Title: Path-integral Optimization and AdS/CFT

Speakers: Tadashi Takayanagi

Collection: Tensor Networks: from Simulations to Holography III

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Abstract: In this talk we will start with a review of path-integraloptimization, which provides a useful description of non-unitary tensor networks for Euclidean path-integrals in CFTs. We will explain an emergence of AdS geometry in this method and an interpretation as a computational complexity. Next we will give its application to analytical calculations of entanglement of purification, which was quite recently reproduced by numerical calculations. Finally, we would like to present a derivation of a path-integral optimization method directly from the AdS/CFT.

TENSOR NETWORKS: FROM SIMULATIONS TO HOLOGRAPHY III

@Perimeter, Nov.16-20, 2020

Path-integral Optimization and AdS/CFT

(A continuous approach to "AdS =Tensor Network")

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YITP YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

It from Qubit Simons Collaboration

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① Introduction: AdS/CFT and Tensor Network

② Path-Integral Optimization in CFTs

* Caputa-Kundu-Miyaji-Watanabe-TT [Phys.Rev.Lett. 119 (2017) 071602]
* Caputa-Kundu-Miyaji-Watanabe-TT [JHEP 11 (2017) 097]
* Bhattacharyya-Caputa-Das-Kundu-Miyaji-Watanabe-TT [JHEP 07 (2018) 086]

③ An Application: Entanglement of Purification

* Caputa-Miyaji-Umemoto-TT [Phys.Rev.Lett. 122 (2019) 11, 111601]
* Camargo-Hackl-Heller-Jahn-TT, arXiv: 2009.11881

④ PI-Opt. from Hartle-Hawking Wave function in AdS

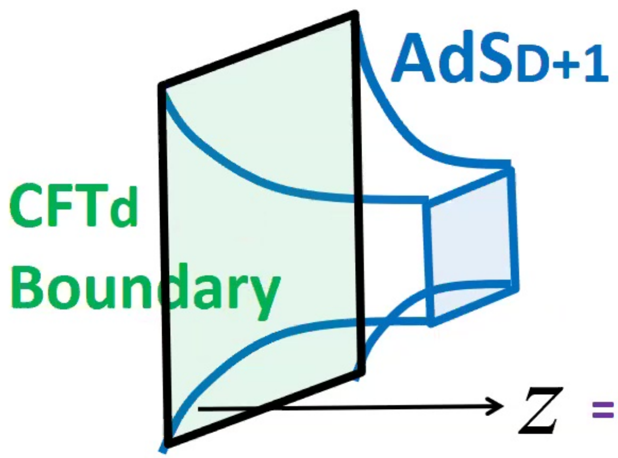
* Boruch-Caputa-TT arXiv:2011.08188

⑤ Conclusions

① Introduction

AdS/CFT [Anti de-Sitter space/Conformal Field Theory] [Maldacena 1997]

Gravity on D+1 dim. AdS = D dim. CFT on bdy



= a length scale of RG flow

Metric of AdSd+1

$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + \sum_{i=1}^{d-1} dx_i^2}{z^2}$$

⇒ ``Geometrization'' of Renormalization Group Flow

[AdS = Real Space Renormalization]

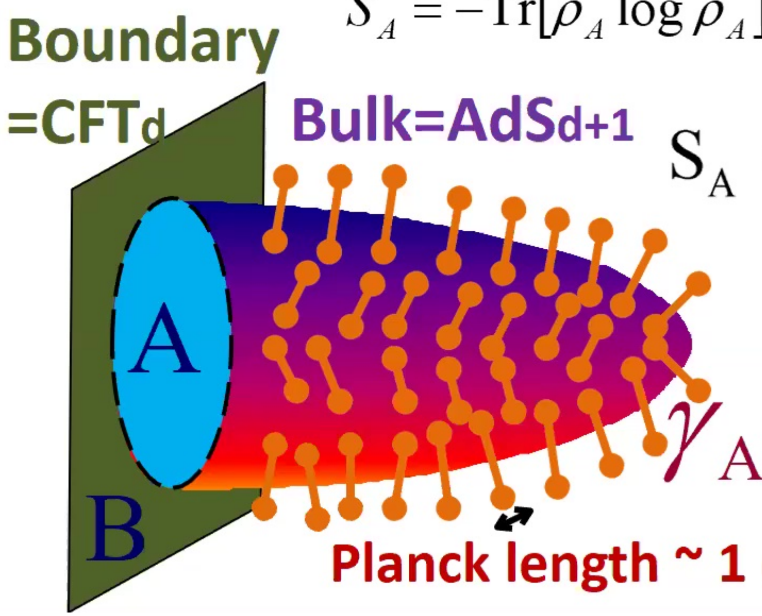
An essence of this emergence of geometry from QFTs is

Quantum entanglement

⇒ ``Geometry'' of Quantum States in many-body system

In AdS/CFT, the entanglement entropy is computed as the area of minimal surface [Ryu-TT 06, Hybeny-Rangamani-TT 07]

$$S_A = -\mathrm{Tr}[\rho_A \log \rho_A], \quad \rho_A = \mathrm{Tr}_B \rho_{tot} \quad .$$



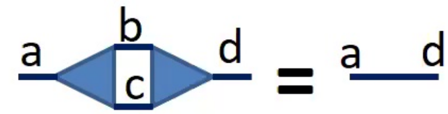
$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \approx \underbrace{\frac{\mathrm{Area}(\gamma_A)}{l_{pl}^{d-1}}}_{\text{Area in the unit of Planck length}}.$$

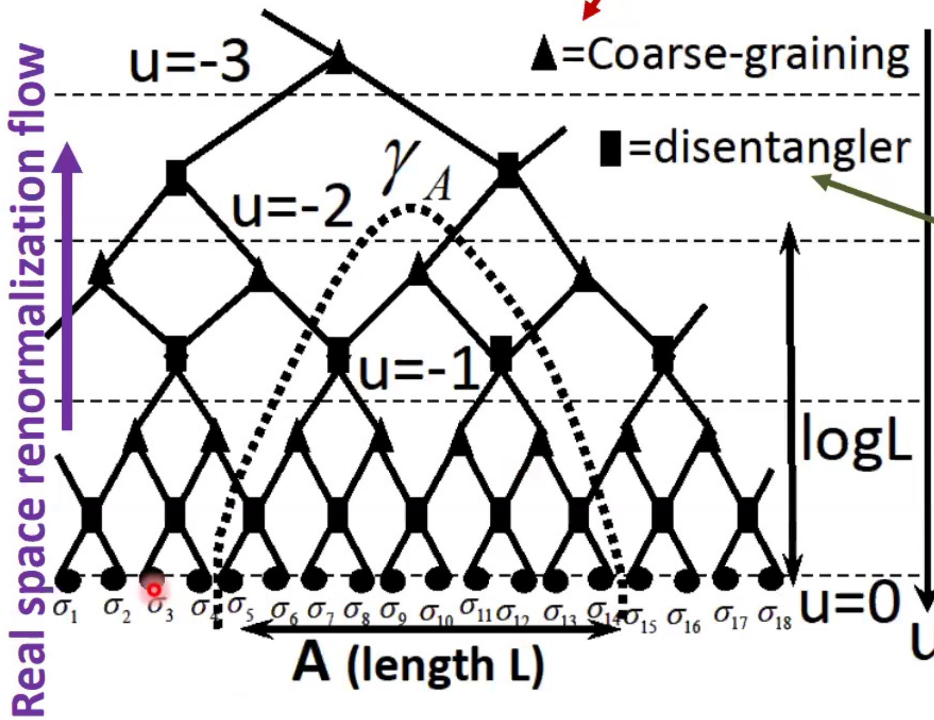
Spacetime in gravity = Collections of bits of entanglement
= Real space renormalization ?
⇒ Tensor Networks !

