Title: Quantum Cellular Automata, Tensor Networks, and Area Laws

Speakers: Ignacio Cirac

Collection: Tensor Networks: from Simulations to Holography III

Date: November 17, 2020 - 8:00 AM

URL: http://pirsa.org/20110022

Abstract: Quantum Cellular Automata are unitary maps that preserve locality and respect causality. I will show that in one spatial dimension they correspond to matrix product unitary operators, and that one can classify them in the presence of symmetries, giving rise to phenomenon analogous to symmetry protection. I will then show that in higher dimensions, they correspond to other tensor networks that fulfill an extra condition and whose bond dimension does not grow with the system size. As a result, they satisfy an area law for the entanglement entropy they can create. I will also define other classes of non-unitary maps, the so-called quantum channels, that either respect causality or preserve locality and show that, whereas the latter obey an area law for the amount of quantum correlations they can create, as measured by the quantum mutual information, theformer may violate it. Additionally, neither of them can be expressed as tensor networks with a bond dimension that is independent of the system size.

QUANTUM CELLULAR AUTOMATA, TENSOR NETWORKS & AREA LAWS

WORKSHOP ON TENSOR NETWORKS AND HOLOGRAHY

Perimeter Institute
Waterloo, November 17, 2020

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Georgios Styliaris (MPQ)
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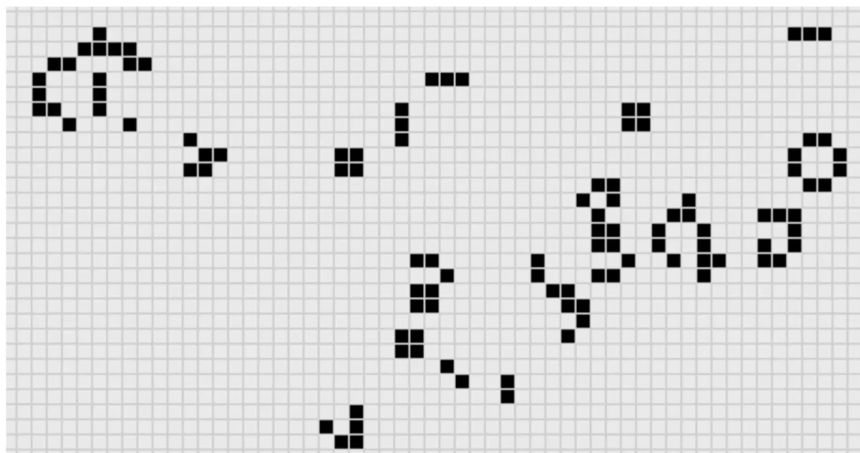
erc

CELLULAR AUTOMATA

Defined by simple local rules

Conway's game of life



http://pi.math.cornell.edu

QUANTUM CELLULAR AUTOMATA

- How to define them
  - Should include Classical Celullar Automata
  - Obey the rules of Quantum Physics
  - For unitary, there exists an accepted definition
  - For general actions (channels)?

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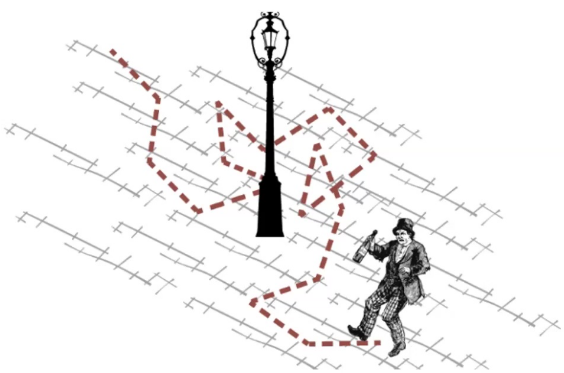
Piroli, JIC, PRL **125**, 190402 (2020)

QUANTUM CELLULAR AUTOMATA

Several definitions:

Feynman (1982)
Deutsch (1985)
Grössing and Zeilinger (1988)
Waltrous (1995)
Richter and Werner (1996)
Schumacher and Werner (2004)
Arrighi, Nesme and Werner (2008)

Random walks: single particle

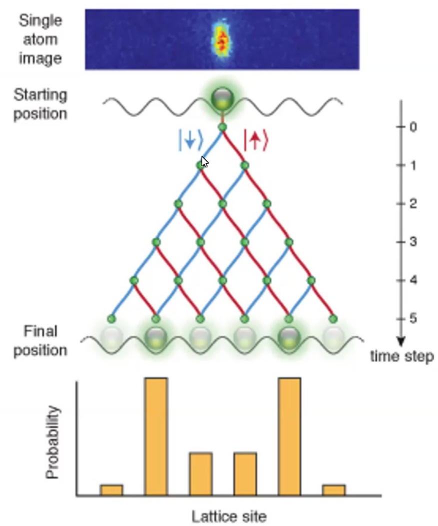


Medium.com

Quantum (random?) walks:

Aharonov, Davidovich, Zagury,PRA **48**, 1687 (1993)

