Title: Query complexity and cutoffs in AdS3/CFT2

Speakers: Bartek Czech

Collection: Tensor Networks: from Simulations to Holography III

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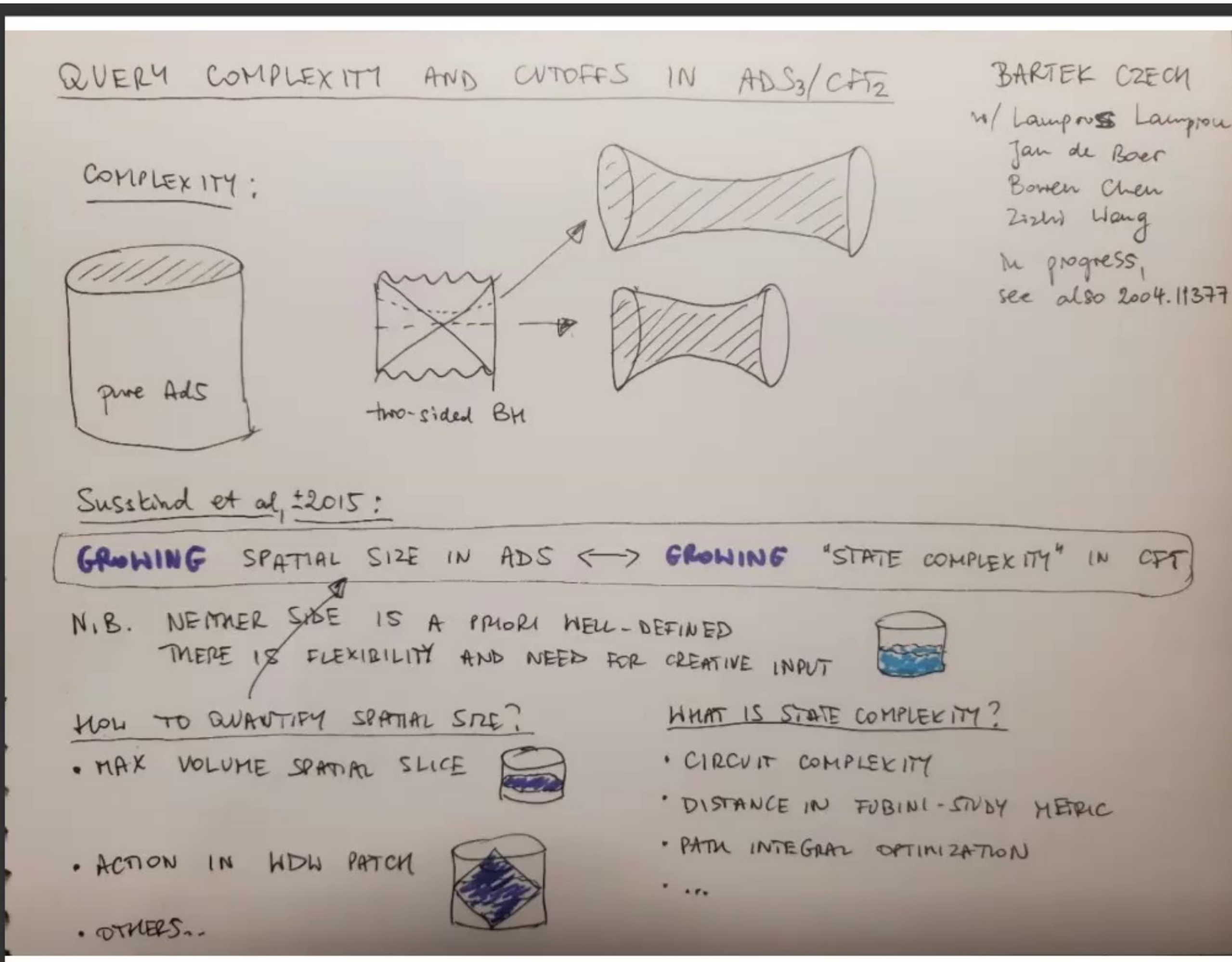
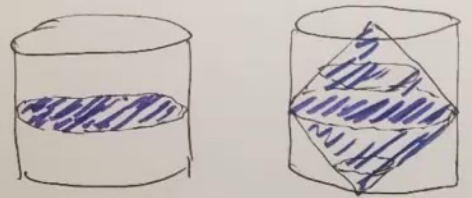
Abstract: A quantum state is a map from operators to real numbers that are their expectation values. Evaluating this map always entails using some algorithm, for example contracting a tensor network. I propose a novel way of quantifying the complexity of a quantum state in terms of "query complexity": the number of times an efficient algorithm for computing correlation functions in the given state calls a certain subroutine. I construct such an algorithm for a general "state at a cutoff" in 1+1-dimensional field theory. The algorithm scans cutoff-sized intervals for operators whose expectation values will be computed. It can be written as a Matrix Product State, with individual matrices performing translations in the space of (cutoff-sized) intervals and reading off consecutive operator inputs. If we take the queried subroutine to be a translation in the space of intervals, query complexity counts "how many" intervals the algorithm visits--a notion of distance in the space of intervals. A unique distance function is consistent with the requisite notion of translations; therefore the query complexity of a state at a cutoff is unambiguously defined. In holographic theories, the query complexity evaluates to the integral of the Ricci scalar on a spatial slice enclosed by the bulk cutoff, which in pure AdS3 agrees with the volume proposal but otherwise departs from it.