MERA [Vidal 05]

Coarse-graining = Isometry

$$[T]^{\dagger}_{abc}[T]_{bcd} = \delta_{ad}$$



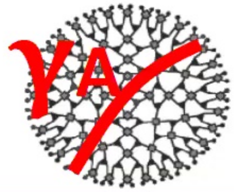


Disentangler
= Unitary trf.

$$S_A \le \mathrm{Min}[\#\mathrm{links}]$$
$$\propto \log L$$
$\Rightarrow$ agrees with results in 2d CFT !

This suggests a connection:
[Swingle 2009]

TN of MERA = A time slice of AdS



?

Hyperbolic Disk !

However, ∃ many difficulties: [Beny 2011, Bao et.al. 2015, Czech et.al. 2015]

- The MERA has a causal structure but the time slice does not ?
- Why is the EE bound saturated ?
- How can we find the sub-AdS locality ? (i.e. approximation by GR ?)

Though there are improvements of models/interpretations:

[Perfect TN: Pastawski-Yoshida-Harlow-Preskill 15]
[Kinematical Space: Czech-Lamprou-McCandlish-Sully 15]
[Random TN: Hayden-Nezami-Qi-Thomas-Walter-Yang 16]
[Matchgate TN: Jahn-Gluza-Pastawski-Eisert 17] [Hyper inv. TN: Evenbly 17]
[P-adicTN: Bhattacharrya-Hung-Lei-Li 17] [TN geometry: Misted-Vidal 18],.....

, the true connection between TN and AdS is not still clear.

Our Key Observations towards AdS/TN

[1] A disadvantage of the original TN approach is the presence of ***discretized structure***. The genuine AdS/CFT is continuous.

[2] To describe a time slice of AdS, it looks better to regard it as a ***Euclidean evolution*** rather than Lorentzian one.

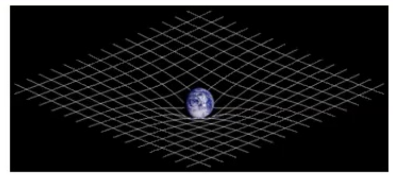
A Time slice of AdS
= Continuous TN from Euclidean CFT path-integrals ?

Motivated by this, we give an alternative approach **based on path-integrals**, related to a continuum limit of TNs.

⇒ **Path-integral Optimization !**

We will describe the discretized lattice structure by **the metric in a continuous way** and remove unnecessary tensors to make the TN efficient.

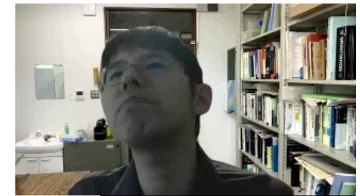
Our guiding principle



Eliminating unnecessary tensors in TN for a given state

= Creating the most efficient TN (= **Optimization of TN**)

⟺ Solving the dynamics of Gravity (i.e. Einstein eq.)



② Path-Integral Optimization and CFT

(2-1) Formulation

A Basic Rule: Simplify a path-integral s.t. it produces the correct UV wave functional.

Consider 2D CFTs for simplicity. **(z=- Euclidean time, x=space)**

Deformation of discretizations in path-integral
= Curved metric such that one cell (bit) = unit length.

$$\boxed{ds^2 = e^{2\phi(x,z)}(dx^2 + dz^2).}$$

Note: The original flat metric is given by (ε is UV cutoff):

$$\boxed{ds^2 = \varepsilon^{-2} \cdot (dx^2 + dz^2).}$$

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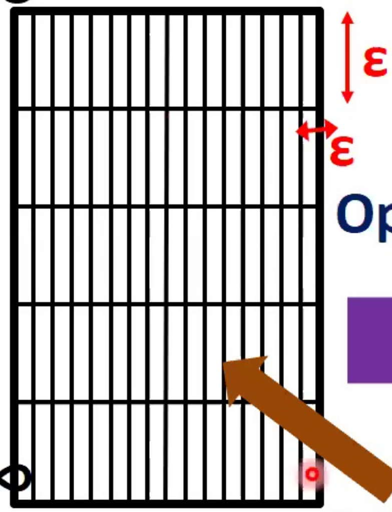
A Sketch: Optimization of Path-Integral

$z=0$ $\Psi^{\text{Flat}}_{UV}[\Phi(x)] \propto \Psi^{g=e^{2\phi}}_{UV}[\Phi(x)]$

Tensor Network (non Unitary)



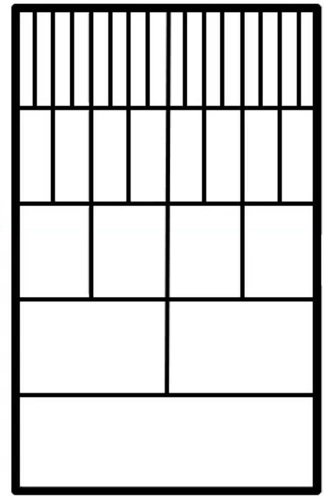
Time

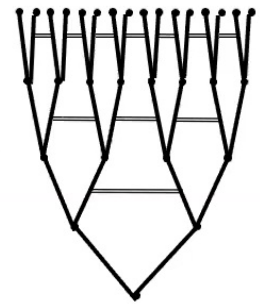
$-Z$

$z=\infty$

Space X

Optimize





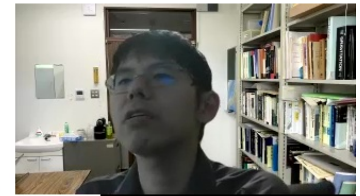
High energy modes $k>1/z$ are not important !

