Title: Fun with replicas and holographic tensor networks

Speakers: Michael Walter

Collection: Tensor Networks: from Simulations to Holography III

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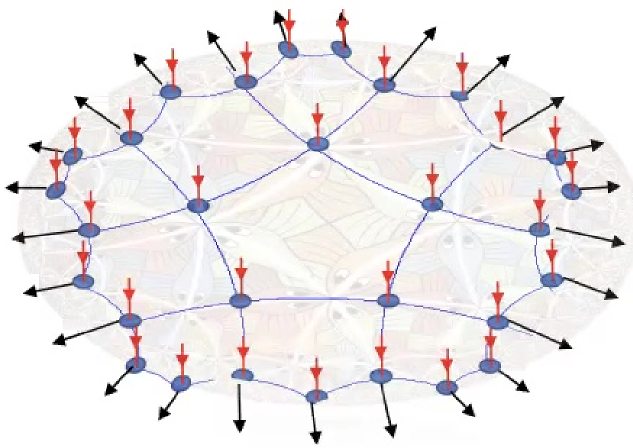
Fun with Replicas & Holographic Tensor Networks

Michael Walter



Tensor Networks Workshop, PI, 2020

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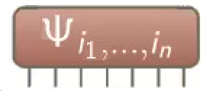
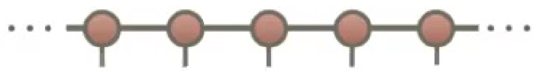
The tensor network toolbox

Tensor network: many-body state defined by contracting network of (local) tensors

$$|\Psi\rangle = \sum_{i_1,\ldots,i_n} \Psi_{i_1,\ldots,i_n} \, |i_1,\ldots,i_n\rangle$$

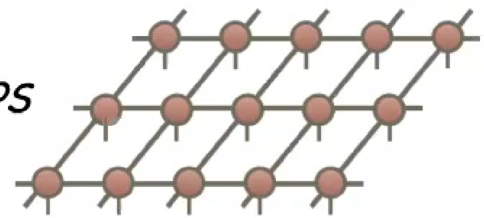


e.g. *MPS*

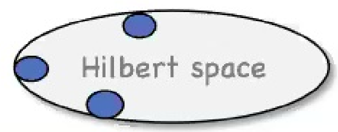


White, Fannes-Nachtergaele-Werner, Östlund-Rommer

PEPS



Verstraete-Cirac



Numerical tool on classical and quantum computers

Analytical tool that provides "dual" descriptions

Conceptual tool to build toy models of complex phenomena

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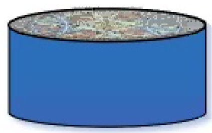
Motivation: Holography

Susskind, 't Hooft
Maldacena

In holography, gravity emerges from complex QM system

boundary: d-dim
QM system (a CFT)



time

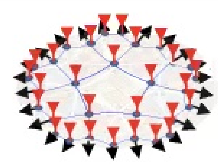
Controlled setup to study quantum gravity; including black holes, wormholes, ...

bulk: (d+1)-dim gravity (in AdS)

Understand this emergence using QIT

Build toy models that reproduce and explain

but also inspiration for other applications in QI, cond-mat, ...



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Outline

Random tensor networks as a versatile toy model of holography.

Key tool: "Replica trick". Explains many interesting features by mapping to simple classical stat mech models.

Three "advanced" features that connect to recent developments:

1. Non-perturbative corrections to the entropy
2. Quantifying entanglement
3. Holographic mapping and "islands"

based on recent joint work with Dong-Qi, Lancien, Penington-Witteveen, ...

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Entropy

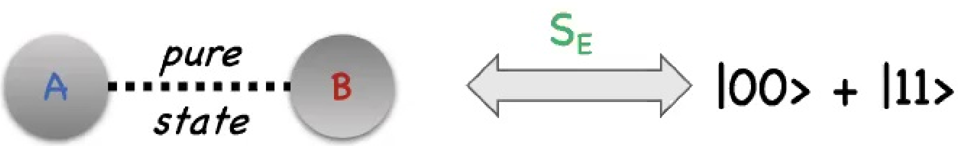
Entropy is a central quantity in quantum information theory:

$$S(A) = -\mathrm{tr}\,\rho_A \log \rho_A$$

quantum system A
density matrix ρ_A

Many interpretations and uses in optimal rates & capacities.