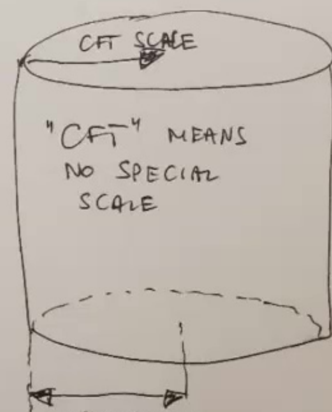

QUERY COMPLEXITY AND CUTOFFS IN ADS3/CFT2
BARTEK CZECH
w/ Lampros Lamprou
Jan de Boer
Bowen Chen
Zizhi Wang
In progress,
see also 2004.11377
COMPLEXITY:
pure AdS
two-sided BH
Susskind et al, ±2015:
GROWING SPATIAL SIZE IN ADS ⟷ GROWING "STATE COMPLEXITY" IN CFT
N.B. NEITHER SIDE IS A PRIORI WELL-DEFINED
THERE IS FLEXIBILITY AND NEED FOR CREATIVE INPUT
HOW TO QUANTIFY SPATIAL SIZE?
• MAX VOLUME SPATIAL SLICE
• ACTION IN WDW PATCH
• OTHERS...
WHAT IS STATE COMPLEXITY?
• CIRCUIT COMPLEXITY
• DISTANCE IN FUBINI-STUDY METRIC
• PATH INTEGRAL OPTIMIZATION
• ...

A PRELIMINARY: COMPLEXITY DEPENDS ON THE CUTOFF:



BOTH ARE STRICTLY SPEAKING INFINITE.

WHY?



BECAUSE IT REPRESENTS ALL POSSIBLE SCALES

- WITHOUT A CUTOFF, STATE COMPLEXITY MUST BE INFINITE BECAUSE WE MUST PREPARE THE STATE OVER ALL SCALES

COMPLEXITY IS ONLY WELL-DEFINED AFTER WE SPECIFY A CUTOFF (CFT SCALE)

PLAN FOR TODAY:

A COMPLETELY NEW IDEA FOR:

- STATE
- CUTOFF
- STATE COMPLEXITY

LOGICAL STARTING POINTS:

- SUBREGION DUALITY
- MODULAR PARALLEL TRANSPORT
- QUERY COMPLEXITY

BIGGEST CONCEPTUAL CHANGE:

THINK OF THE STATE AS AN ALGORITHM, WHICH TAKES:

$$\underset{\text{INPUT}}{\overset{\text{OPERATORS}}{\mathcal{O}_1\,\mathcal{O}_2 \ldots \mathcal{O}_n}} \xrightarrow[\text{SPECIFIED AS AN ALGORITHM}]{\text{COMPUTATION}} \underset{\text{OUTPUT}}{\overset{\mathbb{R}}{\text{EXPECTATION VALUES}}}$$

STATE AS A MAP **OPERATORS** $\rightarrow$ **EXPECTATION VALUES**:

OPERATOR	EXP. VALUE
OP_1	#
OP_2	#
OP_3	#
OP_4	#
OP_5	⋮
⋮	⋮
OP_{100}	
OP_{101}	
OP_{102}	
OP_{103}	
⋮	

CUTOFF?

SPECIFYING THESE NUMBERS CHARACTERIZES THE STATE

QUESTION: HOW EXACTLY DO WE SPECIFY THESE NUMBERS?

ANSWER: WE GIVE AN ALGORITHM FOR COMPUTING THEM

SOME OPERATORS WILL BE **UV** OPERATORS.

WE DON'T SPECIFY THEIR EXPECTATION VALUES IN A STATE AT A CUTOFF.

$\Rightarrow$ CUTOFF RESTRICTS THE DOMAIN OF MAP

OPERATORS $\xrightarrow{\text{STATE}}$ **EXPECTATION VALUES**

STATE AS AN ALGORITHM TO EVALUATE MAP

$$\text{OPERATORS} \xrightarrow[\text{EXHIBITED AS AN ALGORITHM}]{\text{STATE}} \text{EXPECTATION VALUES}$$

EXAMPLE 1: $|0\rangle_{CFT}$

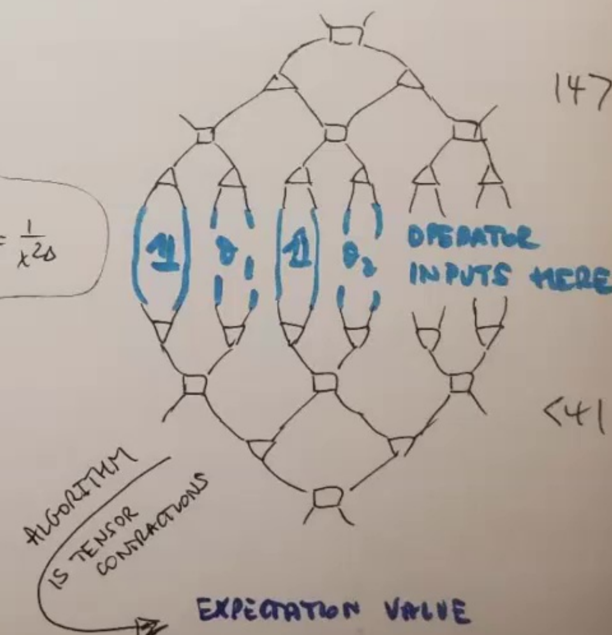
- SPECIFIED BY SYMMETRIES

$$L_{-1}|0\rangle = L_0|0\rangle = L_{+1}|0\rangle = 0$$

- USE THOSE SYMMETRIES TO EVALUATE MAP $\Rightarrow \langle O_\Delta(x) O_\Delta(0)\rangle = \frac{1}{x^{2\Delta}}$ etc.