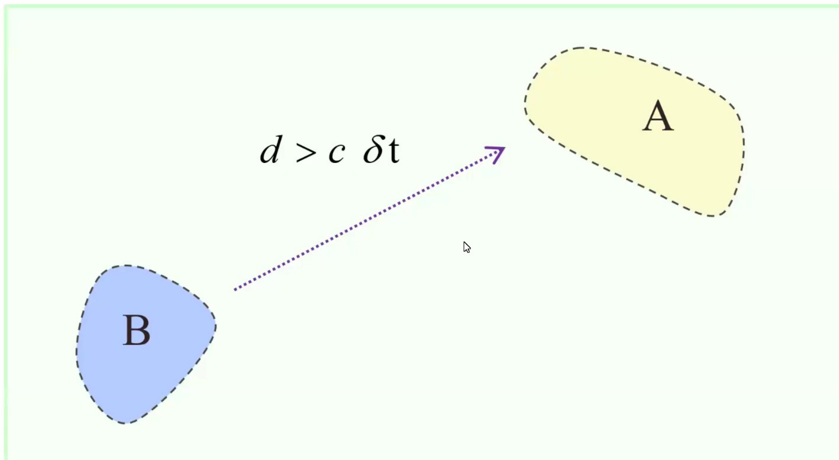
Include the „coin"



Dieter Meschede

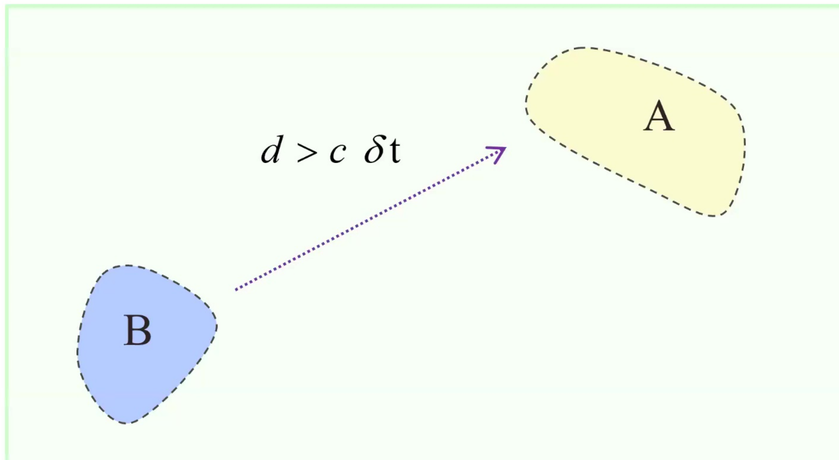
Experiments with photons, atoms, etc

CAUSALITY



For some time δt, the action in B cannot be sensed at A

CAUSALITY



For some time δt, the action in B cannot be sensed at A

Goal: Characterize the unitary operator U, describing the evolution of the whole system for a time δt, and that obeys causality

- Action in B, represented by u_B : $(1 \otimes u_B) | \Psi(0)\rangle$
- Evolution after a time δt: $| \Psi(\delta t)\rangle = U \left(1 \otimes u_B\right) | \Psi(0)\rangle$
- The outcome of any measurement in any region A, separated $d > d_0 = c\ \delta t$ is independent of u_B

CAUSALITY

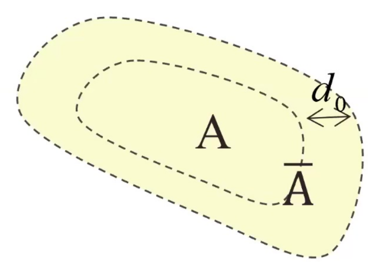
- Action in B, represented by u_B : $\left(1 \otimes u_B\right) | \Psi(0)\rangle$
- Evolution after a time δt: $|\Psi(\delta t)\rangle = U\left(1 \otimes u_B\right) | \Psi(0)\rangle$
- The outcome of any measurement in any region A, separated $d > d_0 = c\ \delta t$, is independent of u_B

$$\langle X_A \rangle = \langle \Psi(0) | u_B^\dagger U^\dagger X_A U u_B | \Psi(0)\rangle = \langle \Psi(0) | U^\dagger X_A U | \Psi(0)\rangle$$

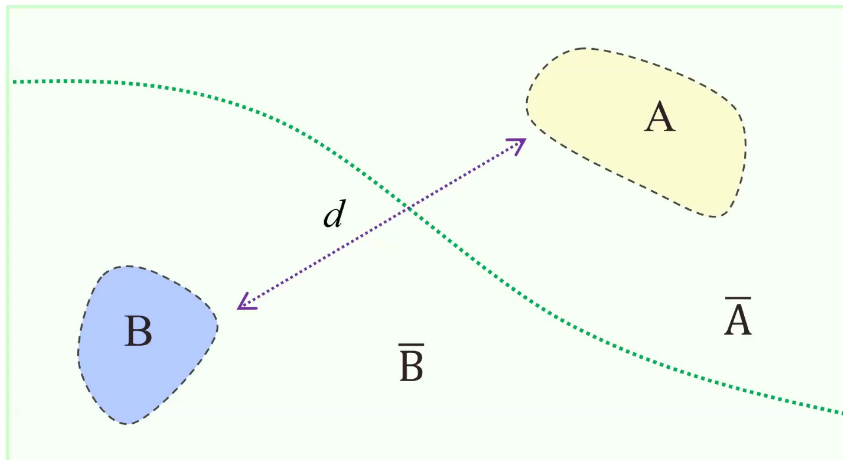
independent of u_B for all X_A

Characterization: $U^\dagger X_A U = \tilde{X}_{\bar{A}}$ for all X_A

- $\tilde{X}_{\bar{A}}$ is supported in $\bar{A}$, the neighborhood of A
  (i.e., acting trivially, like the identity, outside $\bar{A}$)
- is in the Heisenberg picture