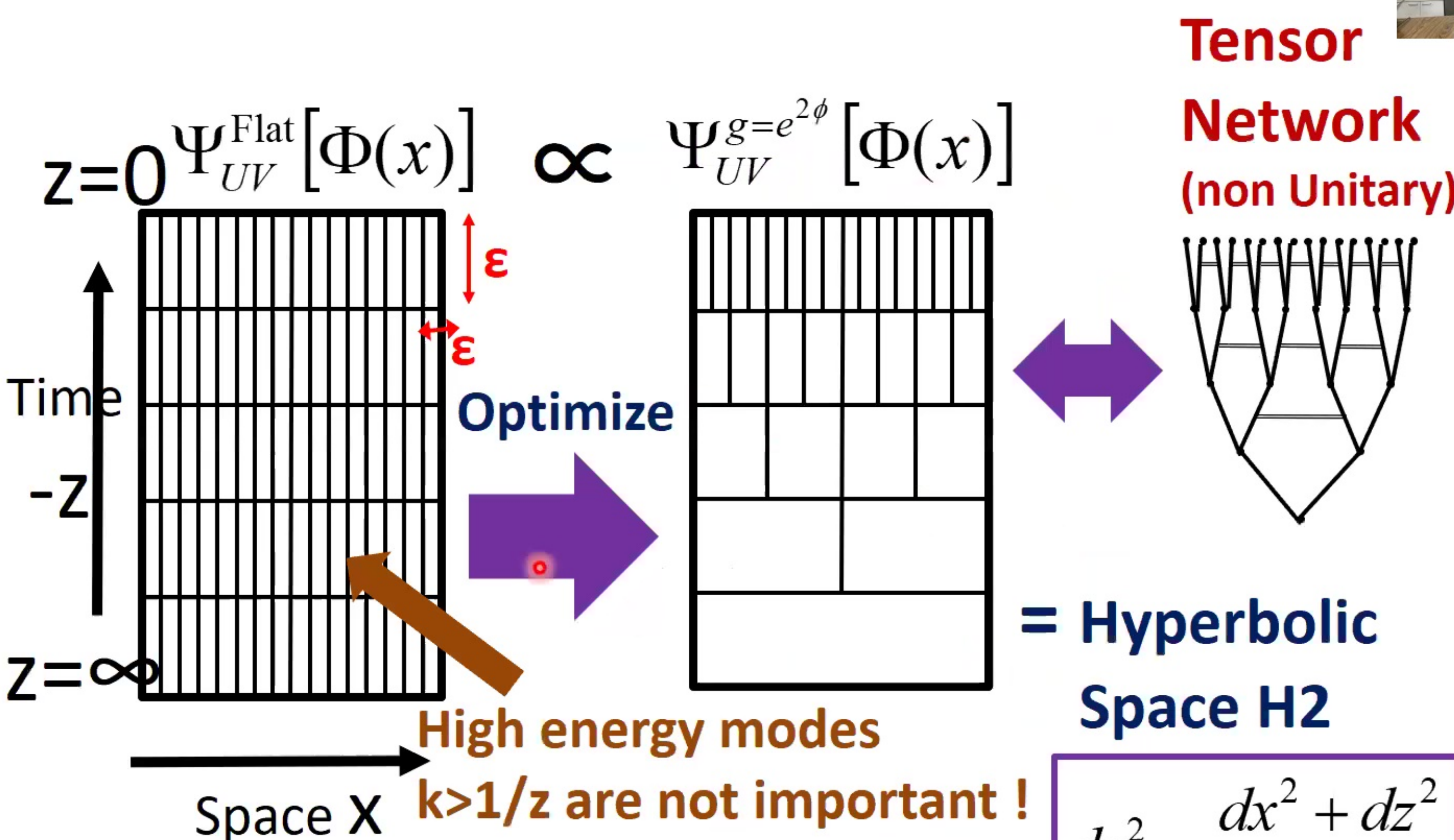
= Hyperbolic Space H2

$$ds^2 = \varepsilon^{-2}\cdot(dx^2+dz^2).$$

$$ds^2 = \frac{dx^2+dz^2}{z^2}$$

A Sketch: Optimization of Path-Integral





$$ds^2 = \frac{dx^2 + dz^2}{z^2}$$

$$ds^2 = \varepsilon^{-2} \cdot (dx^2 + dz^2).$$

The wave functional for CFT vacuum is given by

$g_{ab}(x,z)$: background metric

$$\Psi_{UV}^{g}[\Phi(x)]=\int \prod_{\substack{0<z<\infty \\ -\infty<x<\infty}} D\Phi(x,z)\, e^{-S_{CFT}(\Phi)} \cdot \delta\big(\Phi(x)-\Phi(x,z=0)\big)$$

In CFTs, owing to the Weyl invariance, we have

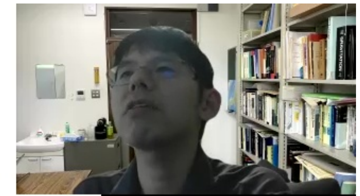
$$\boxed{\Psi_{UV}^{g_{ab}=e^{2\phi}\delta_{ab}}[\Phi(x)]=\exp\big(I[\phi(x,z)]\big)\cdot \Psi_{UV}^{\text{Flat}}[\Phi(x)]\ .}$$

Optimized wf. **Original wf.**

<u>**Our Proposal**</u> (Optimization of Path-integral for CFTs):

Minimize $I[\phi(x,z)]$ w.r.t $\phi(x,z)$

with the boundary condition $e^{2\phi}|_{z=\varepsilon}=\varepsilon^{-2}$.



Motivation of Our Proposal

The normalization N of wave functional estimates repetitions of same operations of path-integration.
→ **Minimize this !**

⇒ **Our conjecture:**

$$\boxed{\begin{aligned}\mathrm{Min}[I[\varphi]] &\sim \mathrm{Min\#\ of\ Tensors\ in\ TN} \\ &\approx \mathrm{Computational\ complexity\ of\ TN\ state}\end{aligned}}$$

[For a justification based on circuit complexity, refer to Camargo-Heller-Jefferson-Knaute 2019]

[cf. Holographic Complexity: Susskind 2014,Brown-Roberts-Susskind-Swingle-Zhao 2015, Lehner-Myers-Poisson-Sorkin 2016, Chapman-Marrochio-Myers 2016,…]

For 2D CFTs, I[ϕ] is given by the Liouville action !

$$I[\phi] = \mathrm{Log}\left[\frac{\Psi_{g=e^{2\phi}\delta_{ab}}}{\Psi_{g=\delta_{ab}}}\right] = S_L[\phi],$$

Liouville Action

# of Isometries 

[Czech 17]