Entanglement entropy: $S_E = S(A) = S(B)$



Mutual information: $I(A{:}B) = S(A) + S(B) - S(AB)$

bounds
correlations

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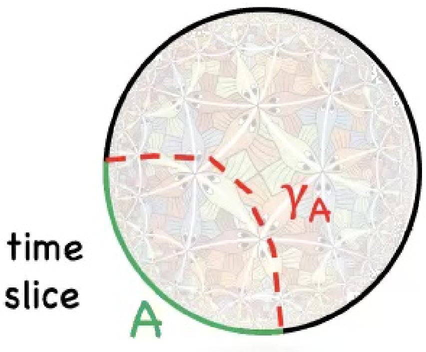
Motivation: Entropy in holography

In holography, there is a remarkably simple formula for entropies:
Boundary entropies are given by areas of bulk minimal surfaces.



$$S(A) = \frac{|\gamma_A|}{4G} + \ldots$$

Ryu-Takayanagi (RT) law

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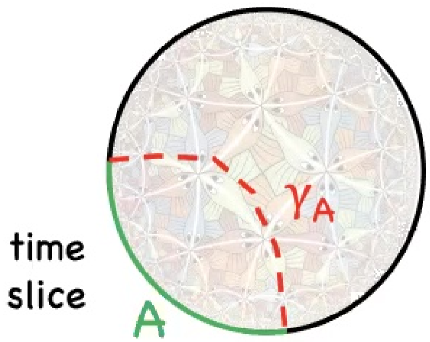
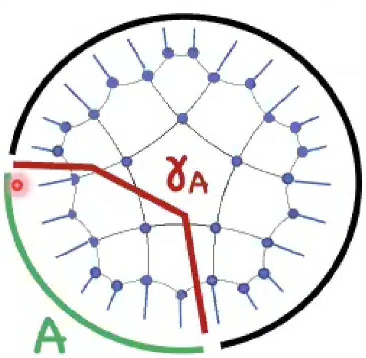


$$S(A) = \frac{|\gamma_A|}{4G} + \dots$$

Ryu-Takayanagi (RT) law

Mysterious? A very similar bound holds in tensor networks!

Swingle



$$S(A) \leq N\,|\gamma_A|$$

N qubits/bond
γ_A = minimal cut

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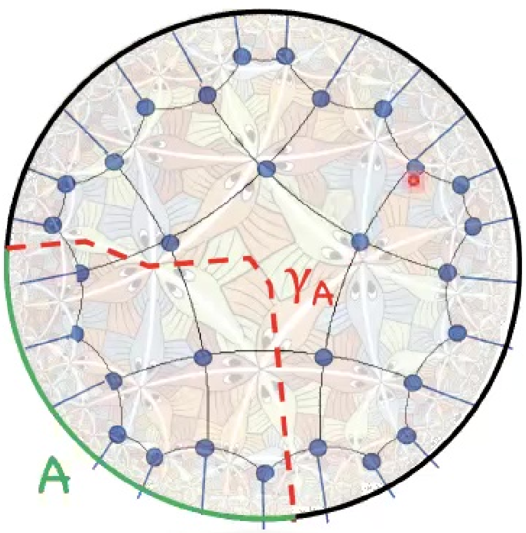
Holographic tensor networks

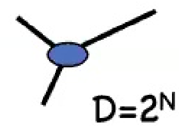
Harlow et al
Hayden-...-W

This suggest using TNS to define "exactly solvable" toy model.



Approach: Define boundary state via tensor network in bulk

simple bulk tensors, e.g. random tensors



For large N random tensor networks, Ryu-Takayanagi law emerges:

$$S(A) \simeq N\,|\gamma_A|$$ ✓

Mostly works in any geometry. By now, many variations known → **talk by Jens Eisert!**

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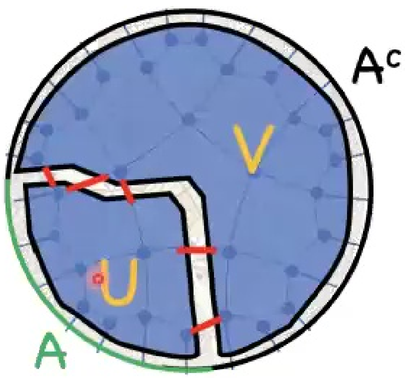
Two interpretations

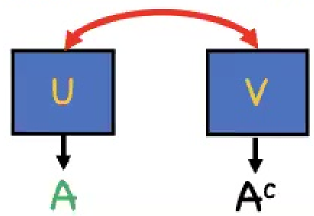
Harlow et al
Hayden-...-W

1. Random tensors ≈ isometries in any direction ("perfect tensors")



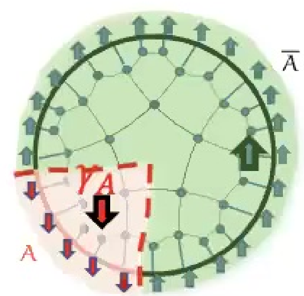
N $|\gamma_A|$ many EPR pairs

$|\Psi\rangle$ =



2. Disorder average → ferromagnetic spin model

large N → low T



We focus on latter interpretation, which is derived using replica trick.

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The replica trick

Any polynomial function of state can be computed as expectation value given sufficiently many "replicas":

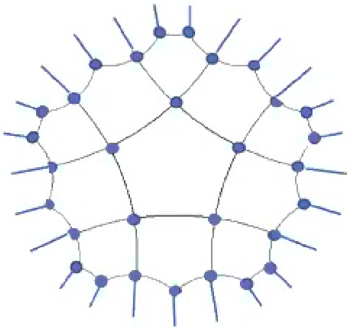
$\mathrm{tr}[\bigcirc \rho^{\otimes n}]$

Example: Rényi entropy

$$S_n(\rho) = \frac{1}{1-n} \log \mathrm{tr}[\rho^n] = \frac{1}{1-n} \log \mathrm{tr}[X \rho^{\otimes n}]$$

X = cyclic permutation

How to apply to random tensor networks?

