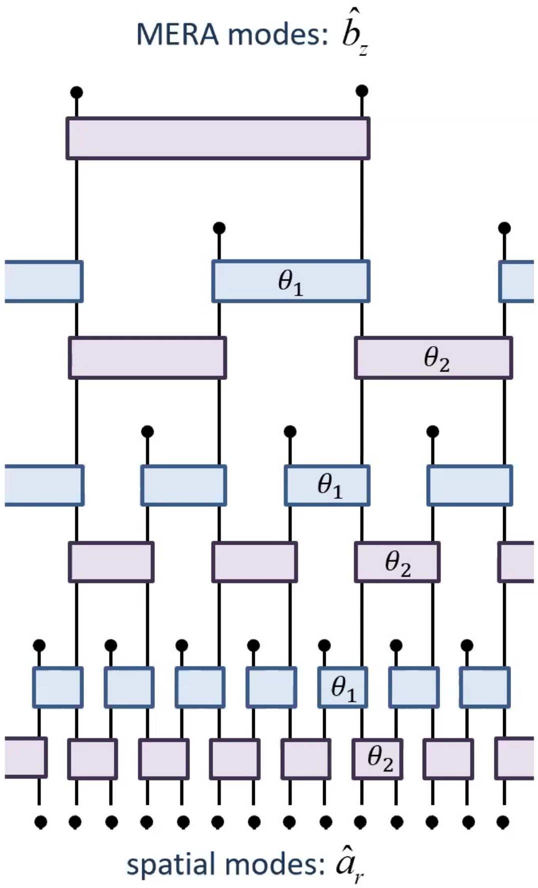
$$\langle \psi_{GS} | \hat{c}^\dagger_k \hat{c}_k | \psi_{GS} \rangle = \begin{cases} 1 & \Lambda_k < 0 \\ 0 & \Lambda_k > 0 \end{cases}$$



Can we represent the free-fermion ground state as a MERA?

Gaussian MERA

"Entanglement renormalization in noninteracting fermionic systems"
G. E. and G. Vidal, **Phys. Rev. B 81**, 235102 (2010)



We can restrict to unitary gates that preserve fermionic mode structure (i.e. that implement Bogoliubov transformations):

Gaussian MERA defines mapping between fermionic modes:

$$\hat{b}_z = \sum_{r=0}^{N-1} U_{zr} \hat{a}_r$$

N-MERA modes

N-spatial modes

Gaussian MERA

"Entanglement renormalization in noninteracting fermionic systems"
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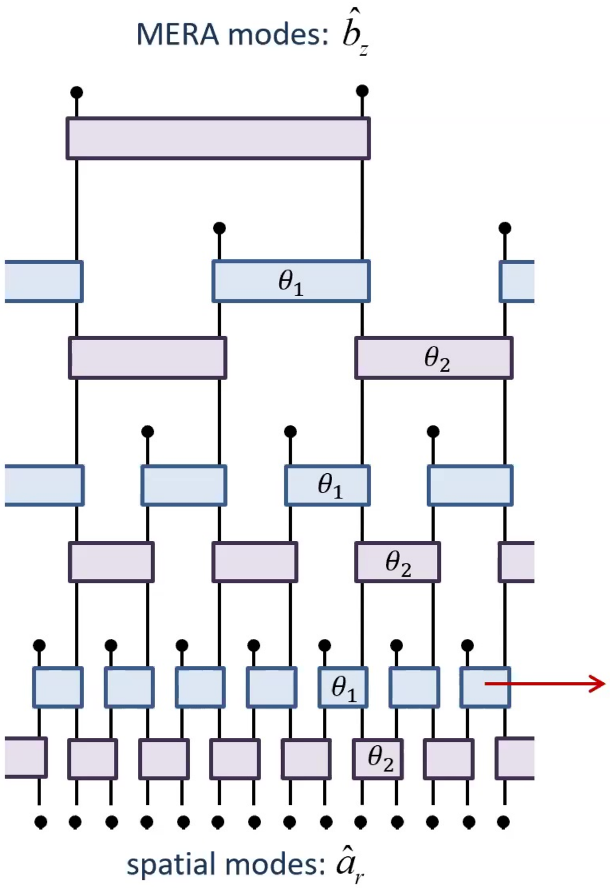

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Gaussian MERA defines mapping between fermionic modes:

$$\hat{b}_z = \sum_{r=0}^{N-1} U_{zr} \hat{a}_r$$

***N*-MERA modes** — ***N*-spatial modes**

Each two-qubit gate maps a pair of fermion modes into two new modes:

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}$$

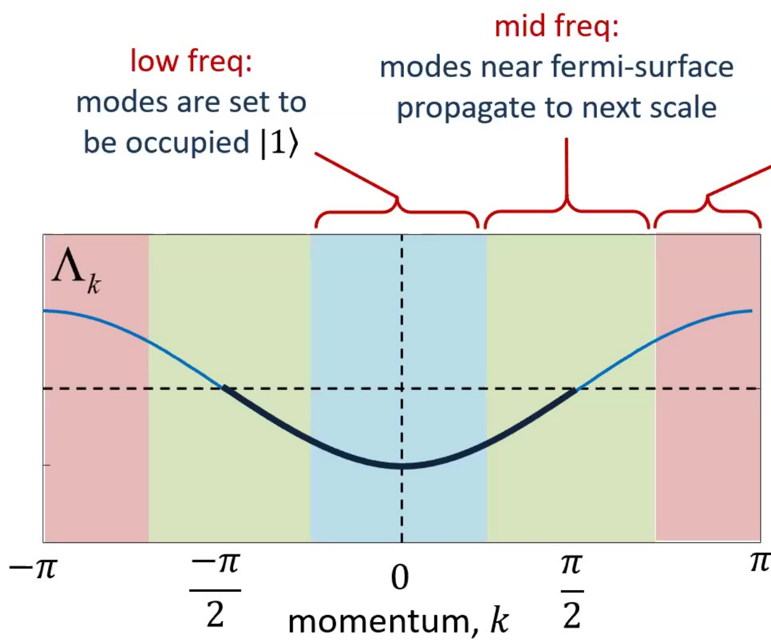
Gaussian MERA is built in the same direct-sum structure as the wavelet unitary circuits!