LOCALITY



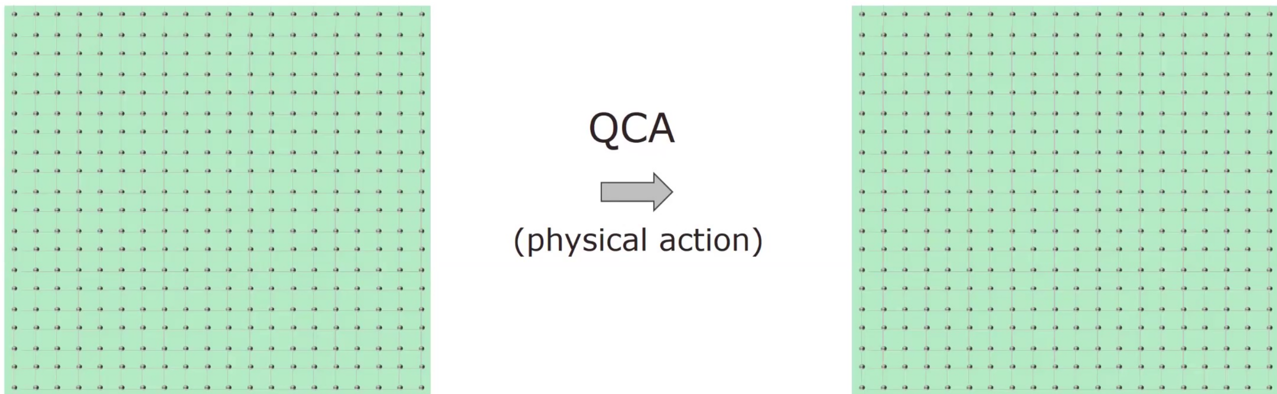
Local action: it „acts in the sorrounding"
it cannot correlate two separated regions

Goal: Characterize the unitary operator U, describing the evolution of the whole system that obeys locality

- If we start with a product state of regions $\bar{A}$ and $\bar{B}$: $|\Psi\rangle = |\Psi_{\bar{A}}\rangle \otimes |\Psi_{\bar{B}}\rangle$
- Evolution after a time step: $|\Psi'\rangle = U|\Psi\rangle$
- We do not create correlations: $\langle X_A \otimes Y_B\rangle = \langle \Psi'|(X_A \otimes Y_B)|\Psi'\rangle = \langle X_A\rangle\langle Y_B\rangle$

LATTICES & QUANTUM CHANNELS

- Discretize space and time

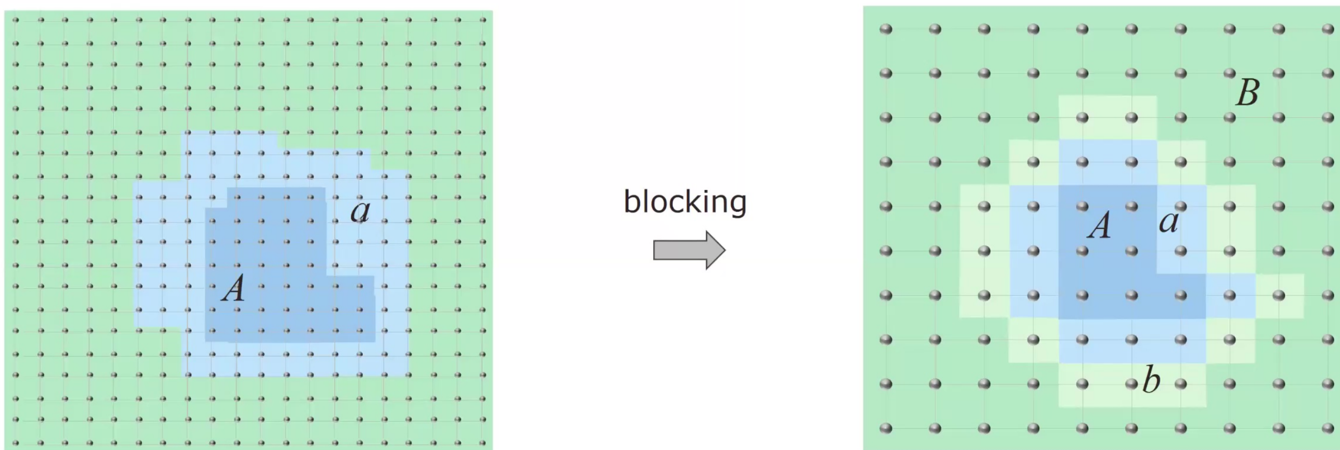


- General physical action: Quantum Channel

$$\rho_0 \xrightarrow[\text{(physical action)}]{\text{QCA}} \rho_1 = \mathrm{E}(\rho_0)$$

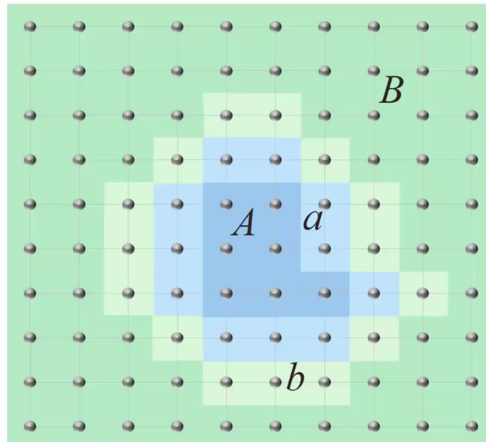
LATTICE

By blocking, we can choose the range=1



A: Region
a: Boundary of A
} $\bar{A}$

b: Boundary of a
B: rest
} $\bar{B}$



Support of operators:

- Operator supported in A: $X_A = X_A \otimes 1_a \otimes 1_b \otimes 1_B$
- Operator supported in $\bar{A}$: $X_{\bar{A}} = X_{\bar{A}} \otimes 1_{\bar{B}} = X_{Aa} \otimes 1_b \otimes 1_B$
- Operator supported in $\bar{B}$: $Y_{\bar{B}} = 1_{\bar{A}} \otimes Y_{\bar{B}} = 1_A \otimes 1_a \otimes Y_{bB}$

QUANTUM CHANNELS

ρ_0 → Quantum channel → $\rho_1 = \mathrm{E}(\rho_0)$

It is a completely positve map

It is trace preserving

- Kraus representation

$$\mathrm{E}(\rho) = \sum_k A_k \rho A_k^\dagger \qquad 1 = \sum_k A_k^\dagger A_k$$

- Adjoint channel:

$$\mathrm{E}^\dagger(X) = \sum_k A_k^\dagger X A_k \qquad \text{(Heisenberg picture)}$$

- Relation: $\mathrm{tr}\left[X\,\mathrm{E}(\rho)\right] = \mathrm{tr}\left[\mathrm{E}^\dagger(X)\rho\right]$

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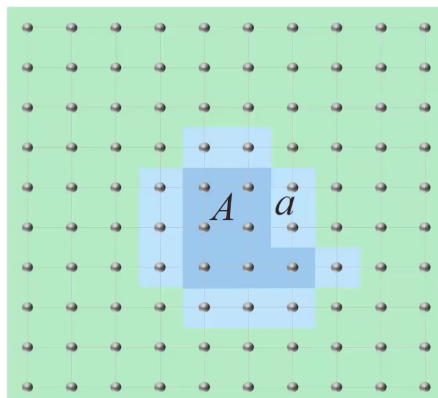
  - Relation: $\mathrm{tr}\left[X\,\mathrm{E}(\rho)\right] = \mathrm{tr}\left[\mathrm{E}^\dagger(X)\rho\right]$
  - Unitary: $\mathrm{E}(\rho) = U\rho U^\dagger$
    $\mathrm{E}^\dagger(X) = U^\dagger X U$

CAUSALITY PRESERVING QUANTUM CHANNELS

E is a CPQC if for any region A, and any X_A

$$\mathrm{E}^{\dagger}(X_A) = X_{\bar{A}}$$

The support is only extended in one unit
(in the Heisenberg picture)



A Quantum Cellular Automaton (QCA) is a unitary CPQC

$$U^{\dagger} X_A U = X_{\bar{A}}$$

This extends the definition of QCA to arbitrary actions

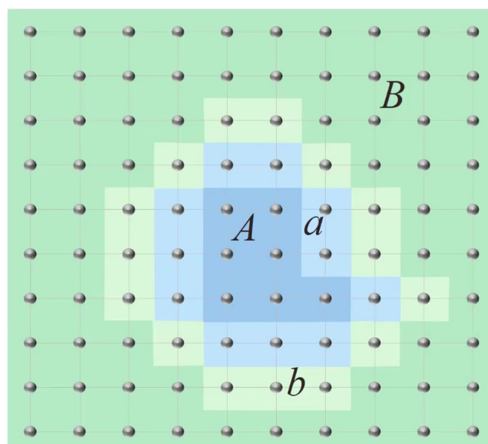
Schumacher and Werner (2004), Arrighi, Nesme and Werner (2008)

LOCALITY PRESERVING QUANTUM CHANNELS

E is a LPQC if for any region A, and any $\rho_{\bar{A},\bar{B}} \geq 0$

$$\mathrm{tr}_{a,b}\left[\mathrm{E}(\rho_{\bar{A}} \otimes \rho_{\bar{B}})\right] = \sigma_A \otimes \sigma_B$$

If there are no correlations between distant regions, A and B,
no correlations are created



CHARACTERIZATION
Choi-Jamiolkowsky states

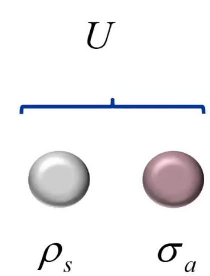
QUANTUM CHANNELS

Quantum channel

$$\rho_0 \Rightarrow \rho_1 = \mathrm{E}(\rho_0)$$

Stinespring dilation

$$\mathrm{E}(\rho_s) = \mathrm{tr}_a\left[U_{s+a}\left(\rho_s \otimes \sigma_a\right)U_{s+a}^{\dagger}\right]$$



A channel can be viewed as interaction with environment

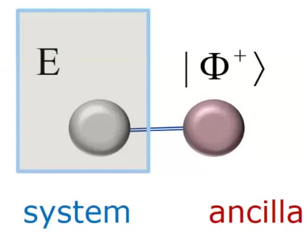
Quantum channel

$\rho_0 \Rightarrow \rho_1 = \mathrm{E}(\rho_0)$

Choi-Jamiolkowsky state

E channel $\Leftrightarrow$ R_{s+a} state

(isomorphism between channels and states)



- State: $R_{s+a} = \left(\mathrm{E} \otimes 1_a\right)(\Phi_{s+a})$ with $\Phi = |\Phi^+\rangle\langle\Phi^+|$

$$|\Phi^+\rangle = \sum_n |n\rangle_s \otimes |n\rangle_a$$

- Channel: $\mathrm{E}(\rho_s) = \mathrm{tr}_a\left[\rho_a^T R_{s+a}\right]$

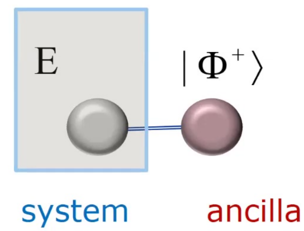
Choi-Jamiolkowski state

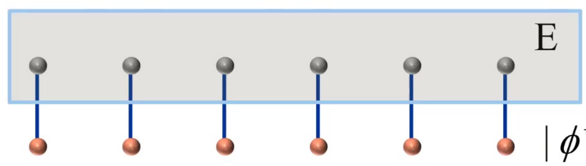
E ⟺ R_{s+a}

channel — state

(isomorphism between channels and states)