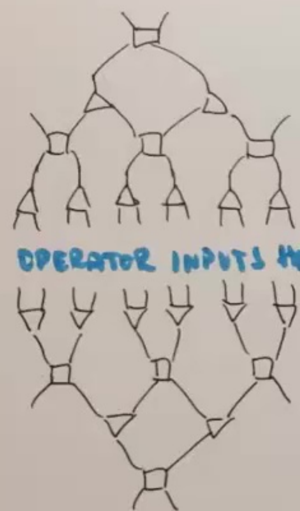
- IT'S OK HERE, BUT IT DOESN'T GENERALIZE TO STATES WITHOUT SPECIAL PROPERTIES / SYMMETRIES

EXAMPLE 2: TENSOR NETWORKS



QUERY COMPLEXITY OF AN ALGORITHM:

- HOW MANY TIMES ALGORITHM CALLS SOME KEY SUBROUTINE



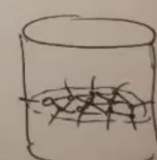
TENSOR NETWORK $\xrightarrow{\text{GENERALIZE}}$ ALGORITHM FOR COMPUTING CORRELATION FUNCTIONS

TENSOR CONTRACTION $\longrightarrow$ SUBROUTINE OF ALGORITHM

#{TENSOR CONTRACTIONS} $\longrightarrow$ $\boxed{\text{QUERY COMPLEXITY}}$

$\Downarrow$ SPECIAL CASE:

- QUERY COMPLEXITY OF A TENSOR NETWORK PROPORTIONAL TO #{TENSORS}
- THIS RECOVERS THE ORIGINAL MOTIVATION FOR THE "VOLUME PROPOSAL": A HYPOTHETICAL SPACE-FILLING TENSOR NETWORK

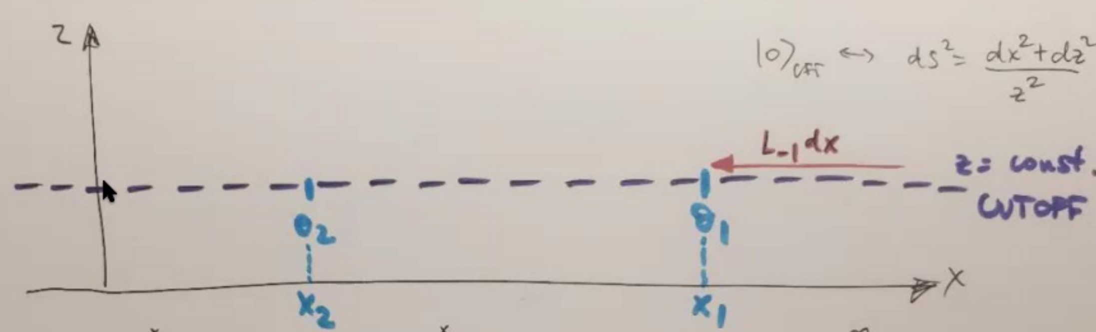


OUR PROPOSAL:

STATE COMPLEXITY IS QUERY COMPLEXITY OF THE OPTIMAL ALGORITHM TO EVALUATE OPERATORS $\xrightarrow{\text{STATE}}$ EXP VALUES