Gaussian MERA from wavelets

Any discrete wavelet transform can be interpreted as a Gaussian MERA

Can we use a standard wavelet transform to build an approximation to the free-fermion ground state?

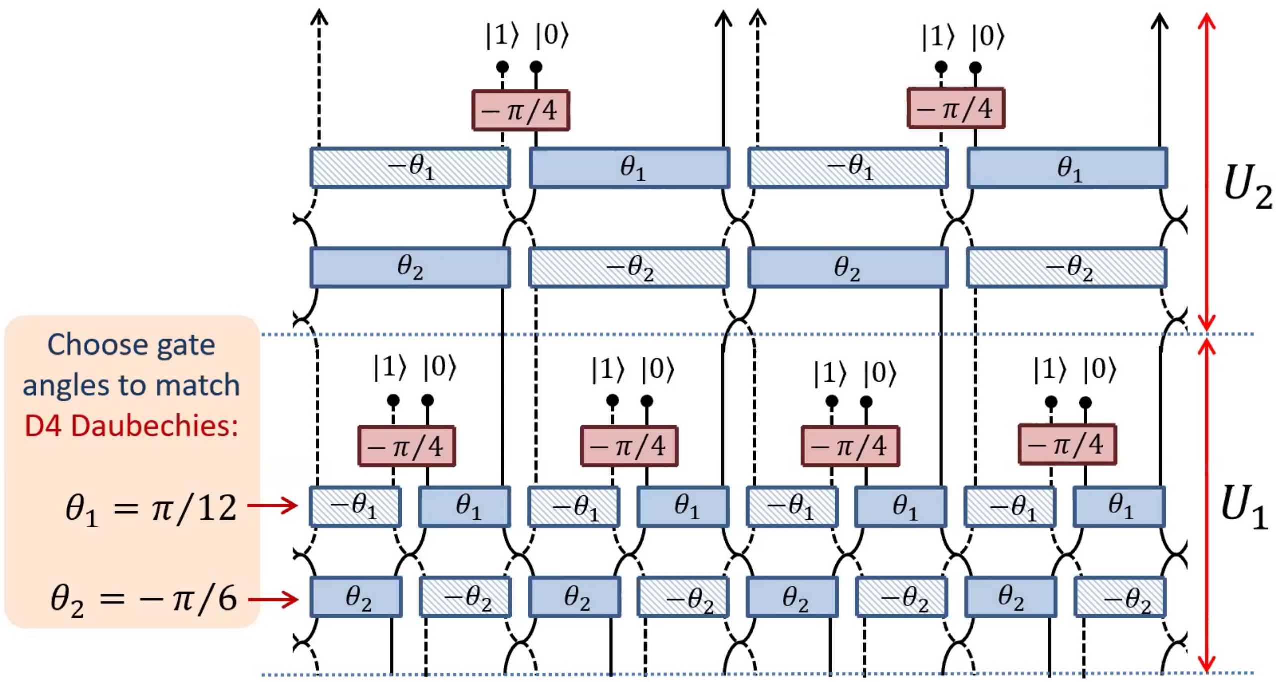
Not so easy! Each step of a standard wavelet transforms acts as a filter between high / low frequency components. Here we need a 3-band filter with high / mid / low frequency components.



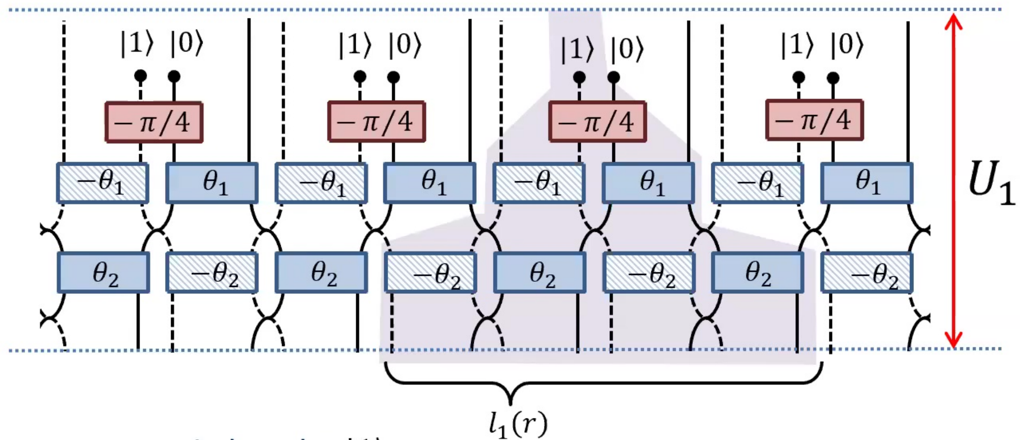
We can make a 3-band filter by coupling together two copies of a Daubechies D4 wavelet.

Gaussian MERA from wavelets

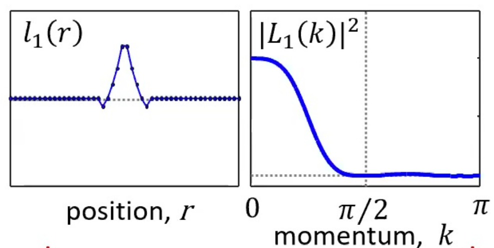
- Build one wavelet transform on the even sub-lattice (solid gates) and one on the odd sub-lattice (striped gates)
- Couple the two sets of wavelets together with $\pi/4$ gates