Multipartite systems:



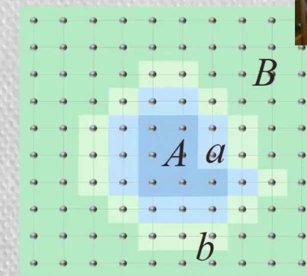


$$|\phi^{+}\rangle = |0,0\rangle + |1,1\rangle$$

Characterization:

R_{s+a} is a Choi state iff $\begin{cases} R_{s+a} \geq 0 \\ \mathrm{tr}_a\left(R_{s+a}\right) = 1_s \end{cases}$

CHARACTERIZATION CHOI STATES



1. Causality Preserving Quantum Channels

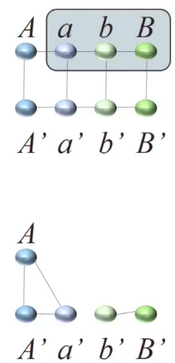
E is a CPQC if for any region A, and any X_A

$$E^{\dagger}(X_A) = X_{\bar{A}}$$

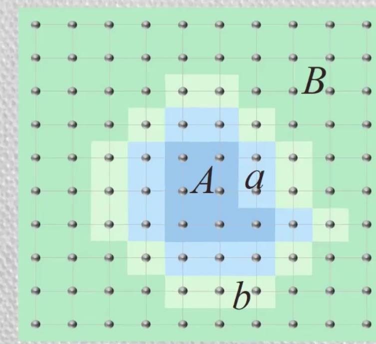
$$\Updownarrow$$

$$\mathrm{tr}_{a,\bar{B}}(R) = \sigma_{A,\bar{A}'} \otimes 1_{\bar{B}'}$$

Choi state

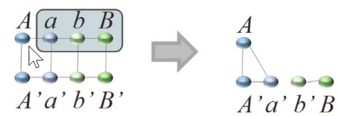


QUANTUM CHANNELS



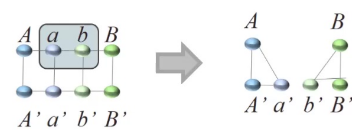
- E is a CPQC if for any region A, and any X_A

$$\mathrm{E}^{\dagger}(X_A) = X_{\bar{A}}$$



- E is a LPQC if for any region A, and any $\rho_{\bar{A},\bar{B}} \geq 0$

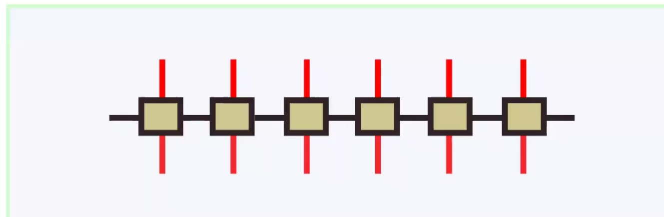
$$\mathrm{tr}_{a,b}\left[\mathrm{E}(\rho_{\bar{A}} \otimes \rho_{\bar{B}})\right] = \sigma_A \otimes \sigma_B$$



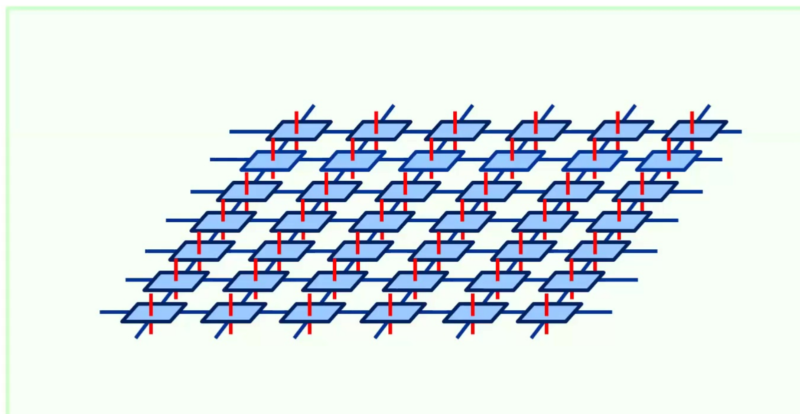
TENSOR NETWORKS
Quantum Channels

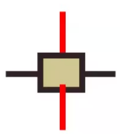
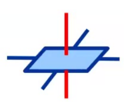
TENSOR NETWORKS FOR UNITARY OPERATORS

MPU



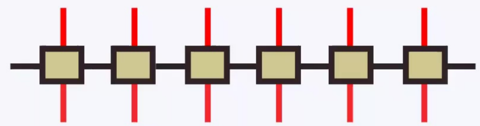
PEPU



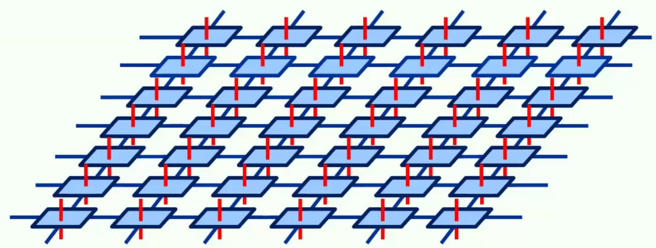
  a single tensor describes the whole operator