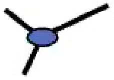


$$|\Psi\rangle = \left(\bigotimes_{\langle x,y \rangle} \langle xy| \right) \left(\bigotimes_{x} |V_x\rangle \right)$$

max. entangled states

$$|xy\rangle = \sum_{\mu=1}^{D} |\mu, \mu\rangle$$

random tensors



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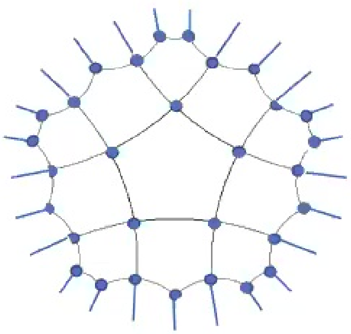
Replica trick in random tensor networks

Any polynomial function of state can be computed as expectation value given sufficiently many "replicas":

$\mathrm{tr}[O\rho^{\otimes n}]$

For random tensor networks:



$$|\Psi\rangle = \Big(\bigotimes_{\langle x,y\rangle} \langle xy|\Big)\Big(\bigotimes_{x} |V_x\rangle\Big)$$

max. entangled states   random tensors

Only need to know how to average random tensor!

$$\overline{|V_x\rangle\langle V_x|^{\otimes n}} \propto \sum_{\pi_x \in S_n} \pi_x$$

Result: "Partition function" of classical S_n spin model!

boundary conditions determined by quantity of interest

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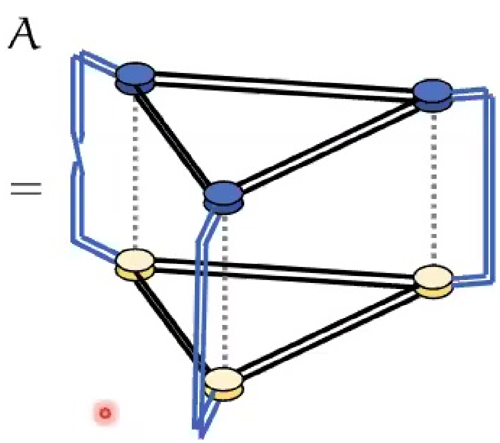
Replica trick for 2nd Rényi

$S_2(A) = -\log \mathrm{tr}[\rho_A{}^2]$

$$|\Psi\rangle = \left(\bigotimes_{\langle x,y\rangle} \langle xy|\right)\left(\bigotimes_{x} |V_x\rangle\right)$$

Replica trick: $\mathrm{tr}\,\rho_A^2 = \mathrm{tr}(\rho \otimes \rho) F_A$

$\mathrm{tr}\,\rho_A^2 =$



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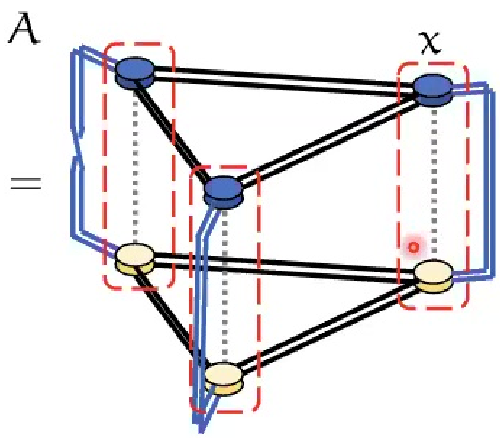
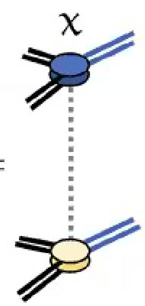


$|V_x\rangle\langle V_x|^{\otimes 2} =$



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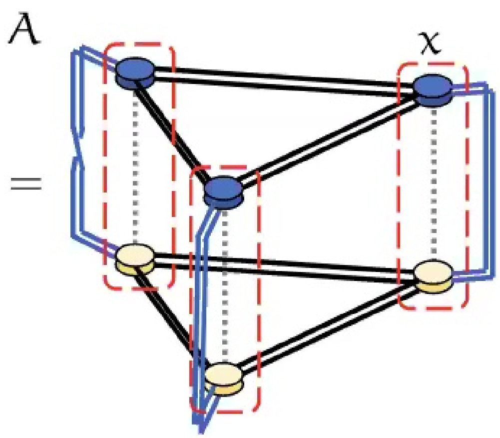
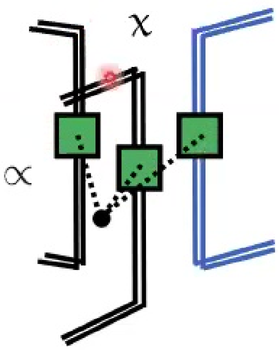