Gaussian MERA from wavelets

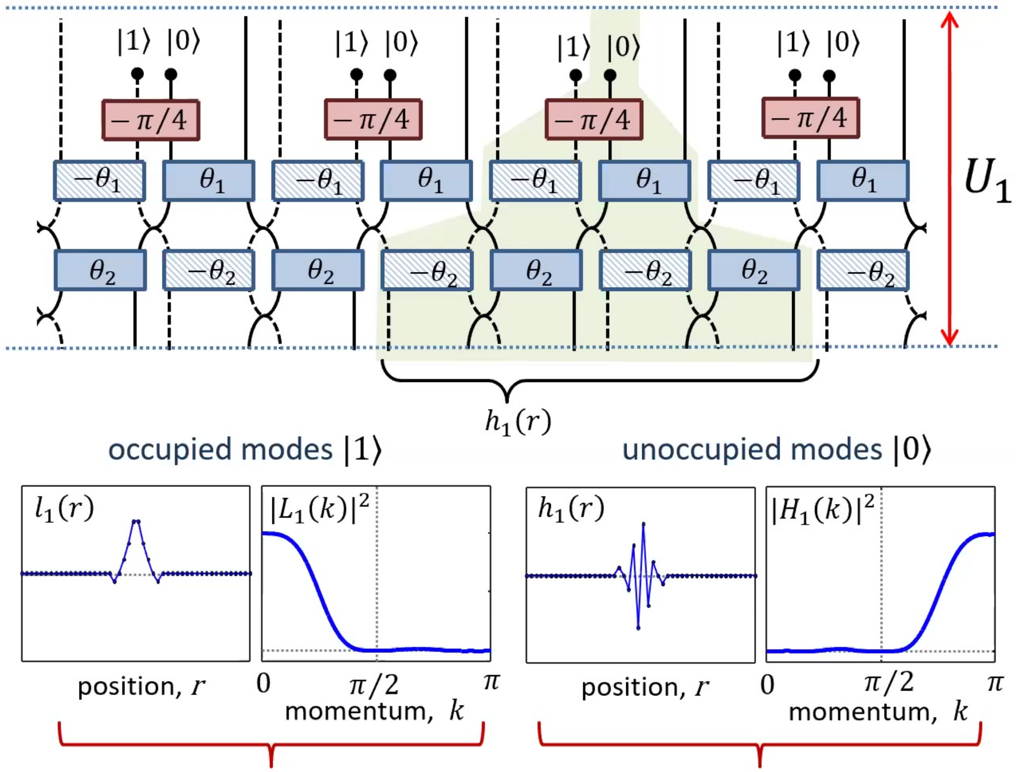


occupied modes $|1\rangle$



well localized in the low-frequency range

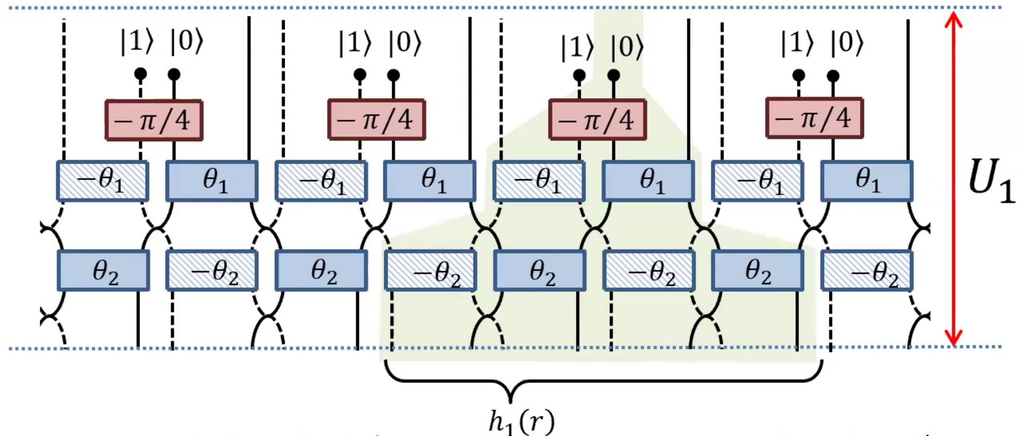
Gaussian MERA from wavelets



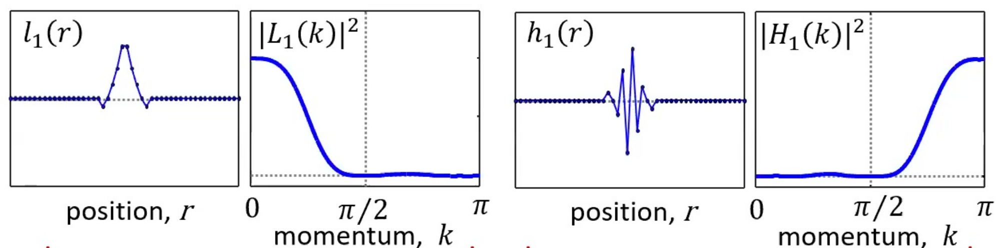
well localized in the low-frequency range

well localized in the high-frequency range

Gaussian MERA from wavelets



occupied modes |1⟩ unoccupied modes |0⟩



well localized in the low-frequency range

well localized in the high-frequency range

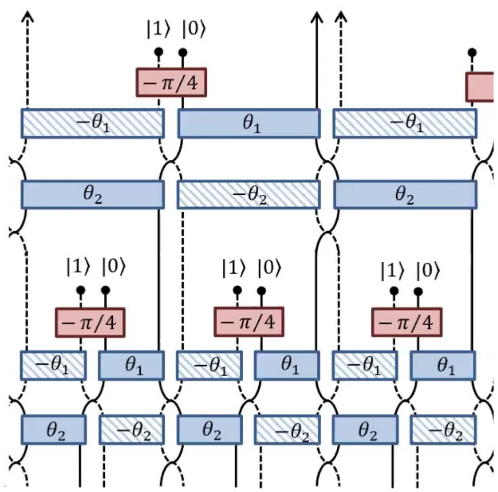
Conclusion: the (doubled) wavelet circuit is preparing an accurate approximation to the free-fermion ground state!

MERA from wavelets

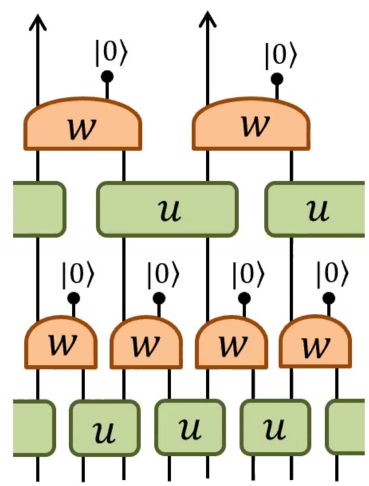
One final trick: the system of free spinless fermions can be transformed into (two copies of) the critical Ising model via some standard mappings.