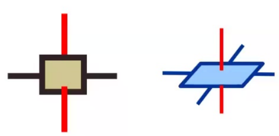
TENSOR NETWORKS FOR UNITARY OPERATORS

MPU



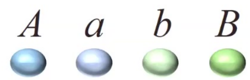
PEPU

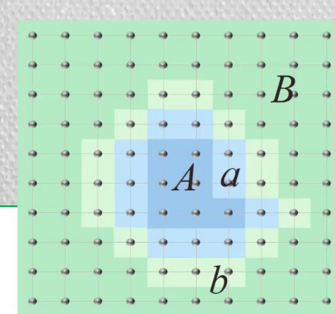


 a single tensor describes the whole operator

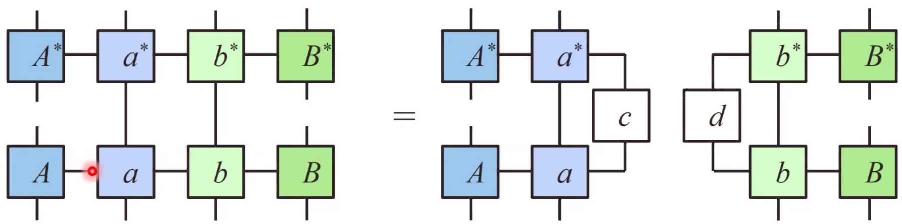
One can similarly define Quantum Channels

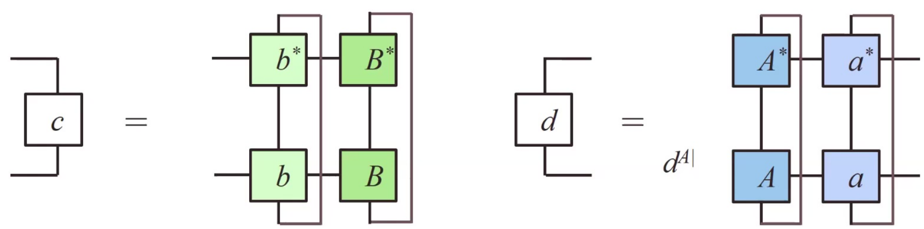
Simple:

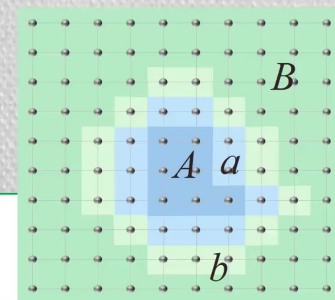




JIC, Perez-Garcia, Schuch, Verstraete
J. Stat. Mech. 083105 (2017)

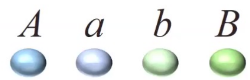


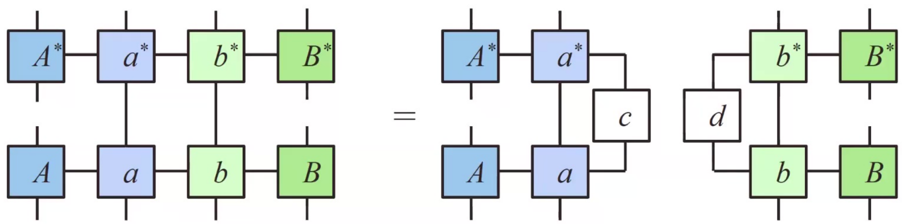


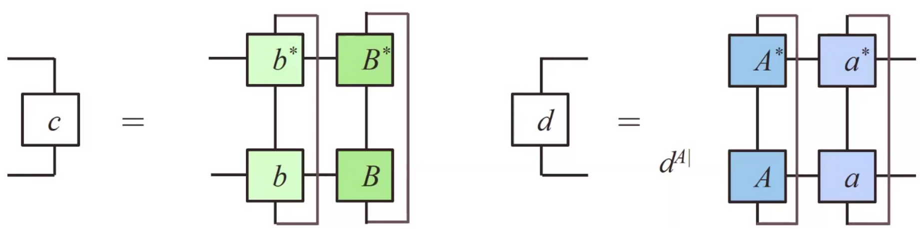


Simple:

JIC, Perez-Garcia, Schuch, Verstraete
J. Stat. Mech. 083105 (2017)





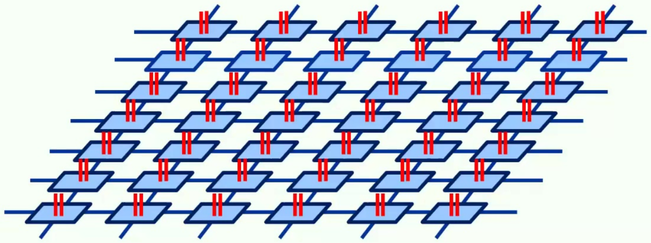


If it is simple, it is unitary (+details)

TENSOR NETWORKS FOR CHOI STATES

PEPS

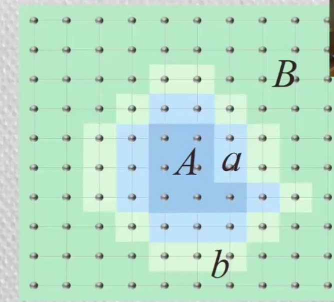
R



R_{s+a} is a Choi state iff $\begin{cases} R_{s+a} \geq 0 \\ \mathrm{tr}_a\left(R_{s+a}\right) = 1_s \end{cases}$