CONSTRUCTING THE ALGORITHM: EXAMPLE

$|0\rangle_{CFT} \leftrightarrow ds^2 = \frac{dx^2+dz^2}{z^2}$



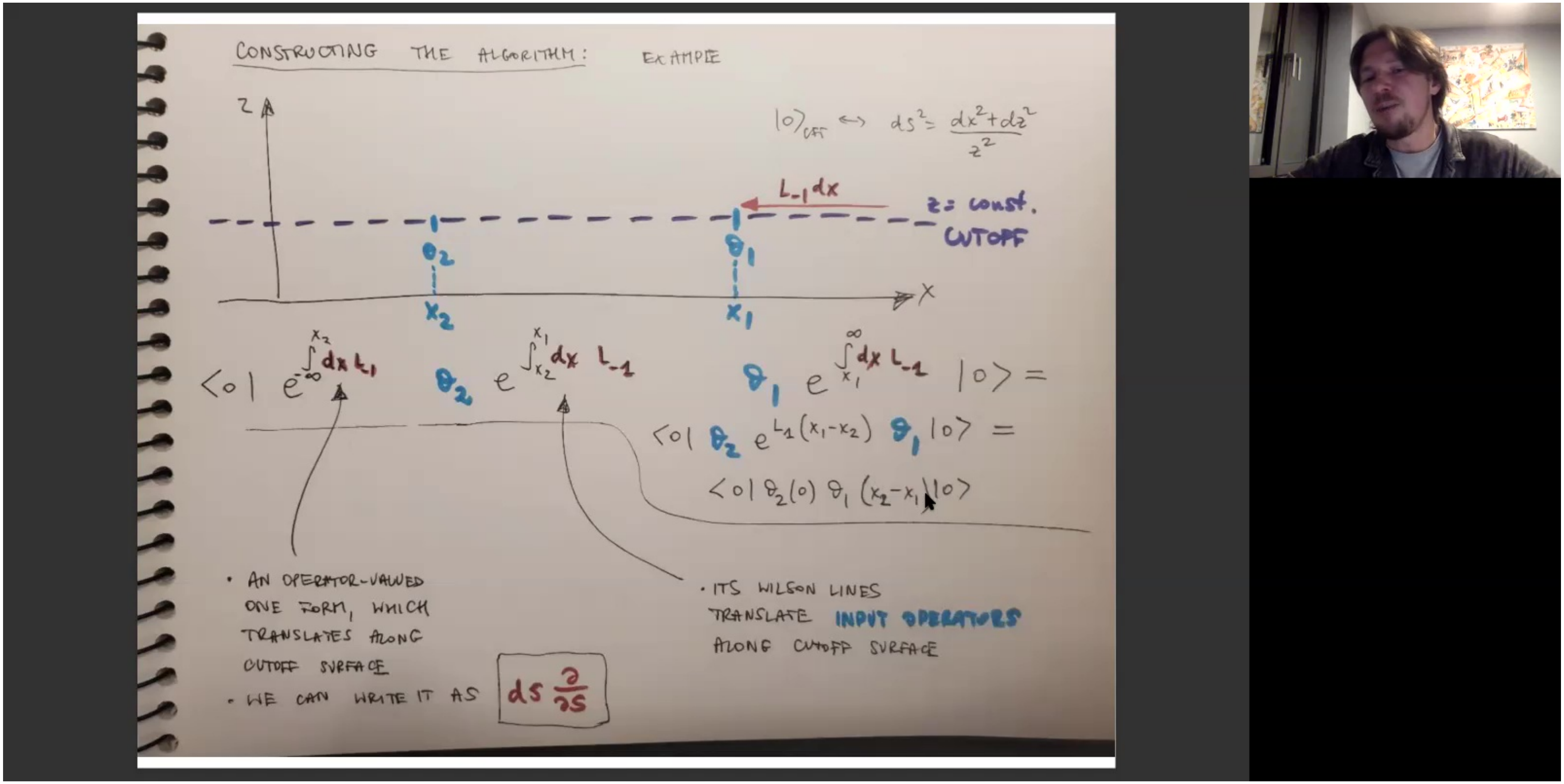
$$\langle 0|\, e^{\int_{-\infty}^{x_2} dx\, L_{-1}}\, \mathcal{O}_2\, e^{\int_{x_2}^{x_1} dx\, L_{-1}}\, \mathcal{O}_1\, e^{\int_{x_1}^{\infty} dx\, L_{-1}}\, |0\rangle =$$

$$\langle 0|\, \mathcal{O}_2\, e^{L_1(x_1-x_2)}\, \mathcal{O}_1 |0\rangle =$$

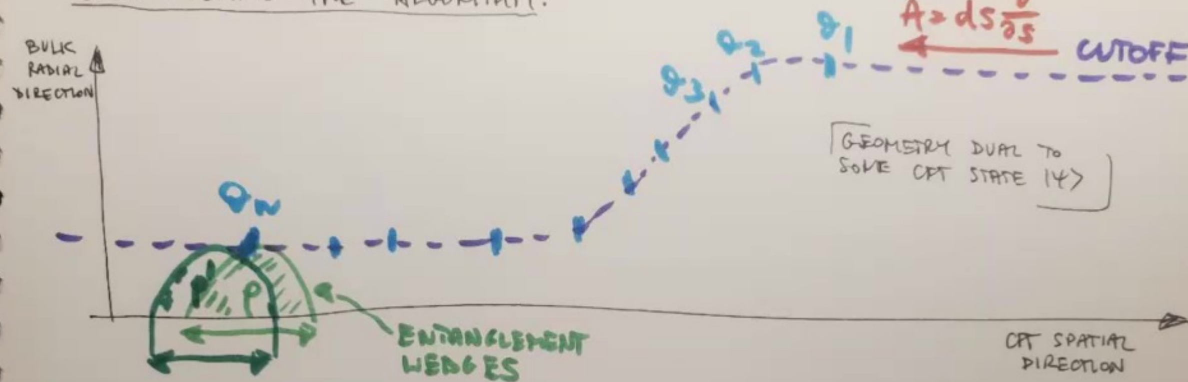
$$\langle 0|\, \mathcal{O}_2(0)\, \mathcal{O}_1(x_2-x_1)|0\rangle$$

- AN OPERATOR-VALUED ONE FORM, WHICH TRANSLATES ALONG CUTOFF SURFACE
- WE CAN WRITE IT AS $\boxed{ds \frac{\partial}{\partial s}}$