Free spinless fermions	Two copies of free majorana fermions	Quantum critical Ising model
$H_{\mathrm{FS}} = \frac{1}{2}\sum_r \left(\hat{a}^\dagger_{r+1}\hat{a}_r + h.c.\right)$ →	$H_{\mathrm{FM}} = \sum_r i\left(\breve{d}_{2r}\breve{d}_{2r+1} - \breve{d}_{2r-1}\breve{d}_{2r+2}\right)$ →	$H_{\mathrm{Ising}} = \sum_r \left(-X_r X_{r+1} + Z_r\right)$

Ground state of free spinless fermions



Ground state of the critical Ising model



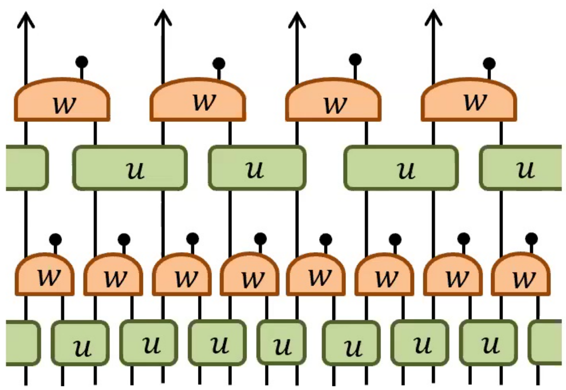
MERA from wavelets

Daubechies D4 construction for the ground state of the critical Ising model:

Isometries: $w_{r,r+1} = \left(\frac{\sqrt{3}+\sqrt{2}}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-\sqrt{2}}{4}\right) Z_r Z_{r+1} + i\left(\frac{1+\sqrt{2}}{4}\right) X_r Y_{r+1} + i\left(\frac{1-\sqrt{2}}{4}\right) Y_r X_{r+1}$

Disentanglers: $u_{r,r+1} = \left(\frac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\frac{i}{4}\right) X_r Y_{r+1} + \left(\frac{i}{4}\right) Y_r X_{r+1}$

define a binary MERA of bond dim $\chi = 2$



- Amazingly simple result! These simple unitary gates somehow manage to approximately encode the Ising CFT.
- We can similarly get closed form expressions for larger bond dimension MERA (e.g. $\chi = 8$) from higher order wavelets.
- Useful example understanding how MERA can encode a CFT (without needing numerics)
- Useful for debugging numeric algorithms…

MERA from wavelets

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Wavelet-based MERA produce a good approximation to the Ising CFT.

		Exact	MERA $\chi = 2$	MERA $\chi = 8$
	Energy	-1.27323...	-1.24212 (2.4% err.)	-1.26774 (0.4% err.)
central charge:	c	0.5	0.4957	0.5041
primary fields:	$\Delta_\mathbb{I}$	0	0	0
	Δ_σ	0.125	0.1402	0.1233
	Δ_ϵ	1	1	1
	Δ_μ	0.125	0.1445	0.1291
	Δ_ψ	0.5	0.5	0.5
	$\Delta_{\bar{\psi}}$	0.5	0.5	0.5
	Δ_H	2	2	2
fusion coeffs:	$C_{\epsilon,\sigma,\sigma}$	0.5	0.4584	0.4957
	$C_{\epsilon,\mu,\mu}$	-0.5	-0.4201	-0.5060
	$C_{\psi,\mu,\sigma}$	$\frac{e^{-i\pi/4}}{\sqrt{2}}$	$\frac{1.1422e^{-i\pi/4}}{\sqrt{2}}$	$\frac{1.0014e^{-i\pi/4}}{\sqrt{2}}$
	$C_{\bar{\psi},\mu,\sigma}$	$\frac{e^{i\pi/4}}{\sqrt{2}}$	$\frac{1.1422e^{i\pi/4}}{\sqrt{2}}$	$\frac{1.0014e^{i\pi/4}}{\sqrt{2}}$
	$C_{\epsilon,\psi,\bar{\psi}}$	i	$1.234i$	$1.0243i$
	$C_{\epsilon,\bar{\psi},\psi}$	$-i$	$-1.234i$	$-1.0243i$

MERA from wavelets

Daubechies D4 construction for the ground state of the critical Ising model:

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Disentanglers: $u_{r,r+1} = \left(\frac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\frac{i}{4}\right) X_r Y_{r+1} + \left(\frac{i}{4}\right) Y_r X_{r+1}$

What do we learn about MERA from wavelets?

(1) MERA of small bond dimension can accurately encode conformal data.

- Previously: numerical optimization via energy minimization always required large bond dimension in order to produce accurate conformal data.
- Can now be understood as a limitation of the optimization (which targets the short-ranged properties), rather than a failing of the ansatz itself.
- This suggests that there could be better optimization strategies for producing accurate conformal data from critical systems.

MERA from wavelets

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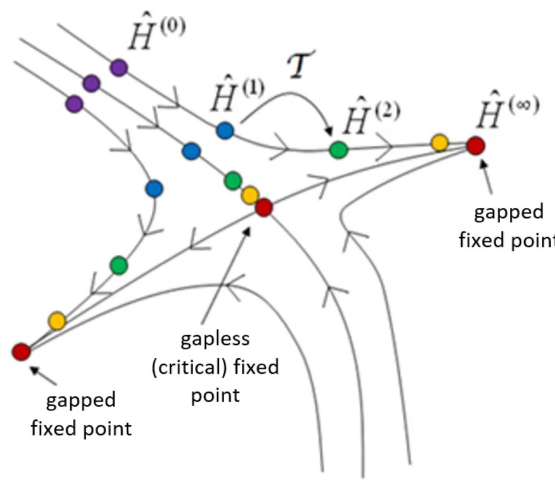
(1) MERA of small bond dimension can accurately encode conformal data.

(2) MERA can produce RG flows to gapless fixed points.