$\overline{|V_x\rangle\langle V_x|^{\otimes 2}} \propto$



$= I + F$

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Replica trick for 2nd Rényi

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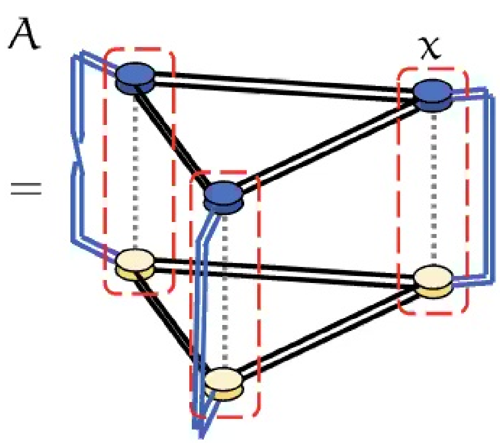
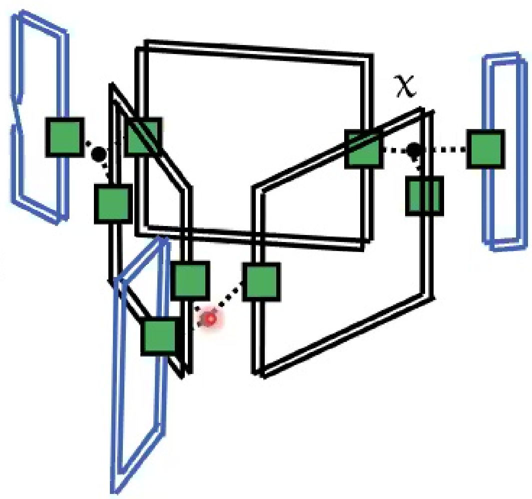


$\overline{\mathrm{tr}\,\rho_A^2} \propto$



Pick I vs F at each vertex.
Each loop is trace: factor $D=2^N$

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Replica trick for 2nd Rényi

$S_2(A) = -\log \mathrm{tr}[\rho_A{}^2]$

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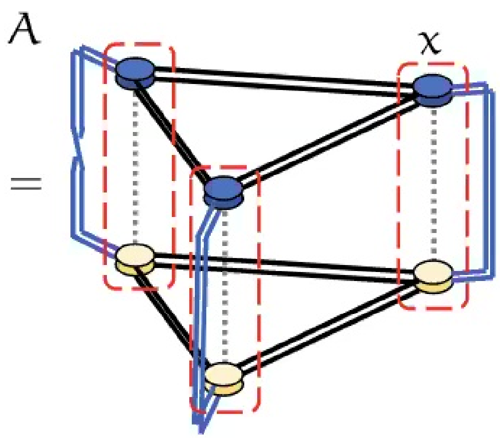
Replica trick: $\mathrm{tr}\,\rho_A^2 = \mathrm{tr}(\rho\otimes\rho)F_A$