- ITS WILSON LINES TRANSLATE **INPUT OPERATORS** ALONG CUTOFF SURFACE


CONSTRUCTING THE ALGORITHM: EXAMPLE
$|0\rangle_{CFT} \leftrightarrow ds^2 = \frac{dx^2+dz^2}{z^2}$
z
x
$L_{-1}\,dx$
z = const.
CUTOFF
$\mathcal{O}_2$
$\mathcal{O}_1$
x_2
x_1
$\langle 0|\, e^{\int_{-\infty}^{x_2} dx\, L_{-1}}\, \mathcal{O}_2\, e^{\int_{x_2}^{x_1} dx\, L_{-1}}\, \mathcal{O}_1\, e^{\int_{x_1}^{\infty} dx\, L_{-1}}\, |0\rangle =$
$\langle 0|\, \mathcal{O}_2\, e^{L_{-1}(x_1-x_2)}\, \mathcal{O}_1\, |0\rangle =$
$\langle 0|\, \mathcal{O}_2(0)\, \mathcal{O}_1(x_2-x_1)\, |0\rangle$
• AN OPERATOR-VALUED ONE FORM, WHICH TRANSLATES ALONG CUTOFF SURFACE
• WE CAN WRITE IT AS $\boxed{ds \frac{\partial}{\partial s}}$
• ITS WILSON LINES TRANSLATE INPUT OPERATORS ALONG CUTOFF SURFACE

CONSTRUCTING THE ALGORITHM:



$$\langle\psi|\left(P\exp\int_{-\infty}^{s_N} A\right)\mathcal{O}_n(1+A)\dots(1+A)\mathcal{O}_3(1+A)\mathcal{O}_2(1+A)\mathcal{O}_1\left(P\exp\int_{s_1}^{\infty} A\right)|\psi\rangle$$

- IF WE CAN FIND $A = ds\frac{\partial}{\partial s}$, WE WILL HAVE AN ALGORITHM
- IF A CAN BE CONSTRUCTED USING ONLY CFT INGREDIENTS, THIS WILL BE A CFT ALGORITHM, NOT A BULK COMPUTATION

COMMENT ON **OPERATOR INPUTS:**

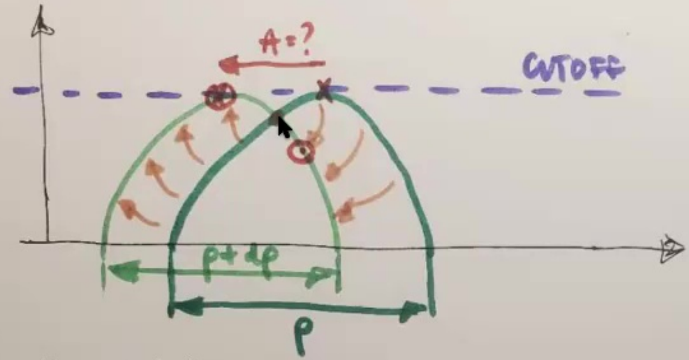
- THEY SHOULD LIVE ON THE CUTOFF SURFACE $\longleftrightarrow$
- THEY SHOULD COMMUTE WITH ρ AND ρ':

BULK $\xleftrightarrow{\text{SUBREGION DUALITY}}$ BOUNDARY

$[\mathcal{O}, \rho] = 0$

$[\mathcal{O}, \rho + d\rho] = 0$

CONSTRUCTING $A = ds\frac{\partial}{\partial s}$



- WE KNOW A CFT OPERATION THAT DOES WHAT THE **BROWN** ARROWS DO:

idV

MODULAR PARALLEL TRANSPORT
(1903.04493)