RG flow: $H^{(0)} \to H^{(1)} \to H^{(2)} \to \cdots \to H^{(\infty)}$



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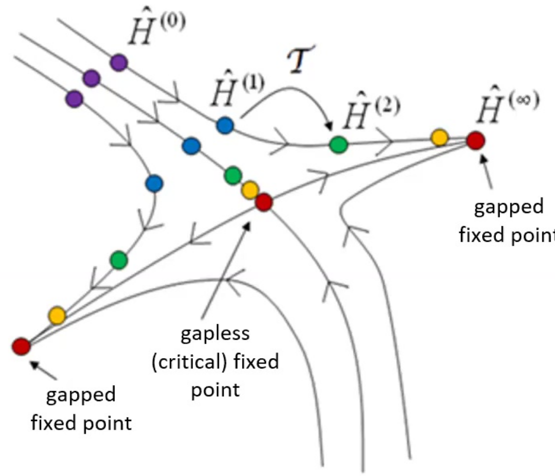
Previous understanding: MERA can approximate a gapless RG flow… but will always eventually flow to a gapped fixed point!

- finite bond dim MERA always has some finite truncation error.
- finite truncation error will always introduce a gap (or relevant RG perturbation).



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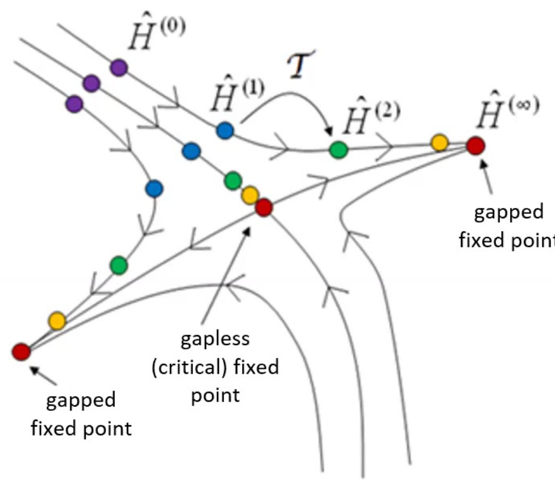
wavelet constructions show that this statement is false!



MERA from wavelets

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What do we learn about MERA from wavelets?

(1) MERA of small bond dimension can accurately encode conformal data.

(2) MERA can produce RG flows to gapless fixed points.

(3) MERA are provably accurate representations for (certain) lattice CFTs.

"Rigorous Free-Fermion Entanglement Renormalization from Wavelet Theory", J. Haegeman, B. Swingle, M. Walter, J. Cotlet, **G.E.**, V. Scholz, **Phys. Rev. X 8,** 011003 (2018).

- Provides a more general constructions of MERA using Selesnick wavelets.
- Formulates rigorous error bounds for the accuracy of MERA correlation functions.
- Also demonstrates that branching MERA can encode a system of 2D fermions (with 1D fermi-surface).

MERA from wavelets

Daubechies D4 construction for the ground state of the critical Ising model:

Isometries: $w_{r,r+1} = \left(\frac{\sqrt{3}+\sqrt{2}}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-\sqrt{2}}{4}\right) Z_r Z_{r+1} + i\left(\frac{1+\sqrt{2}}{4}\right) X_r Y_{r+1} + i\left(\frac{1-\sqrt{2}}{4}\right) Y_r X_{r+1}$

Disentanglers: $u_{r,r+1} = \left(\frac{\sqrt{3}+2}{4}\right) I_r I_{r+1} + \left(\frac{\sqrt{3}-2}{4}\right) Z_r Z_{r+1} + \left(\frac{i}{4}\right) X_r Y_{r+1} + \left(\frac{i}{4}\right) Y_r X_{r+1}$

What do we learn about MERA from wavelets?

(1) MERA of small bond dimension can accurately encode conformal data.

(2) MERA can produce RG flows to gapless fixed points.

(3) MERA are provably accurate representations for (certain) lattice CFTs.

"Rigorous Free-Fermion Entanglement Renormalization from Wavelet Theory", J. Haegeman, B. Swingle, M. Walter, J. Cotlet, **G.E.**, V. Scholz, **Phys. Rev. X 8,** 011003 (2018).

"Quantum circuit approximations and entanglement renormalization for the Dirac field in 1+1 dimensions", F. Witteveen, V Scholz, B. Swingle, M. Walter, **arXiv:1905.08821** (2019).

"Bosonic entanglement renormalization circuits from wavelet theory", F. Witteveen, M. Walter, **arXiv:2004.11952** (2020).

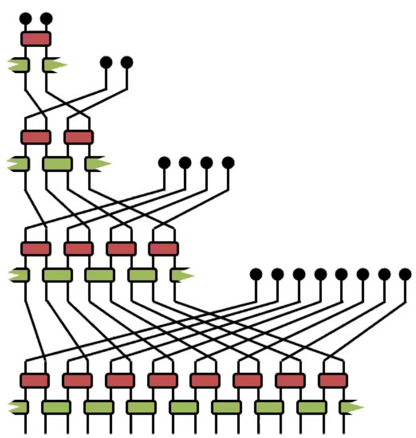
"Scaling limits of lattice quantum fields by wavelets", V. Morinelli, G. Morsella, A. Stottmeister, Y. Tanimoto, **arXiv:2010.11121** (2020).

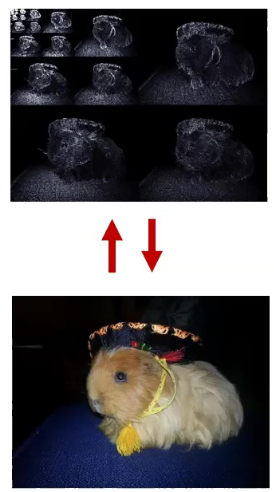
+ many more

A lot of other exciting work going on using wavelet-based constructions of renormalization flows and MERA (e.g. for quantum fields, for bosons…)

Overview

- What are wavelets? What are they useful for?
- How to wavelets related to tensor networks (in particular to unitary circuits and MERA)
- What can we learn about MERA from wavelets?
- What can we learn wavelets from MERA?





Can tensor networks improve upon wavelet-base methods in the setting of signal/image analysis?