# of Unitaries 

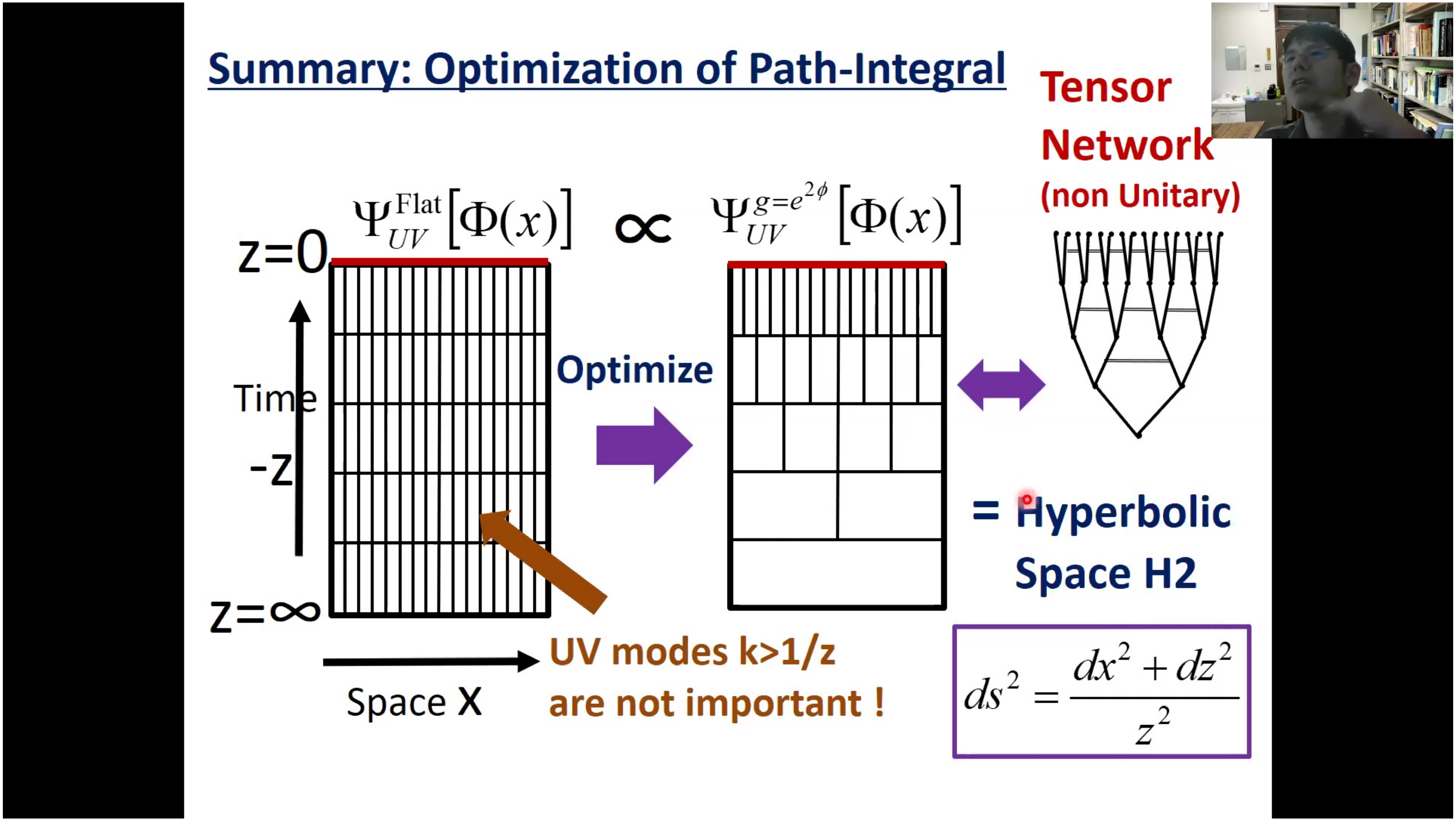
$$S_L[\phi] = \frac{c}{24\pi}\int dxdz\left[(\partial_x\phi)^2 + (\partial_z\phi)^2 + e^{2\phi}\right]$$

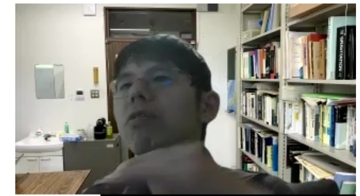
$$= \frac{c}{24\pi}\int dxdz\left[(\partial_x\phi)^2 + \left(\partial_z\phi + e^{\phi}\right)^2\right] + (\text{surface term})$$

$$\Rightarrow \text{Minimum: } e^{2\phi} = \frac{1}{z^2}.$$

Hyperbolic plane (H2) = Time slice of AdS3

$$ds^2 = (dx^2 + dz^2)/z^2.$$

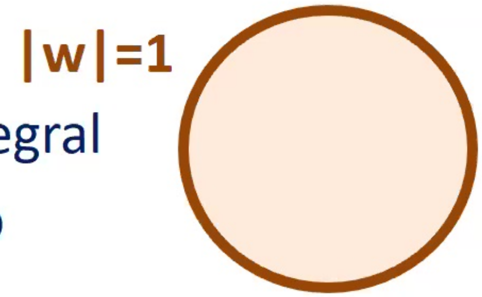

Summary: Optimization of Path-Integral
Tensor Network
(non Unitary)
$\Psi_{UV}^{\mathrm{Flat}}[\Phi(x)] \propto \Psi_{UV}^{g=e^{2\phi}}[\Phi(x)]$
z=0
Time
-Z
z=∞
Optimize
= Hyperbolic Space H2
Space X
UV modes k>1/z
are not important !
$ds^2 = \frac{dx^2 + dz^2}{z^2}$



(2-2) Excited States and Back-reactions

Consider the vacuum state of 2d CFT on a circle dual to a global AdS3.

The optimization of the path-integral of 2d CFT on a disk with leads to

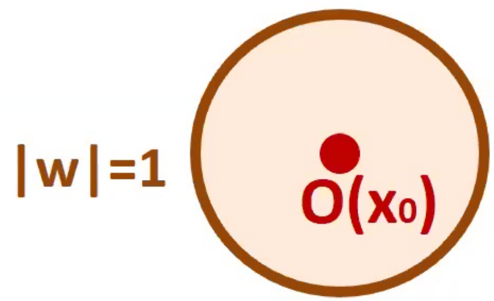


$$ds^2 = \frac{4dwd\bar{w}}{(1-|w|^2)^2}. \Rightarrow$$

Hyperbolic Disk H2
= time slice of global AdS3

Now we insert an operator O(x) in the center of the disk x=x0.
O(x): conformal dim. hL=hR=h

$$\Rightarrow O(x) \sim e^{-2h\cdot\varphi}.$$



$$\frac{\Psi_{g=e^{2\varphi}}}{\Psi_{\text{Flat}}} \propto e^{\frac{c}{24\pi}S_L} \cdot e^{-2h\varphi(x_0)}.$$

Minimize

$$\boxed{\partial_w \partial_{\overline{w}} \varphi - \frac{1}{4} e^{2\varphi} + \frac{6\pi h}{c} \delta^2(w) =}$$

Solve

Metric: $ds^2 = \dfrac{4d\zeta d\overline{\zeta}}{(1-|\zeta|^2)^2}, \qquad \zeta = re^{i\theta}$

$\Rightarrow$ Deficit angle geometry $\theta \sim \theta + 2\pi a.$

This agrees with the expected gravity dual if h/c<<1.