UNITARY CHANNELS
Quantum Cellular Automata

QUANTUM CELLULAR AUTOMATA



Result: Given a unitary channel, acting as

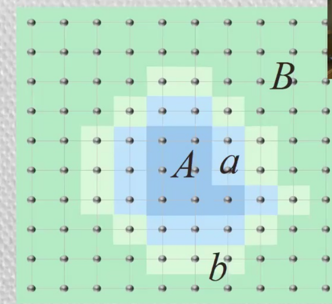
$$\mathrm{E}(\rho) = U \rho U^{\dagger}$$

the following statements are equivalent:

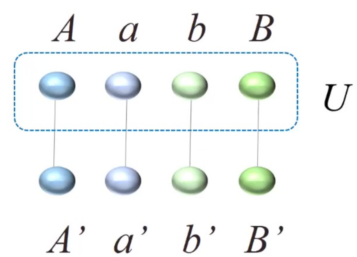
i) E is a CPQC (Quantum Cellular Automaton)

ii) E is a LPQC

iii) E can be represented by a "Simple" PEPU
where the bond dimension only depends on the local dimensions and the coordination number of the lattice



Idea of the proof:



- Choi state $|R_{s+a}\rangle = (U \otimes 1_a)|\Phi_{s+a}\rangle$
- Maximally entangled state is a product state

$$|\Phi_{s+a}\rangle = \bigotimes_n |\Phi_{s_n+a_n}\rangle$$

- Each state is annihilated by a Projector

$$P_n|\Phi_{s_n+a_n}\rangle = 0$$

- The Choi state is the unique ground state of the Hamiltonian

$$H = U\left(\sum_n P_n\right)U^\dagger = \sum_n U P_n U^\dagger = \sum_n h_n$$

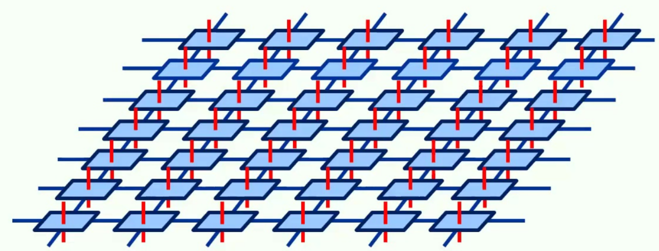
Local (U is LPQC)

Commuting $[h_n, h_m] = 0$

$$H|R_{s+a}\rangle = U^\dagger \sum_n P_n |\Phi_{s+a}\rangle^{\otimes N} = 0$$

Verstraete, Wolf, Perez-Garcia, JIC, PRL 96, 220601 (2006)

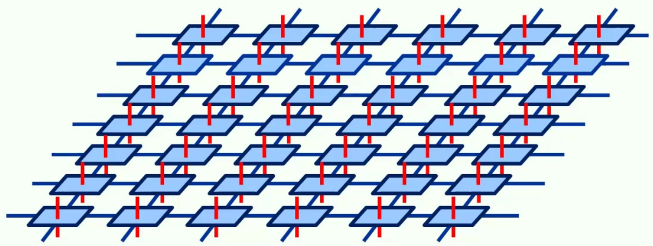
QUANTUM CELLULAR AUTOMATA



Remarks:

- Not all PEPUs are QCA $$U = \frac{1}{\sqrt{2}}\left(1^{\otimes N} + i\sigma_x^{\otimes N}\right)$$

QUANTUM CELLULAR AUTOMATA



Remarks:

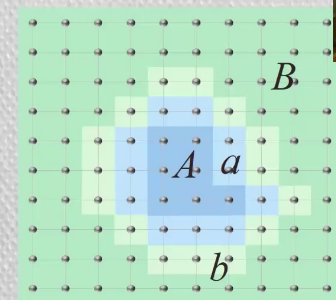
- Not all PEPUs are QCA $U = \frac{1}{\sqrt{2}}\left(1^{\otimes N} + i\sigma_x^{\otimes N}\right)$
- They must be „simple"

In 1D JIC, Perez-Garcia, Schuch, Verstraete, J. Stat. Mech. 083105 (2017)

This is true in any dimension

GENERAL CHANNELS

LOCALITY vs CAUSALITY PRESERVING



Result: For general channels, CPQC and LPQC are different

CPQC

$$\mathrm{tr}_{a,\bar{B}}(R) = \sigma_{A,\bar{A}'} \otimes 1_{\bar{B}'}$$

is convex

$$\mathrm{tr}_{a,\bar{B}}(p_1 R_1 + p_2 R_2) = \tilde{\sigma}_{A,\bar{A}'} \otimes 1_{\bar{B}'}$$

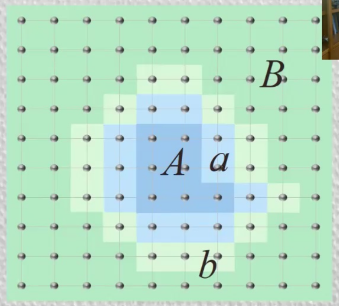
LPQC

$$\mathrm{tr}_{a,b}(R) = \sigma_{A,\bar{A}'} \otimes \sigma_{B,\bar{B}'}$$

is not convex

$$\mathrm{tr}_{a,\bar{B}}(p_1 R_1 + p_2 R_2) \neq \tilde{\sigma}_{\bar{A},\bar{A}'} \otimes \tilde{\sigma}_{\bar{B}'}$$

LOCALITY vs CAUSALITY PRESERVING



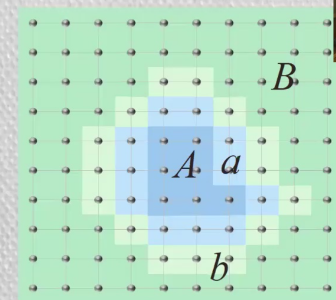
Result: For general channels, CPQC and LPQC are different