Wavelet Applications

Wavelets are useful for signal/image/video compression:

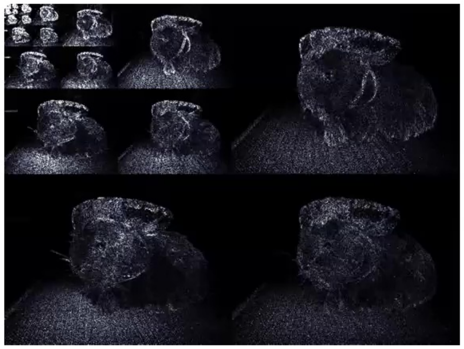


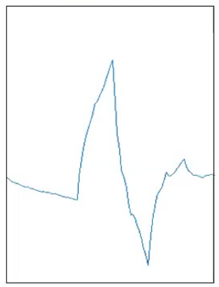
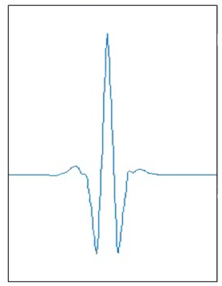
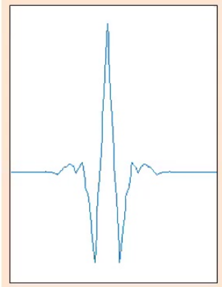
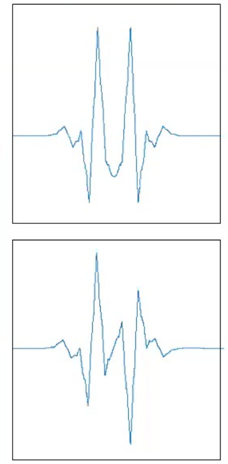
image → transform to wavelet basis → **truncate** (keep only largest 2% of coefficients) → inverse transform → recovered image

peak signal to noise: PSNR: 37.0 dB

- Most coefficients of the transformed image are close to zero (as wavelets are orthogonal to smooth functions)
- Threshold the transformed image as to store only the largest wavelet coefficients (and discard the rest).
- This is the key part of JPEG2000 format, and many other standards for image, audio and video compression

Wavelets for Image Compression

Many times of wavelet are known. What makes a good wavelet for image compression?

desirable properties	Daubechies	Coiflets	CDF wavelets	Scale-3 symmetric
				
orthogonality?	**yes**	**yes**	**near orthogonal**	**yes**
symmetric?	**no**	**near symmetric**	**yes**	**yes**
compression ratio?	**okay**	**good**	**good**	**bad**
			JPEG + other algorithms	

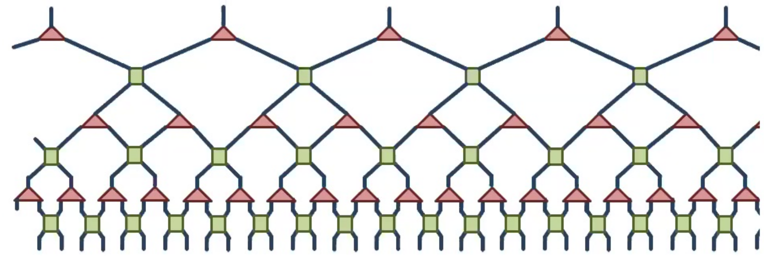
Wavelet Design using Tensor Networks

Can we use tensor networks to build new classes of wavelets with desirable properties?

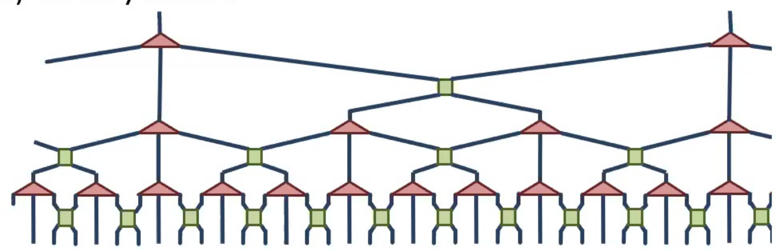
Formulating wavelets in terms of tensors networks greatly simplifies their construction!

- many different types of MERA are known, each can be associated to a family of wavelet
- we know how to incorporate spatial and global internal symmetries into tensor networks

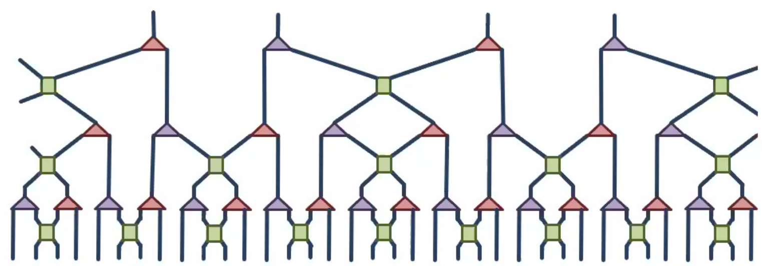
(i) Binary MERA



(ii) Ternary MERA:



(iii) Modified Binary MERA:



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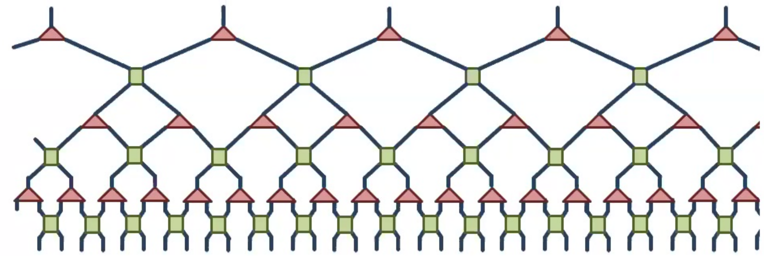
Many new wavelet families:

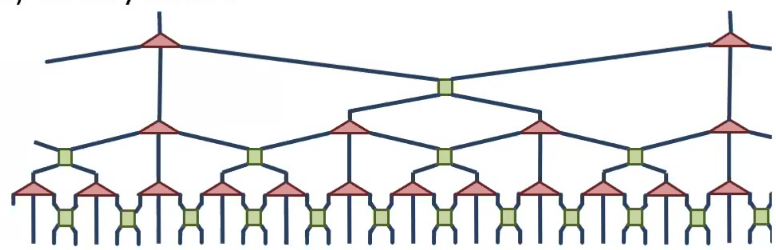
"Representation and design of wavelets using unitary circuits"
G.E., Steven. R. White, **Phys. Rev. A 97,** 052314 (2018)

Example: design of wavelets with orthogonality and reflection symmetry

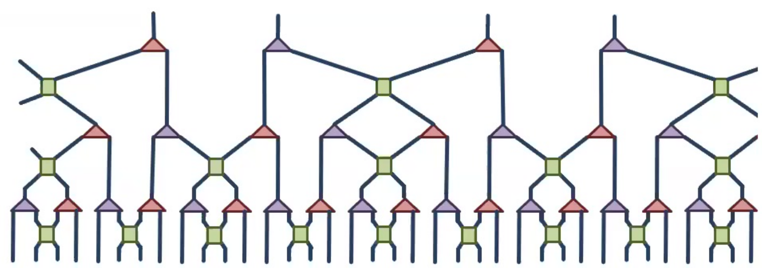
(i) Binary MERA



(ii) Ternary MERA:



(iii) Modified Binary MERA:



Wavelet Design using Tensor Networks

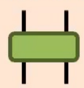
Q: How to we incorporate a symmetry on a tensor network?