$\mathrm{tr}\,\rho_A^2 =$



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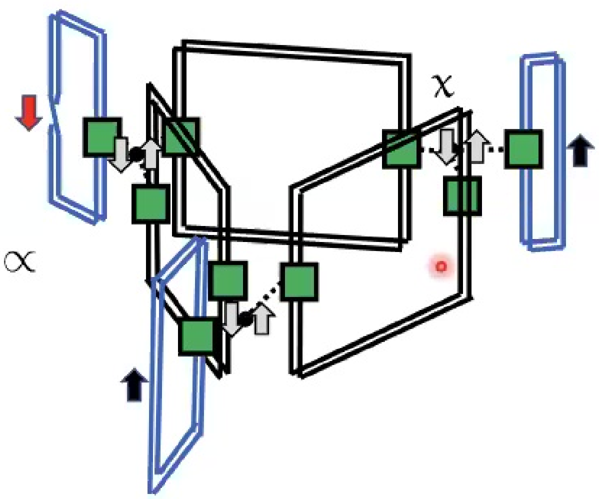
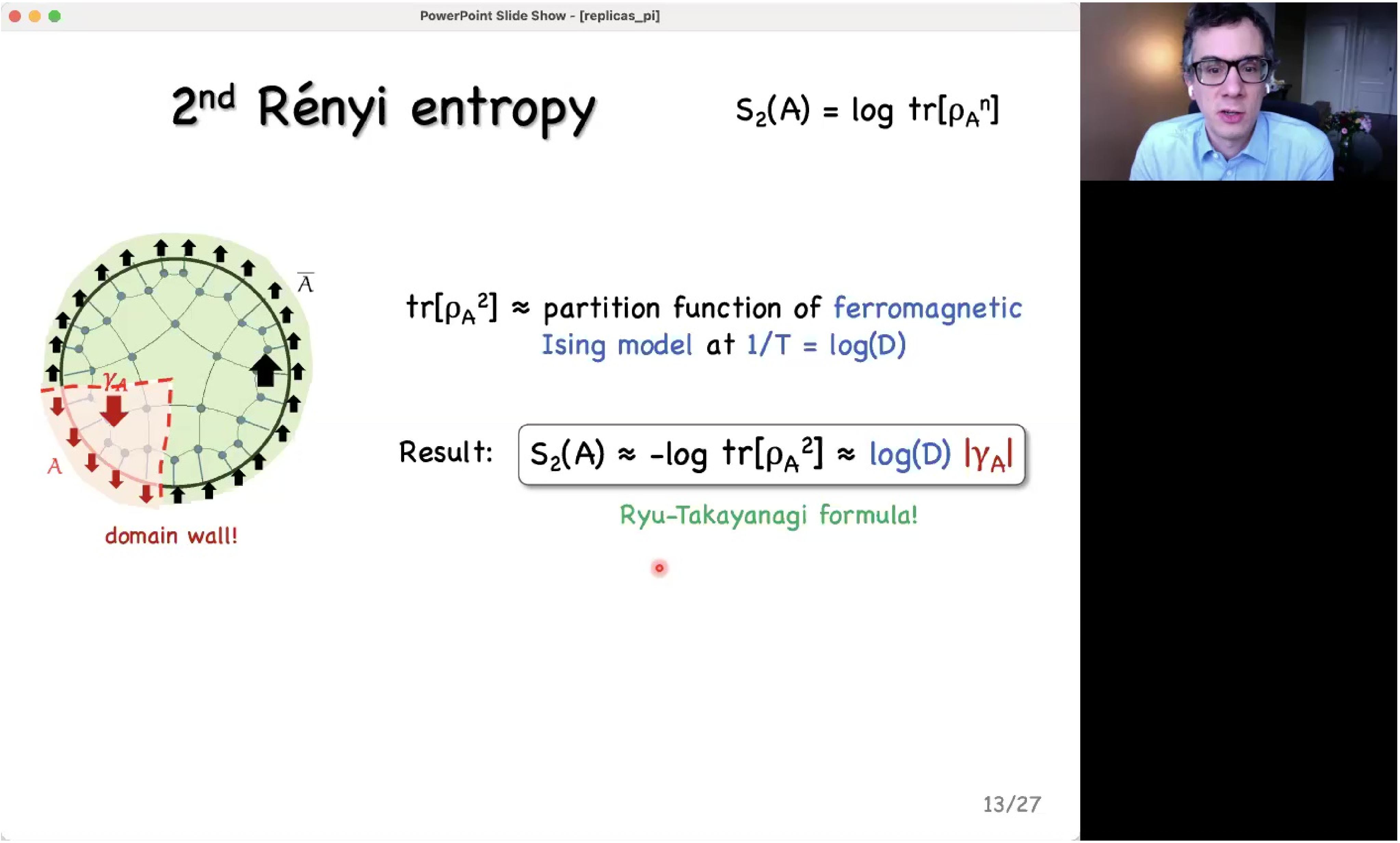