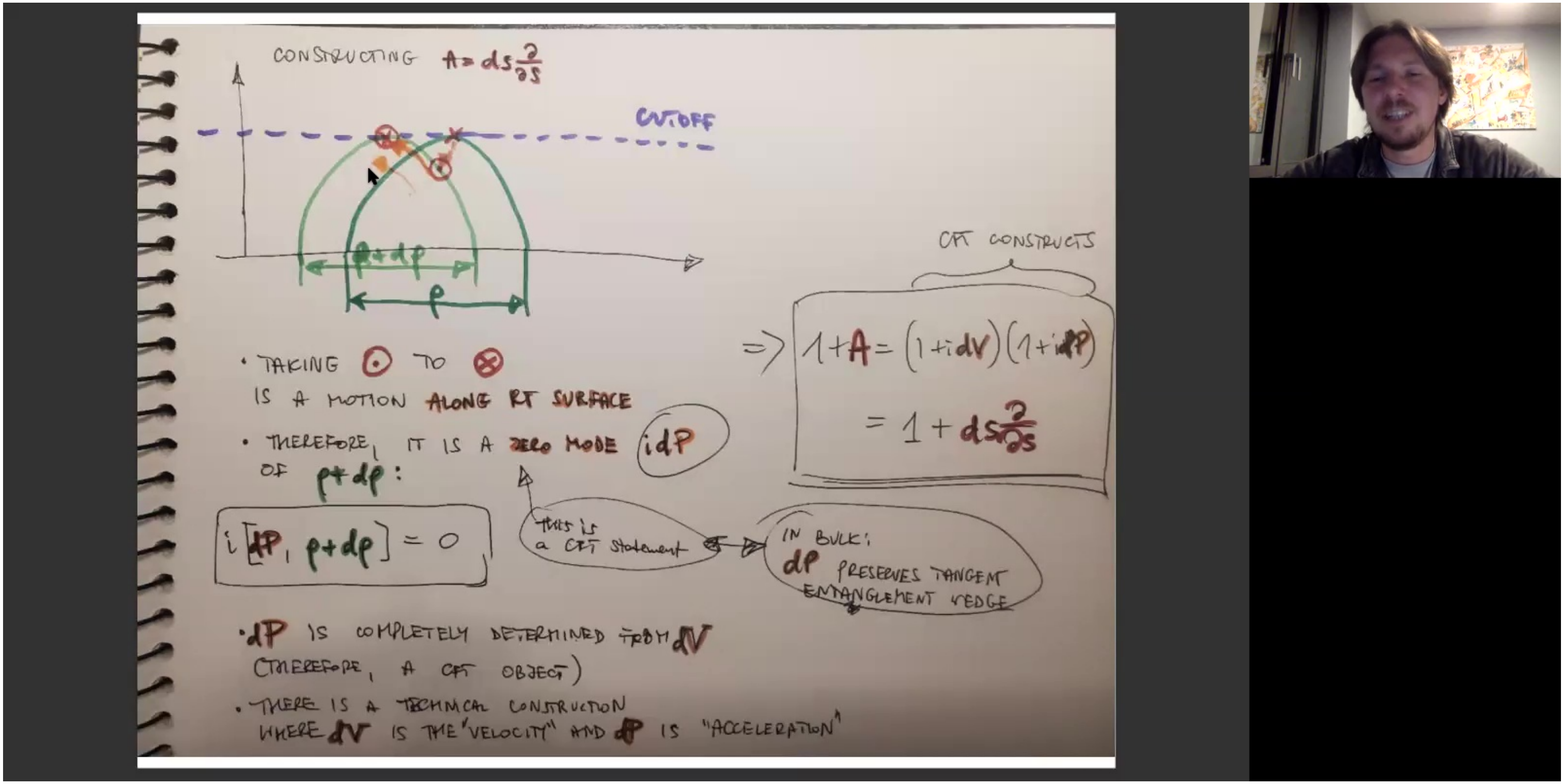
- IN THE BULK, THIS IS A ROTATION ABOUT THE COMMON POINT OF RT - SURFACES
- BUT THIS SENDS × to ⊙
- HOW TO SEND ⊙ to ⊗ ?

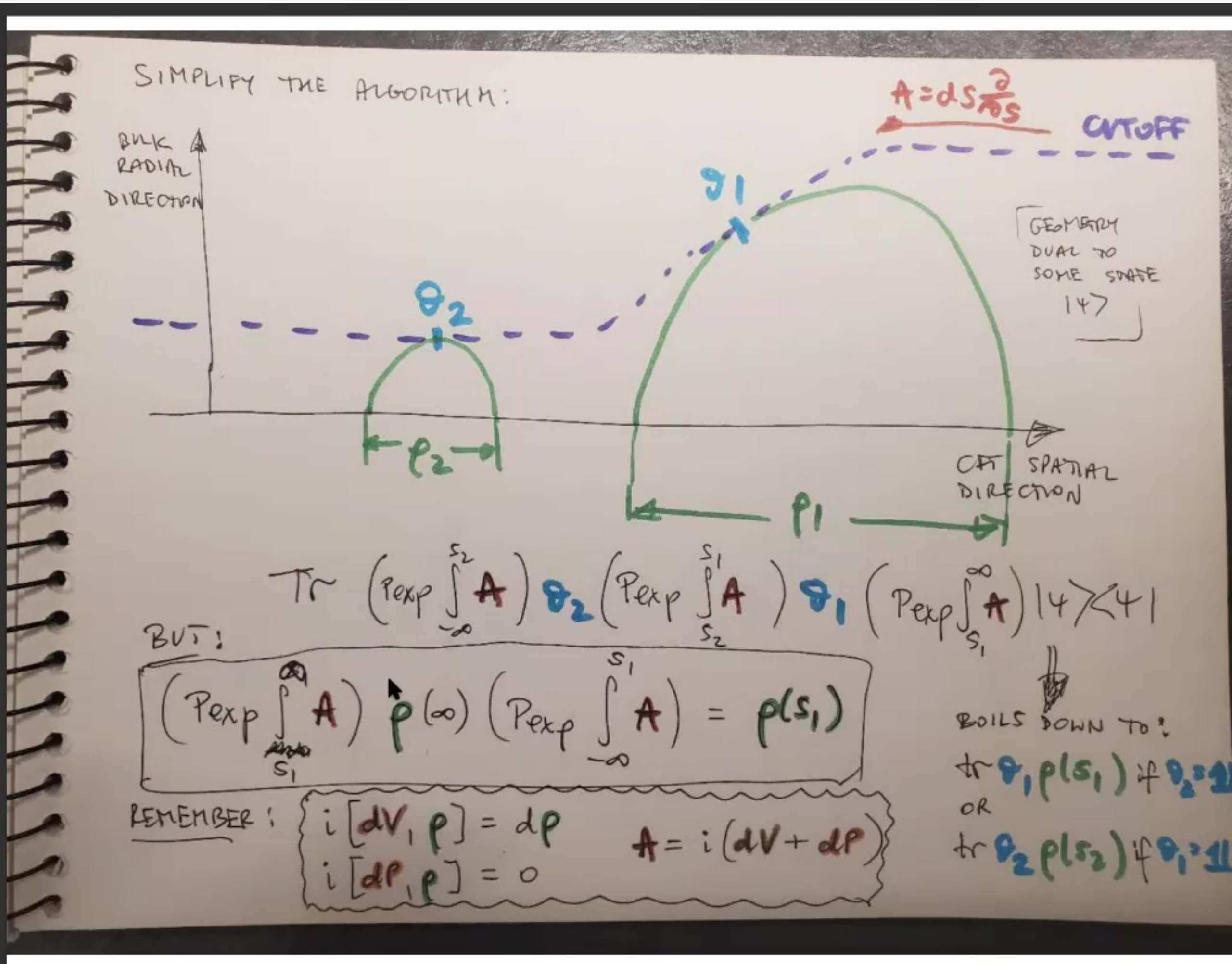
- THIS OPERATION ~~HAS~~ TRANSPORTS ENTANGLEMENT WEDGES TO ENTANGLEMENT WEDGES:

$$\boxed{i[dV, \rho] = d\rho}$$

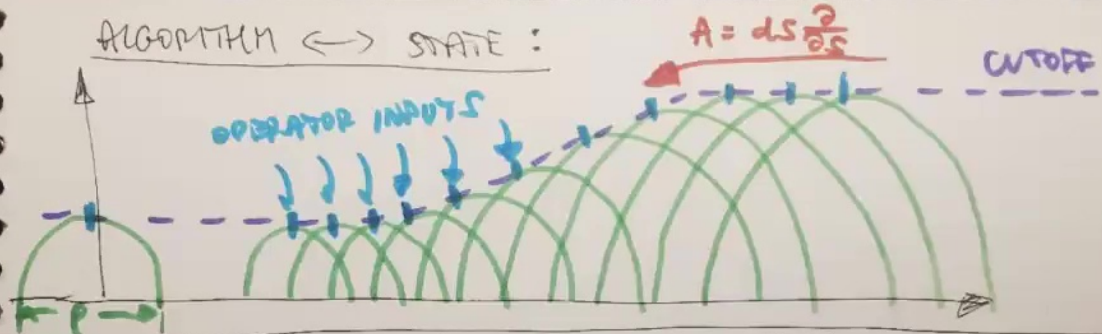
- FOR NOW, WE WRITE:

$$\boxed{1 + A = (1 + idV) \times (\ldots)}$$


CONSTRUCTING A = ds ∂/∂s
CUTOFF
p+dp
p
• TAKING ⊙ TO ⊗
IS A MOTION ALONG RT SURFACE
• THEREFORE, IT IS A ZERO MODE idP
OF p+dp:
i [dP, p+dp] = 0
this is
a CFT statement
IN BULK:
dP PRESERVES TANGENT
ENTANGLEMENT WEDGE
CFT CONSTRUCTS
=> 1+A = (1+idV)(1+idP)
= 1 + ds ∂/∂s
• dP IS COMPLETELY DETERMINED FROM dV
(THEREFORE, A CFT OBJECT)
• THERE IS A TECHNICAL CONSTRUCTION
WHERE dV IS THE "VELOCITY" AND dP IS "ACCELERATION"


SIMPLIFY THE ALGORITHM:
BULK RADIAL DIRECTION
$A = ds\frac{\partial}{\partial s}$
CUTOFF
θ_1
θ_2
GEOMETRY DUAL TO SOME STATE $|\psi\rangle$
ρ_2
ρ_1
CFT SPATIAL DIRECTION
$\mathrm{Tr}\left(\mathrm{Pexp}\int_{-\infty}^{s_2} A\right)\theta_2\left(\mathrm{Pexp}\int_{s_2}^{s_1} A\right)\theta_1\left(\mathrm{Pexp}\int_{s_1}^{\infty} A\right)|\psi\rangle\langle\psi|$
BUT:
$\boxed{\left(\mathrm{Pexp}\int_{s_1}^{\infty} A\right)\rho(\infty)\left(\mathrm{Pexp}\int_{-\infty}^{s_1} A\right) = \rho(s_1)}$
BOILS DOWN TO:
$\mathrm{tr}\,\theta_1\rho(s_1)$ if $\theta_2 = \mathbb{1}$
OR
$\mathrm{tr}\,\theta_2\rho(s_2)$ if $\theta_1 = \mathbb{1}$
REMEMBER:
$i[dV, \rho] = d\rho$
$i[dP, \rho] = 0$
$A = i(dV + dP)$

ALGORITHM $\longleftrightarrow$ STATE :



$$\mathrm{Tr}\ \rho(-\infty)\,(1+A)(\dots)(1+A)(\dots)(1+A)(\dots)(1+A)(\dots)\ \dots\ (\dots)(1+A)(\dots)$$