A: We impose the symmetry on each individual tensor within the network

Orthogonality of wavelets $\Rightarrow$ every tensor should be unitary

Reflection symmetry wavelets $\Rightarrow$ every tensor should be reflection symmetric

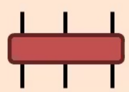
2x2 matrices:



only a single unique matrix: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

corresponds to a swap gate:

3x3 matrices:



1-parameter family of matrices:

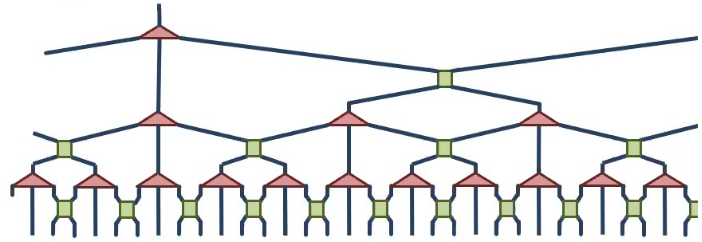
$$\frac{1}{2}\begin{bmatrix} \cos(\theta)+1 & \sqrt{2}\sin(\theta) & \cos(\theta)-1 \\ -\sqrt{2}\sin(\theta) & 2\cos(\theta) & -\sqrt{2}\sin(\theta) \\ \cos(\theta)-1 & \sqrt{2}\sin(\theta) & \cos(\theta)+1 \end{bmatrix}$$

Can we build a circuit using only these two types of gate?
Yes! This is a generalization of the ternary MERA.

Wavelet Design using Tensor Networks

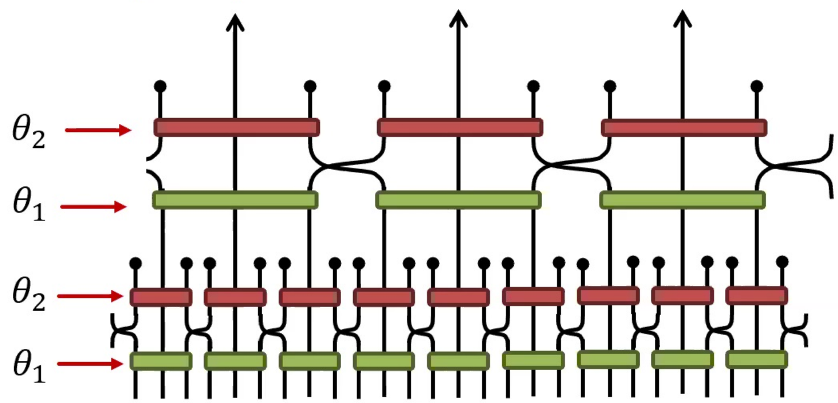
- result is a ternary unitary circuit (3-to-1 scaling) with some free variational parameters $\{\theta_1, \theta_2, \dots\}$.
- however, the resulting wavelets are exactly orthogonal and (anti-) symmetric for any choice of parameters.
- we choose parameters as to maximise the vanishing moments of the wavelets. (but other choices are possible!)

Ternary MERA:



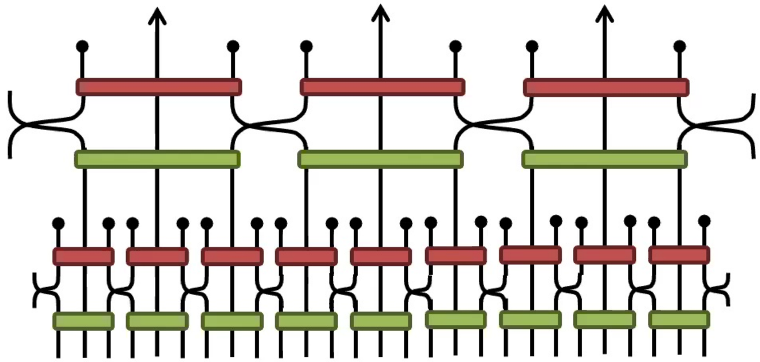
generalization

Ternary unitary circuit:



Wavelet Design using Tensor Networks

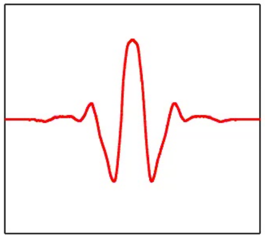
Ternary unitary circuit:



symmetric wavelet

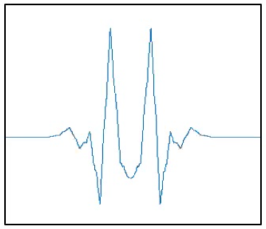
antisymmetric wavelet

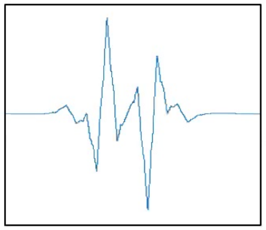
New orthogonal, symmetric wavelets: smooth with good compactness.



Very handsome!

Previous orthogonal, symmetric wavelets: jagged and wide support.





Butt ugly!

Wavelet Design using Tensor Networks

How well do these new wavelets work in practice?

We have been doing some large scale testing and experimentation of new wavelet designs for image compression.