Ising variables & boundary conditions!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2nd Rényi entropy
$S_2(A) = \log \mathrm{tr}[\rho_A{}^n]$
$\mathrm{tr}[\rho_A{}^2] \approx$ partition function of ferromagnetic Ising model at $1/T = \log(D)$
Result: $S_2(A) \approx -\log \mathrm{tr}[\rho_A{}^2] \approx \log(D)\,|\gamma_A|$
Ryu-Takayanagi formula!
$\bar{A}$
γ_A
A
domain wall!
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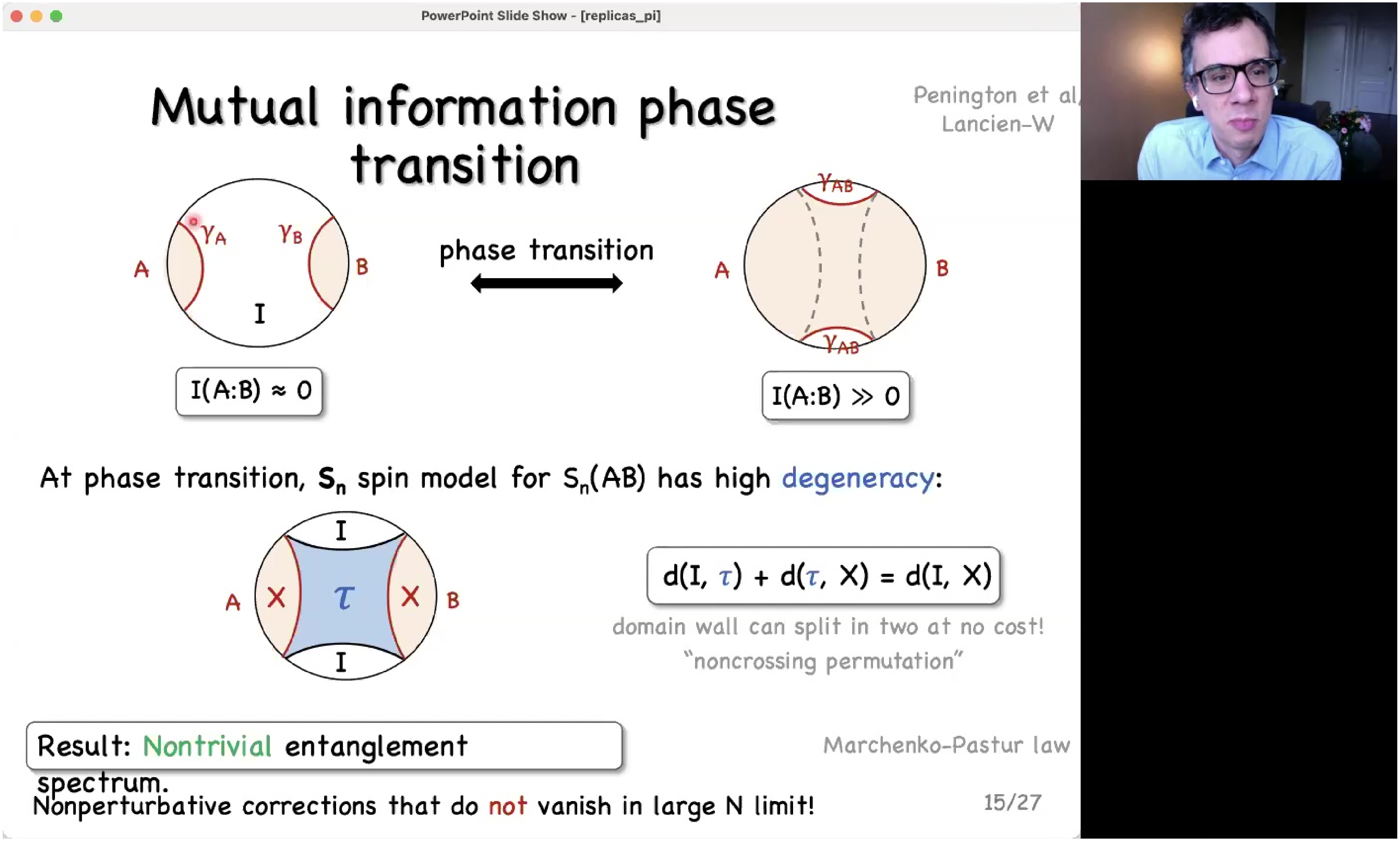
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Mutual information phase transition
Penington et al,
Lancien-W
A
γA
γB
B
I
I(A:B) ≈ 0
phase transition
γAB
A
B
γAB
I(A:B) ≫ 0
At phase transition, Sn spin model for Sn(AB) has high degeneracy:
I
A
X
τ
X
B
I
d(I, τ) + d(τ, X) = d(I, X)
domain wall can split in two at no cost!
"noncrossing permutation"
Result: Nontrivial entanglement spectrum.
Marchenko-Pastur law
Nonperturbative corrections that do not vanish in large N limit!
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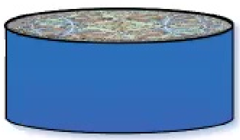
What does it mean?

Random tensor networks (RTN) provide intuitive toy model. Reproduce Ryu-Takayanagi formula (+ much more). Analyzed using replica trick.

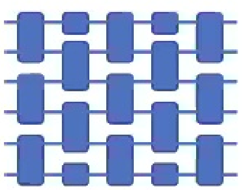


Relevance for holography? Ryu-Takayanagi formula is proved similarly. But: Einstein equations ➔ nontrivial spectrum!

Is all hope lost? No! Remarkably, RTN match precisely so-called fixed-area states in holography. ✓

Dong-Harlow-Marolf
Penington et al

Moreover, general states can be expanded in terms of fixed-area states.
Under certain "diagonal approximations", can lift results!



Similarly, random quantum circuit models have recently been studied, exhibit interesting phenomenology.

relevant to "quantum supremacy" proposals etc

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Entanglement in holographic tensor networks

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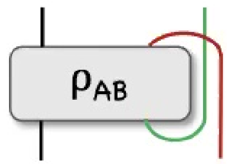
Partial transpose and negativity

Partial transpose: $\rho_{AB}^{\Gamma} =$ 

If ρ_{AB} is not entangled, then this operator has eigenvalues $\lambda_i \geq 0$
Thus negative eigenvalues $\lambda_i < 0$ diagnose entanglement. Peres

Logarithmic negativity: $E_N = \log \Sigma_i |\lambda_i|$ ≥ 0 Plenio

This is an entanglement measure. If $E_N > 0$ then ρ_{AB} is entangled.

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Negativity in random tensor networks

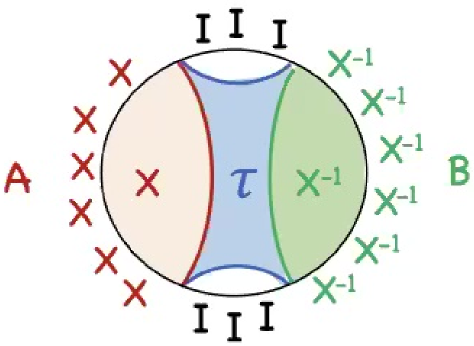
Dong-Qi-W

Approach: Compute even Rényi negativities

$$N_{2k} = \sum_i |\lambda_i|^{2k} = \mathrm{tr}\,(\rho_{AB}^{\Gamma})^{2k} = \mathrm{tr}\,\rho_{AB}^{\otimes 2k} X_A X_B^{-1}$$

using replica trick and analytically continue to k=1/2.