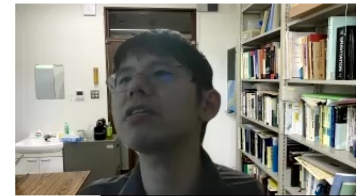
Note: the AdS/CFT predicts $a = \sqrt{1 - 24h/c}.$

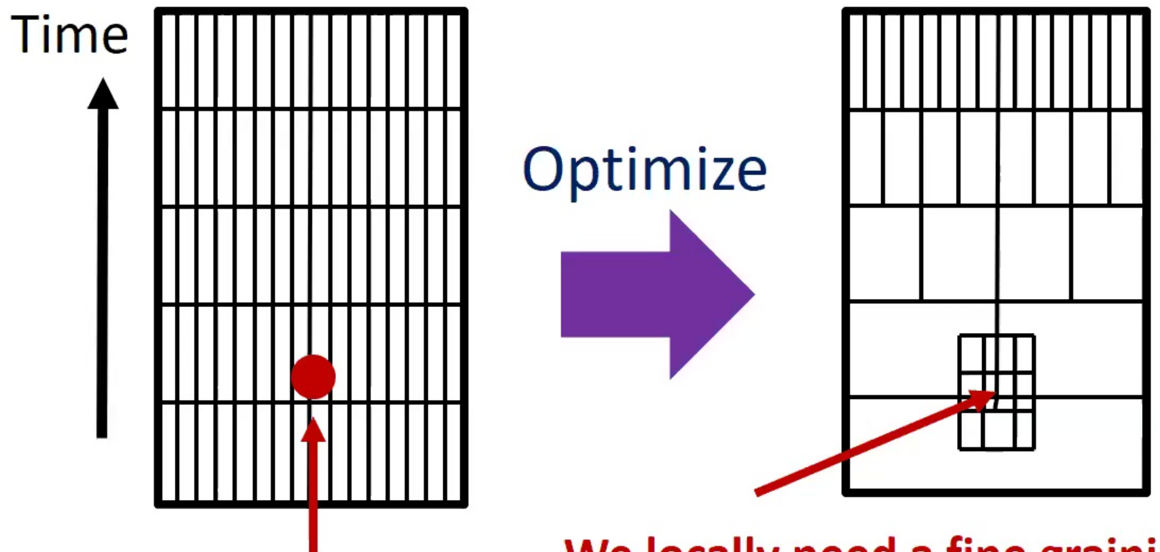
Interestingly, if we consider **the quantum Liouville CFT**,

then $h = \dfrac{\gamma\alpha}{4}(Q - \alpha\gamma/2.), \quad c = 1 + 3Q^2, \quad (Q \equiv 2/\gamma + \gamma).$

$\Rightarrow$ We get $a = \sqrt{1 - 24h/c}.$

Heuristic Summary

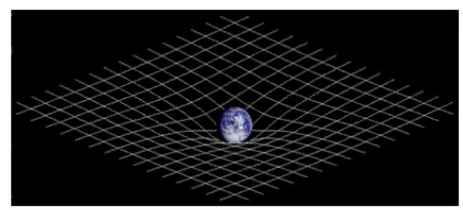




Local excitation
(energy source)

We locally need a fine graining
⇒ The metric gets larger !

This agrees with
general relativity !



Information Source = Back-reactions in GR

③ An Application: Entanglement of Purification

(3-1) Entanglement of Purification

When we divide a total system into A and B: $H_{tot} = H_A \otimes H_B$
the entanglement entropy (EE) is defined by

$$S_A = -\mathrm{Tr}[\rho_A \log \rho_A]. \qquad \rho_A = \mathrm{Tr}_B[|\Psi\rangle\langle\Psi|]$$

The entanglement entropy is known to be a measure of quantum entanglement only for pure states.

We know that mixed states can always be purified:

$$\rho_C = \sum_i \lambda_i |i\rangle_C \langle i| \xrightarrow{H_C \to H_C \otimes H_D} |\Psi\rangle_{CD} = \sum_i \sqrt{\lambda_i}\, |i\rangle_C |i\rangle_D$$

$$\rho_C = \mathrm{Tr}_D[|\Psi\rangle\langle\Psi|]$$

Definition of EoP [Terhal-Horodecki-Leung-Divincenzo 2002]

Consider all purifications $|\Psi\rangle_{A\tilde{A}B\tilde{B}}$ of ρ_{AB} in the extended Hilbert space: $H_A \otimes H_B \to H_A \otimes H_B \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$.

Then, **Entanglement of Purification (EoP)** is defined by

$$\boxed{E_P(\rho_{AB}) = \underset{\text{All purifications } |\Psi\rangle \text{of } \rho_{AB}}{\mathrm{Min}} S_{A\tilde{A}}\left(|\Psi\rangle_{A\tilde{A}B\tilde{B}}\right)}$$

$$\rho_{AB} = \mathrm{Tr}_{\tilde{A}\tilde{B}}\left[|\Psi\rangle\langle\Psi|\right]$$

Entanglement Entropy

Note: $E_p(\rho_{AB}) \geq 0$ and $E_p(\rho_{AB}) = 0 \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B$.

(3-2) Holographic EoP

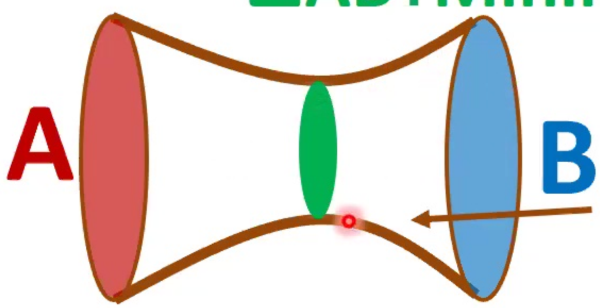
[Umemoto-TT 17, Nguyen-Deva
-Halbasch-Zaletel-Swingle 17]

Conjecture

$$\boxed{E_P(\rho_{AB}) = \frac{\text{Area}(\Sigma_{AB})}{4G_N}}$$

Entanglement Wedge Cross Section

Σ_{AB} : Minimal Surface of M_{AB}



M_{AB}: Minimal surface connecting ∂A and ∂B

Note: When ρ_{AB} is pure,
we simply have $E_P(\rho_{AB}) = S_A = S_B$.

➡ **Standard Hol. EE**

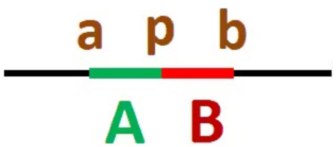
(3-3) EoP from Path-Integral Optimization

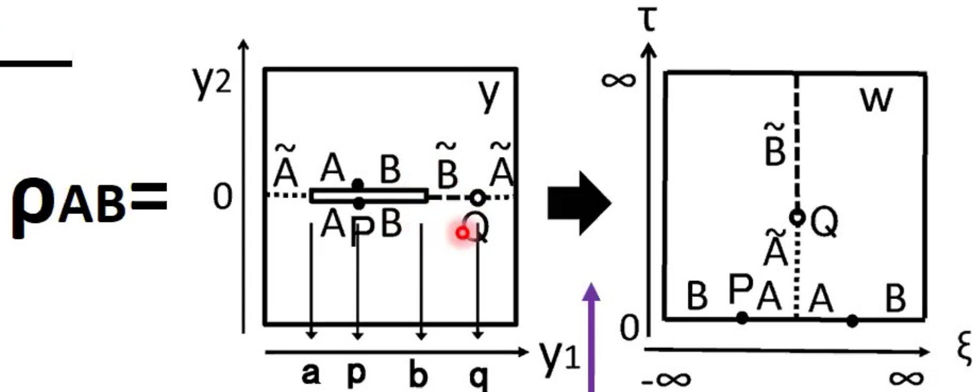
[Caputa-Miyaji -Umemoto-TT 18]

Key Idea: Min over all purifications is realized by PI-opt !