OPERATOR INPUTS

- MAP: OPERATORS $\longrightarrow$ EXPECTATION VALUES
- IN THE FORM OF AN ALGORITHM
- CONSTRUCTED ENTIRELY IN CFT LANGUAGE
- BUT IT "KNOWS" ABOUT THE CUTOFF
- DEPENDS CRUCIALLY ON THE ~~OLD~~ MICROSCOPIC STATE THROUGH A

$$i\,[dV, \rho] = d\rho$$
$$i\,[dP, \rho] = 0$$

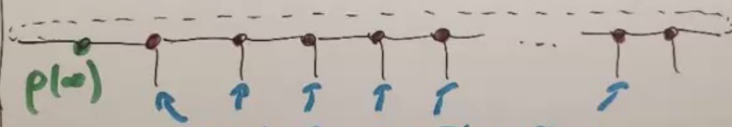
QUERY COMPLEXITY OF THIS ALGORITHM:

$$\mathrm{tr}\, \rho(\infty)\,(1+A)(\dots)(1+A)(\dots)(1+A)(\dots)\ \dots\ (\dots)(1+A)(\dots)$$

COUNTING SUBROUTINES...

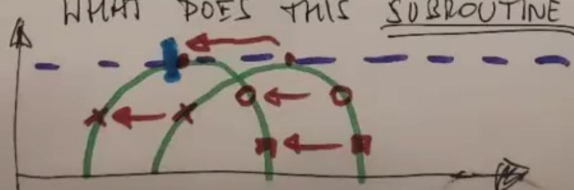
HERE THE SUBROUTINE IS: $\boxed{(1+A)(\dots)}$

BTW: YOU CAN REPRESENT THIS AS A MATRIX PRODUCT STATE:



THEN COUNTING $\boxed{(1+A)(\dots)}$ IS AGAIN COUNTING TENSORS IN A TENSOR NETWORK

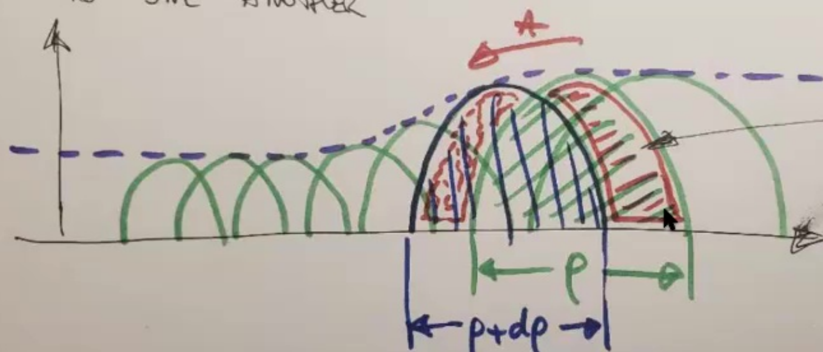
WHAT DOES THIS SUBROUTINE DO?



- IT MAPS CUTOFF-(IZED) ONE ENTANGLEMENT WEDGE TO THE NEXT
- THEN READS OFF AN INPUT OPERATOR

QUERY COMPLEXITY OF OUR ALGORITHM:

• SHOULD BE A MEASURE OF "HOW DIFFICULT" IT IS TO MAP CONSECUTIVE **CUTOFF-SIZED** ENTANGLEMENT WEDGES TO ONE ANOTHER



"**DIFFICULTY**" PROPORTIONAL TO ANGULAR WIDTH OF THESE **CRESCENTS**

• THERE IS AN ESSENTIALLY UNIQUE METRIC IN THE SPACE OF DENSITY MATRICES, WHICH IS CONSISTENT WITH $\boxed{i[A,\rho] = d\rho}$

"THE KIRILLOV-KOSTANT-SOURIAU METRIC FOR THE SJØKVIST CONNECTION A"

AGAIN: DEFINED PURELY IN CFT LANGUAGE

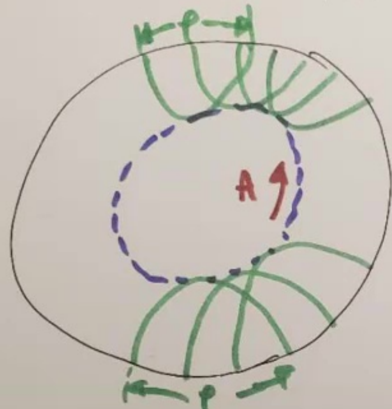
• IF THERE IS A BULK:

⇒ PROPORTIONAL TO: $\boxed{K\, ds_{BULK}}$

EXTRINSIC CURVATURE OF CUTOFF SURFACE

OUR QUERY COMPLEXITY IN THE BULK*:

* IF THERE IS A BULK.
IT IS DEFINED EVEN IN THE ABSCENCE OF A BULK



$$\boxed{\text{QUERY COMPLEXITY} = \oint_{\text{CUTOFF}} K\,ds}$$

• IN PURE AdS_3: $QC = \oint K\,ds = \boxed{(-R)\int_{\text{inside}} dV + 2\pi}$

"VOLUME PROPOSAL"

• OTHERWISE, IT'S DIFFERENT

QUALITATIVELY:

• $QC = \oint K\,ds$ GIVES SOME MEASURE OF SPATIAL SIZE
• IT IS A COMPLEXITY OF A STATE AT A CUTOFF
• DEFINED PURELY IN CFT LANGUAGE

I DECLARE SUCCESS.

THANKS!