Resulting statistical model has $\mathbf{S}_{2k}$ permutation degrees. Ground state:



$$d(I, \tau) + d(\tau, X) = d(I, X)$$
$$d(I, \tau) + d(\tau, X^{-1}) = d(I, X^{-1})$$
$$d(X, \tau) + d(\tau, X^{-1}) = d(X, X^{-1})$$

allows τ domain to form at no cost

Interestingly, such τ exist and we can count them! e.g. $\tau = (1\ 2)(3\ 4)$ ...

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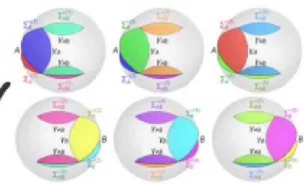
Holographic negativity

Dong-Qi-W

In large D limit: $E_N \approx \frac{1}{2} I(A:B) - \log(3\pi/8)$ entanglement is not maximal!

can also compute "negativity spectrum"

Relevance for holography? Similar replica trick possible to evaluate in fixed area states; agreement to leading order. ✓



Surprising (in both calculations): Solution is "replica symmetry breaking", i.e. does not respect $\mathbf{Z}_{2n}$ symmetry of boundary conditions.

How to further dissect entanglement of holographic states? Nezami-W, Cui-...-W

Other interesting "permutation observables"?

$\mathrm{tr}\, \rho_{ABC}^{\otimes n}\, \pi_A\, \pi_B\, \pi_C$

Penington-W-Witteveen

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Holographic mapping and "islands"

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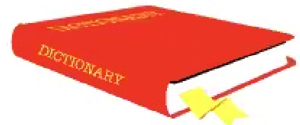
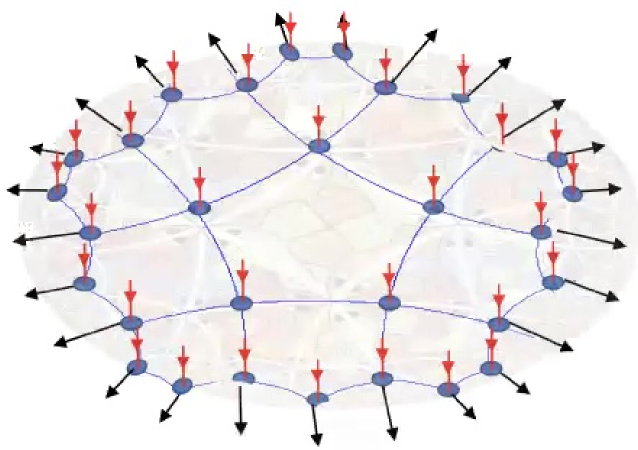
Holographic mappings



AdS/CFT is duality between two theories = "dictionary" that maps states & observables. How to incorporate into toy model?

Approach: Define bulk-boundary mapping via tensor network



red legs: bulk degrees
black legs: boundary degrees

"logical" bulk states are encoded in "physical" boundary Hilbert space

Toy model of how bulk quantum fields get encoded in CFT.

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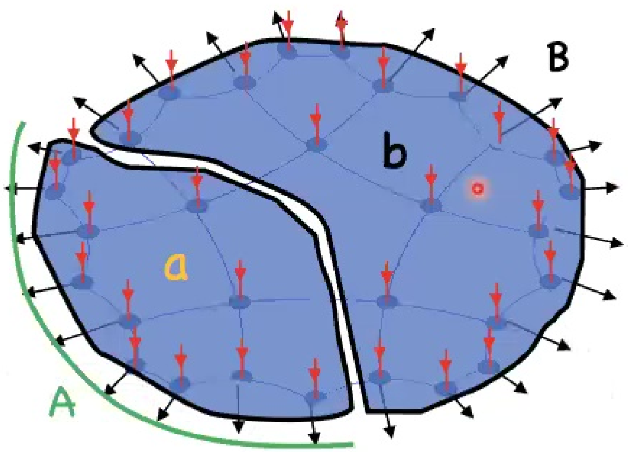
Holographic codes

If bulks legs have small dimension d << D, obtain error correcting code that satisfies "subregion duality", a key QI feature of AdS/CFT:



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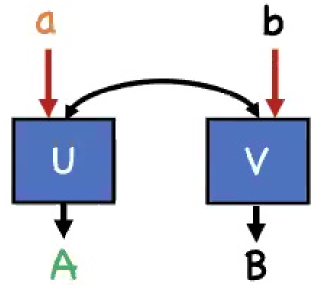

…for isometries U, V.