Consider the setup when AB= a single interval:

$A=[a,p]$ $B=[p,b]$: $\Rightarrow$ $\tilde{A}=[-\infty,a]\cup[q,\infty]$ $\tilde{B}=[b,q]$





Conformal Map

$$\boxed{w=\sqrt{\frac{y-a}{b-y}}}$$

The final optimized metric looks like

$$ds^2 = \frac{\epsilon^2}{\tau^2} \cdot dw d\bar{w} = \frac{\epsilon^2}{\tau^2} \cdot \frac{(b-a)^2}{4|b-y|^3|y-a|} \cdot dy d\bar{y} \equiv e^{2\tilde{\phi}} \cdot dy d\bar{y}.$$

In this setup, we obtain $e^{2\tilde{\phi}_P} = 1$ and $e^{2\phi_Q} = \frac{\epsilon^2 (b-a)^2}{4(q-a)^2(q-b)^2}$. The entanglement entropy $S_{A\tilde{A}} = S_{B\tilde{B}}$ is found to be

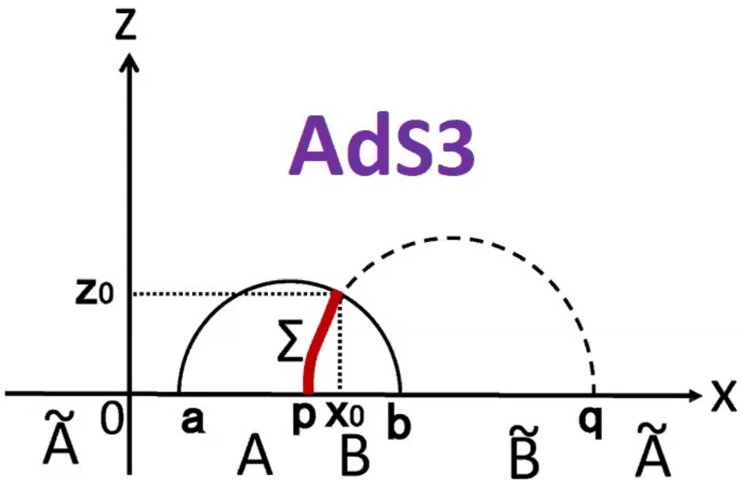
$$\begin{aligned} S_{A\tilde{A}} &= \frac{c}{3} \log\left(\frac{q-p}{\epsilon}\right) + \frac{c}{6}\tilde{\phi}_P + \frac{c}{6}\tilde{\phi}_Q \\ &= \frac{c}{6} \log\left[\frac{(b-a)(q-p)^2}{2\epsilon(q-a)(q-b)}\right], \end{aligned}$$

Minimize w.r.t q

$$S^{min}_{A\tilde{A}} = \frac{c}{6} \log\left[\frac{2(p-a)(b-p)}{\epsilon(b-a)}\right],$$

$$\text{at} \quad q = \frac{2ab-(a+b)p}{a+b-2p}.$$

Calculations of Hol EoP in AdS3/CFT2



$$ds^2 = \frac{dx^2 + dz^2}{z^2}$$

$$L(\Sigma) = \frac{q-p}{2}\int_{\epsilon}^{z_0} \frac{dz}{z\sqrt{\frac{(q-p)^2}{4} - z^2}} = \log\left[\frac{2(p-a)(p-b)}{\epsilon(b-a)}\right].$$

➡ The Hol EoP $L(\Sigma)/4G$ agrees with the CFT result !

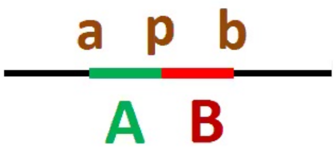
(3-3) EoP from Path-Integral Optimization

[Caputa-Miyaji -Umemoto-TT 18]

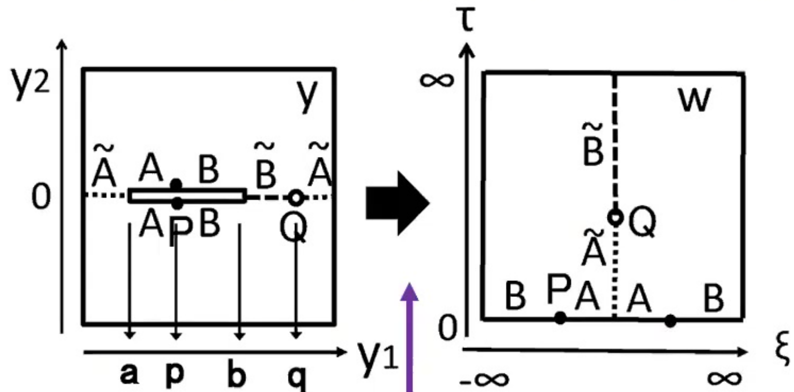
Key Idea: Min over all purifications is realized by PI-opt !

Consider the setup when AB= a single interval:

A=[a,p] B=[p,b] : $\Rightarrow$ $\tilde{A}$=[-∞,a]U[q,∞] $\tilde{B}$=[b,q]



$\rho_{AB}=$



Conformal Map $\boxed{w=\sqrt{\frac{y-a}{b-y}}}$

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The entanglement entropy $S_{A\tilde{A}} = S_{B\tilde{B}}$ is found to be

$$S_{A\tilde{A}} = \frac{c}{3}\log\left(\frac{q-p}{\epsilon}\right) + \frac{c}{6}\tilde{\phi}_P + \frac{c}{6}\tilde{\phi}_Q$$

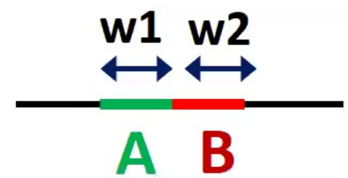
$$= \frac{c}{6}\log\left[\frac{(b-a)(q-p)^2}{2\epsilon(q-a)(q-b)}\right],$$

Minimize w.r.t q

$$\boxed{\begin{aligned} S^{min}_{A\tilde{A}} &= \frac{c}{6}\log\left[\frac{2(p-a)(b-p)}{\epsilon(b-a)}\right], \\ \text{at} \quad q &= \frac{2ab-(a+b)p}{a+b-2p}. \end{aligned}}$$

Holographic EoP → **Path-integral Optimization**

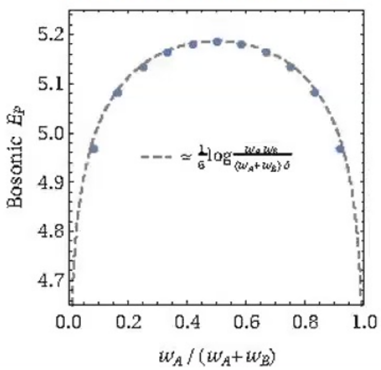
$$\boxed{E_{\mathrm{p}}(\rho_{AB}) = \frac{c}{6}\log\frac{w_1 w_2}{(w_1+w_2)\delta}}$$



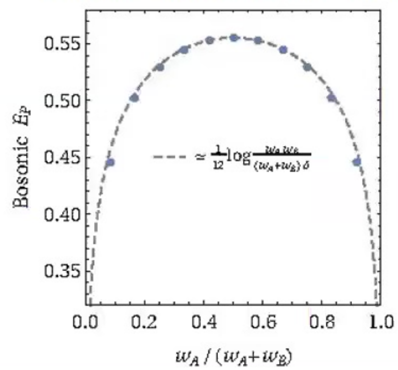
Confirmed by direct Gaussian numerical calculations in free scalar/fermion CFT !