James McCord

"Improved wavelet designs for image compression"
J. C. McCord, **G.E.**, ***in preparation***

symmetric wavelet

antisymmetric wavelet

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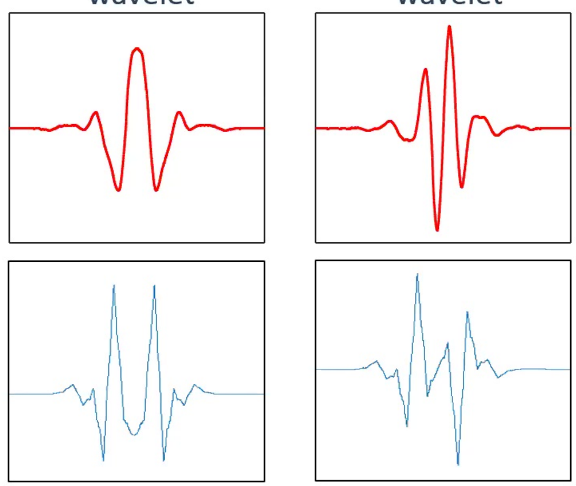

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Butt ugly!

Wavelet Design using Tensor Networks

sample code at:
www.tensors.net/research

CDF 9/7 wavelets:

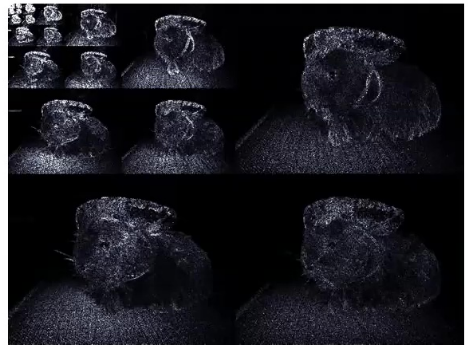
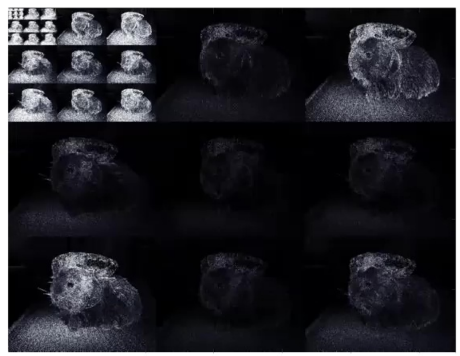


image → transform to wavelet basis → **truncate** (keep only largest 2% of coefficients) → inverse transform → compressed image

New scale-3 wavelets:



Wavelet Design using Tensor Networks

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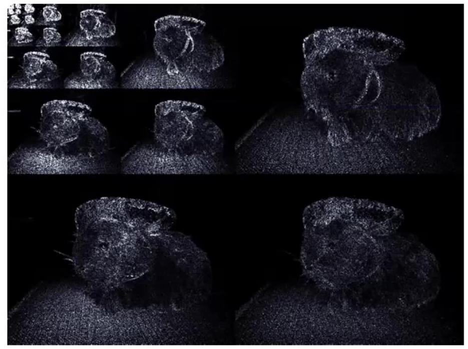

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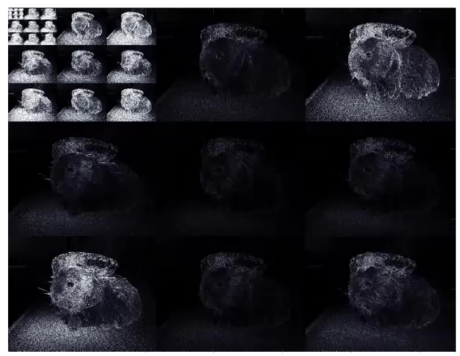

Wavelet Design using Tensor Networks

- We compare the new wavelets against the CDF 9/7 wavelets, which are regarded as the best known wavelets for practical applications.
- We compare compressed images based on multi-scale structural similarity (SSIM), which is a good measure of perceived image quality.
- Calculate the minimum number of coefficients that to be retained (in the transformed image) in order to achieve fixed quality: SSIM > 0.9.

Test database: 1285 images, mainly colour photographs.

	% Coeffs SSIM = 0.9		Ratio
Image set	**cdf**	**new**	**cdf/new**
aerials	14.1471	13.2413	1.068407181
australia	4.3669	4.0321	1.083033655
barcelona	5.7386	5.3671	1.069218013
campusinfall	6.2923	5.862	1.073404981
cannonbeach	9.5341	8.8726	1.074555373
cherries	6.0954	5.7992	1.05107601
columbia	6.0941	5.5082	1.106368687

⋮

new wavelets give 7% improved compression on average

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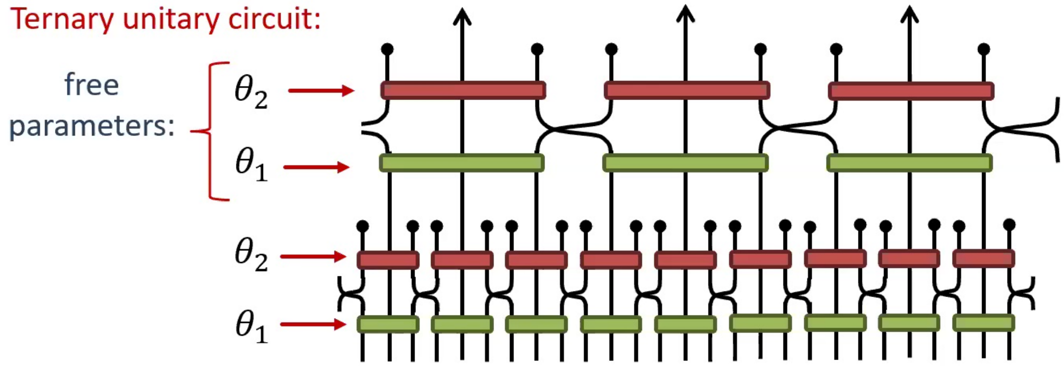
⋮

- 7% is not a huge improvement, but it is significant!
- Some specialised datasets (e.g. a database of fingerprints) gave up to 25% improvement.

new wavelets give 7% improved compression on average

Wavelet Design using Tensor Networks



In progress: **finely tuned wavelets**

Can we tune the free angles $\{\theta_1, \theta_2, \ldots\}$ as to construct optimally efficient wavelets for certain data-types? (e.g. fingerprints, medical images)

This is the real power of the tensor network formalism!

More generally: tensor networks have many connections between existing ideas in data science. There are myriad potential applications of tensor networks outside of physics!

Thanks!