Bulk degrees of freedom in a (b) get encoded into A (B)! ✓

In particular, bulk corrections to entropy: $S(A) \approx N\,|\gamma_A| + S(a)$ ✓

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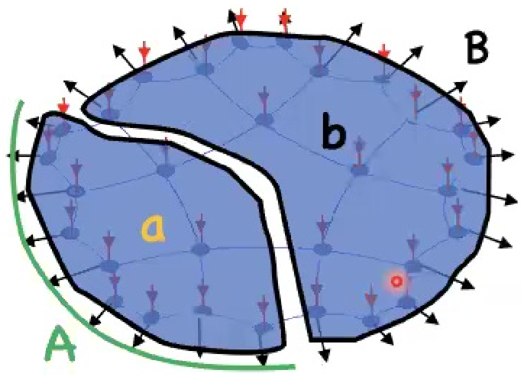
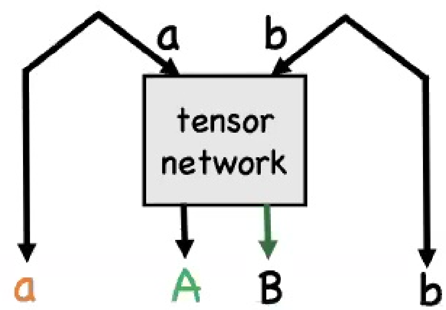
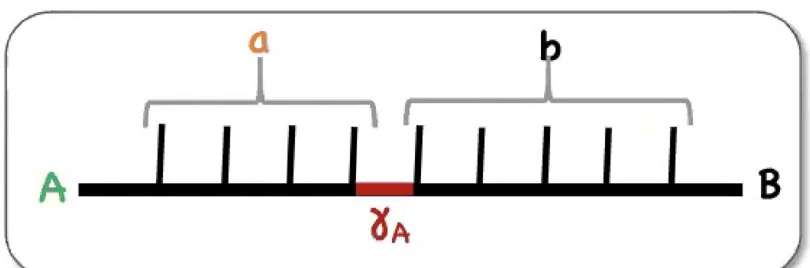
Subregion duality from decoupling

By decoupling, suffices to prove that $I(a:bB) \approx 0$ in Choi state:





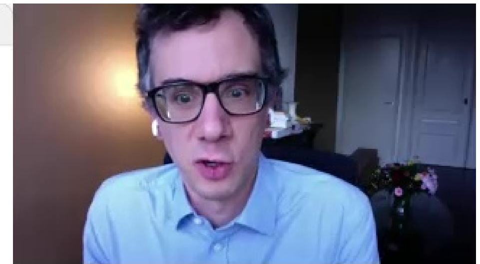
Schematically:



$$S(a) = \log(d)\,|a|$$
$$S(bB) = \log(D)\,|\gamma_A|$$
$$S(abB) = \log(D)\,|\gamma_A| + \log(d)\,|a|$$

Assume bulk legs have small dimension $d \ll D$.

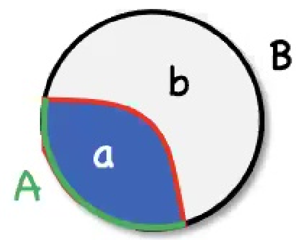
PowerPoint Slide Show - [replicas_pi]



Quantum minimal surfaces and islands 🏝️

What if bulk entropy is not small?

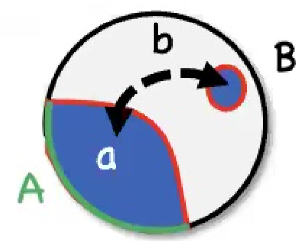
$$S_2(A) \simeq \min \{ N\,|\gamma_A| + S_2(a) \}$$ ✓



"Quantum minimal surface", minimizes "generalized entropy".

Proof using replica trick (additional action from bulk state)!

E.g., if we add highly entangled state between distant bulk sites, obtain "island" disconnected from boundary.



Holographic counterparts feature crucially in very recent developments on black hole information paradox that seek to give a bulk picture of black hole evaporation.

Penington
Almheira et al

Surprising that the simple RTN model reproduces these features!?

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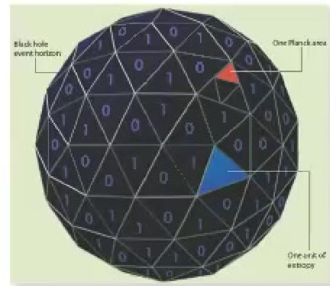
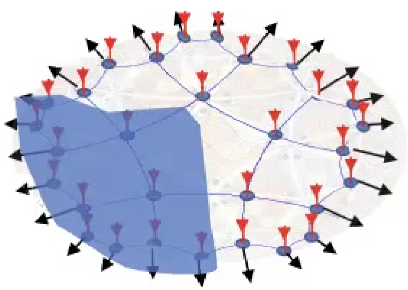
Summary



Holography offers remarkable connection between geometry and entanglement.



fixed area states

Tensor networks offer tools, models, mechanisms.
RTN versatile & easy to analyze via replica trick.

Ongoing research to exploit connections.

Motivation ranges from trying to understand the emergence of space-time from quantum mechanics to learning how dualities can help simulate complex quantum systems on (quantum) computers...

Thank you for your attention!

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