[Camargo-Hackl-Heller-Jahn-TT, 2020]

Free Scalar CFT



Free Fermion CFT



This provides a further support of path-integral optimization as well as Hol. EoP conjecture !

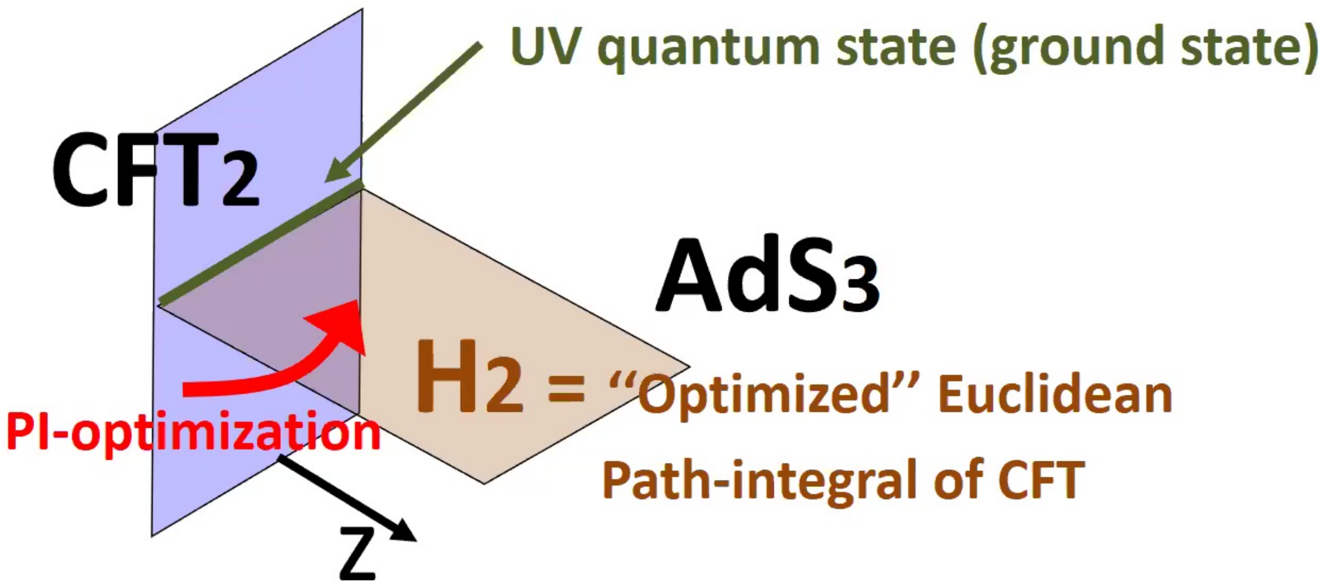
④ Connection between PI-Optimization and AdS/CFT

[Boruch-Caputa-TT, 2020]

We observed that a time slice of AdS emerges from our path-integral optimization. This raises the question:

Q. What does optimization mean in AdS/CFT ?

<u>Heuristic Expectation</u>



PI-optimization = $$\underset{\phi}{\mathrm{Min}}\left[\Psi_{UV}^{g_{ab}=e^{2\phi}\delta_{ab}}[\Phi(x)]\right]$$

One might think the optimization corresponds to ***a saddle point approximation*** of some quantum gravity.

➡ However, a saddle point approximation does not give a minimum but a maximum !

This is important when we would like to take into account quantum gravity (=1/c) corrections in AdS/CFT.

Consider **Hartle-Hawking wave function** on a time slice (

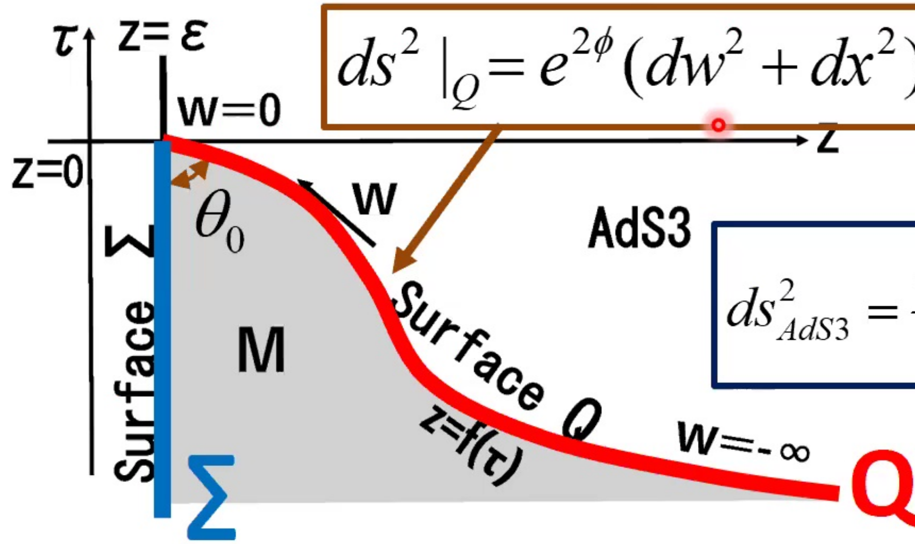
in AdS3 gravity, with an initial condition on the AdS bdy ∑.

$$\Psi^{(T)}_{HH}[e^{2\phi}\delta_{ab}] = \int Dg_{\mu\nu}\, e^{-I_G[g] - T\int_Q e^{2\phi}}\, \delta\!\left(g|_Q - e^{2\phi}\delta_{ab}\right)$$

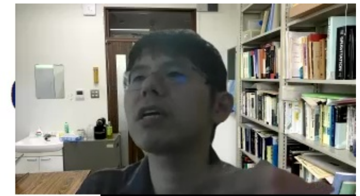
Metric on Q

“Tension” term
~bdy cosmological const.

$$ds^2|_Q = e^{2\phi}(dw^2 + dx^2)$$



$$ds^2_{AdS3} = \frac{dz^2 + d\tau^2 + dx^2}{z^2}$$



By evaluating the HH wave function using classical gravity action, we obtain (we assume $\partial_x \phi = 0$ and write $\dot{\phi} \equiv \partial_w \phi$

$$\Psi_{HH}^{(T)}[e^{2\phi}\delta_{ab}] \approx e^{-I^{(T)}[\phi]+I^{(T)}[\phi_0]}, \qquad (e^{\phi_0} = 1/\varepsilon)$$

Minus sign

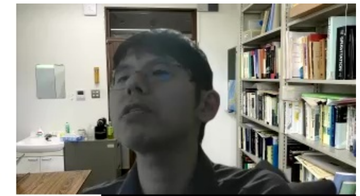
$$I^{(T)}[\phi] = \frac{c}{12\pi}\int dxdw\left[e^{2\phi}\sqrt{1-\dot{\phi}^2 e^{-2\phi}} + \dot{\phi}\,\mathrm{ArcSin}\left[\dot{\phi}e^{-\phi}\right]\right] - \frac{c}{12\pi\varepsilon}\int dx\theta_0$$

When the UV cut off is dominant $\dot{\phi}^2 << e^{2\phi}$, we can trust the continuum limit of field theory. In this case, we have

$$I^{(T)}[\phi] \approx \frac{c}{24\pi}\int dxdw\left[\dot{\phi}^2 + 2e^{2\phi}\right] - \frac{c}{12\pi}\int dx\,\mathrm{ArcSin}\sqrt{1-T^2}.$$

Indeed we reproduce the Liouville action !
The HH wave function gives finite cut off corrections.

Therefore, we can interpret the PI-optimization as an optimization of HH wave function $\Psi^{(T)}_{HH}[e^{2\phi}\delta_{ab}]$:



$$\langle O_1 O_2 \cdots O_n \rangle = \int D\phi \left|\Psi^{(T)}_{HH}[\phi]\right|^2 O_1 O_2 \cdots O_n .$$

$$\Psi^{(T)}_{HH} \propto e^{-I^{(T)}[\phi]}]$$

$$\Rightarrow \mathrm{Min}[I^{(T)}[\phi]]$$

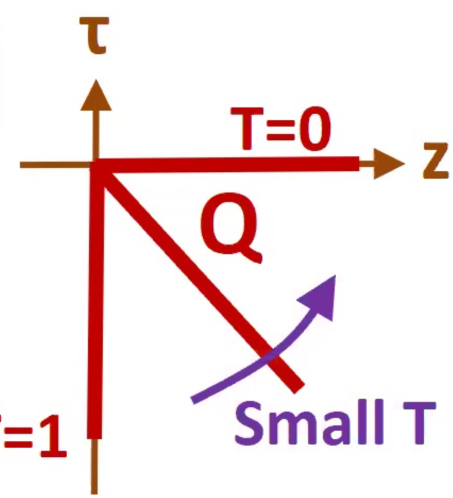
Indeed, its saddle point approximation lead to the maximization of $\Psi^{(T)}_{HH}[e^{2\phi}\delta_{ab}]$, leading to the EOM:

$$\ddot{\phi} + \dot{\phi}^2 - 2e^{2\phi} = 2Te^{2\phi}\sqrt{1 - e^{-2\phi}\dot{\phi}^2}$$

Solution

$$e^{2\phi} = \frac{1}{(1-T^2)w^2}$$

Agree with PI-optimization



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$$\boxed{\langle O_1 O_2 \cdots O_n \rangle = \int D\phi \left| \Psi^{(T)}_{HH}[\phi] \right|^2 O_1 O_2 \cdots O_n .} \Rightarrow \boxed{\begin{array}{l} \Psi^{(T)}_{HH} \propto e^{-I^{(T)}[\phi]}] \\ \Rightarrow \mathrm{Min}[I^{(T)}[\phi]] \end{array}}$$

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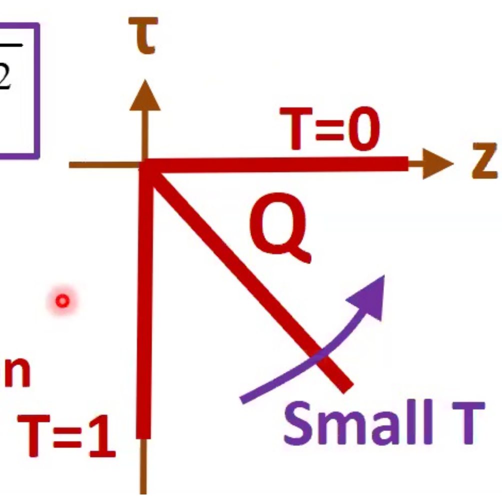
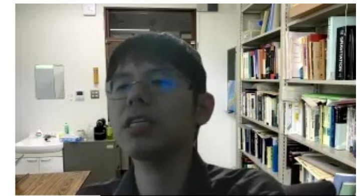


The Liouville potential and the tension of the brane Q is related via $\mu = 1 - T^2$.

This allows us to rewrite the AdS3 metric as follows:

$$\boxed{ds^2 = \underbrace{\frac{d\mu^2}{4\mu^2(1-\mu)}}_{\substack{\textbf{Emergent Time direction} \\ \text{= the energy scale } (\mu) \text{ of} \\ \text{path-integral optimization}}} + \underbrace{e^{2\phi(w)}(dw^2 + dx^2)}_{\substack{\textbf{Optimized CFT metric} \\ \textbf{at the scale } \boldsymbol{\mu}}}}$$

$$\boxed{e^{2\phi} = \frac{1}{\mu w^2}}$$



Generalizations

We can also generalize the HH-wave function derivation of PI-optimization to

- Higher Dim. CFTs
- BTZ black holes
- JT gravity

CFT analysis looks difficult because we need more than the Weyl transformation to optimize PI.
But for Holographic CFTs,
HH-wave function gives an answer !

For details refer to arXiv:2011.08188 [Boruch-Caputa-TT]

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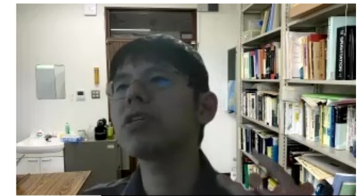
Emergent Time direction
= the energy scale (μ) of path-integral optimization

Optimized CFT metric at the scale μ

$$\boxed{e^{2\phi} = \frac{1}{\mu\, w^2}}$$



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⑤ Conclusions

- We expect that the AdS/CFT provides a geometrization of real space renormalization. ⇒ Conjectured connection to TN.

  But, the direct TN approaches suffer from lattice artifacts etc.

- We argued that the path-integral optimization gives a definite framework which covers also a continuum limit.

- We found : An PI-optimized geometry = a time slice of AdS

- We gave the explicit connection between the AdS/CFT and PI-optimization by considering the Hartle-Hawking wave function.

⇒ PI-optimization = saddle point calculation of HH wave function.

⇒ We can consider quantum fluctuations !

Future problems

- Meaning of Emergence of Time ? Metric in the time direction ?
- Time-dependent b.g. ?
- Implications of large N strongly coupled CFTs and Bulk locality ?
- dS/CFT version ? (Formally, $\mu<0$)

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