CPQC	LPQC
$\mathrm{tr}_{a,\bar{B}}(R) = \sigma_{A,\bar{A}'} \otimes 1_{\bar{B}'}$	$\mathrm{tr}_{a,b}(R) = \sigma_{A,\bar{A}'} \otimes \sigma_{B,\bar{B}'}$
is convex	is not convex
$\mathrm{tr}_{a,\bar{B}}(p_1 R_1 + p_2 R_2) = \tilde{\sigma}_{A,\bar{A}'} \otimes 1_{\bar{B}'}$	$\mathrm{tr}_{a,\bar{B}}(p_1 R_1 + p_2 R_2) \neq \tilde{\sigma}_{\bar{A},\bar{A}'} \otimes \tilde{\sigma}_{\bar{B}'}$

Example: $\mathrm{E}(\rho) = \frac{1}{2}\left(\rho + \sigma_z^{\otimes N} \rho \sigma_z^{\otimes N}\right)$

Requires classical communication

OTHER CHANNELS



Subclasses of CPQC:

- Factorized Channels (fnQC)

  For unitary channels: $U^{\dagger} X_A Y_B U = U^{\dagger} X_A U \ U^{\dagger} Y_B U$

  $$\mathrm{E}^{\dagger}(X_A Y_B) = \mathrm{E}^{\dagger}(X_A)\mathrm{E}^{\dagger}(Y_B)$$

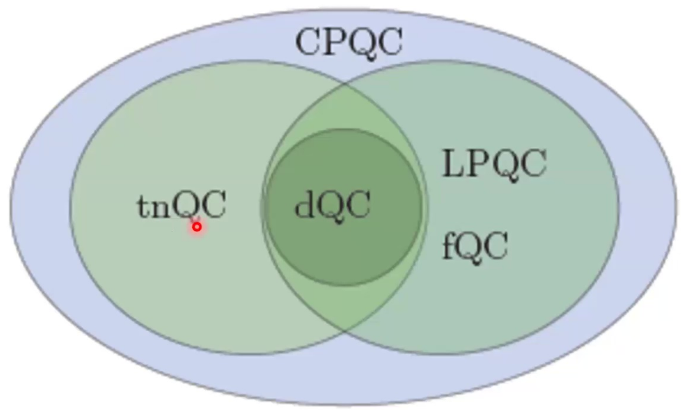
- Tensor Network Channels (tnQC)

  They can be written as tensor networks

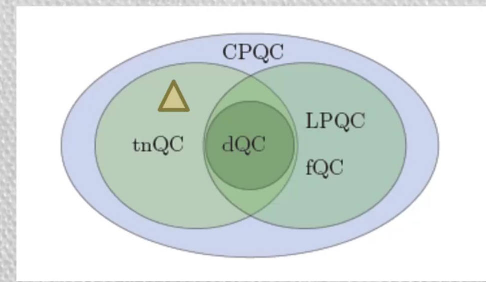
- Steinspring dilation Channels (dQC)

  They can be obtained through a Stinespring dilation of a QCA

RELATION BETWEEN CHANNELS



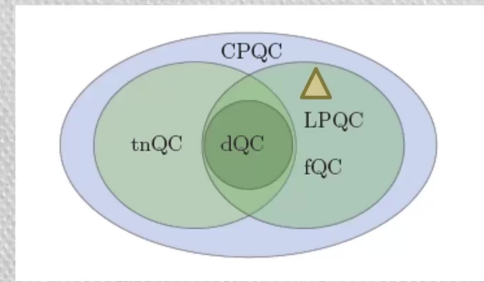
EXAMPLES



Example 1: $E(\rho) = \frac{1}{2}\left(\rho + \sigma_z^{\otimes N} \rho \sigma_z^{\otimes N}\right)$

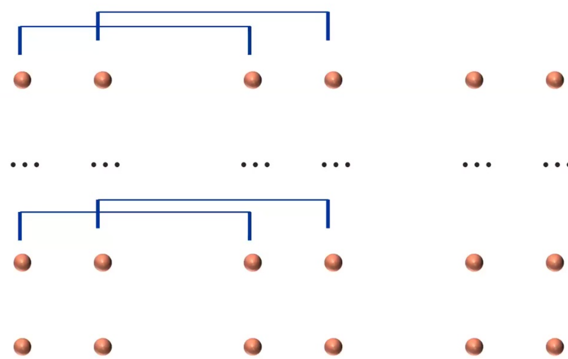
It is CPQC and TnQC but not LPQC nor dQC

EXAMPLES

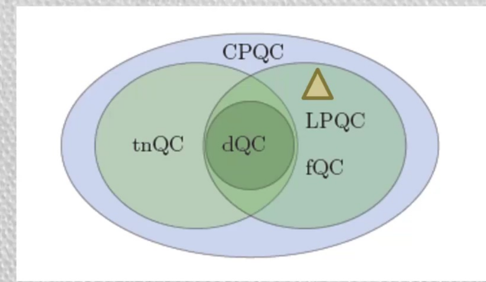


Example 2: Consider a state $|\Psi\rangle = \sum_{s_1,\dots,s_N=0,1} c_{s_1,\dots,s_N} |s_1,\dots,s_N\rangle$

in any spatial dimension



EXAMPLES



Example 2: Consider a state $|\Psi\rangle = \sum_{s_1,\ldots,s_N=0,1} c_{s_1,\ldots,s_N} |s_1,\ldots,s_N\rangle$

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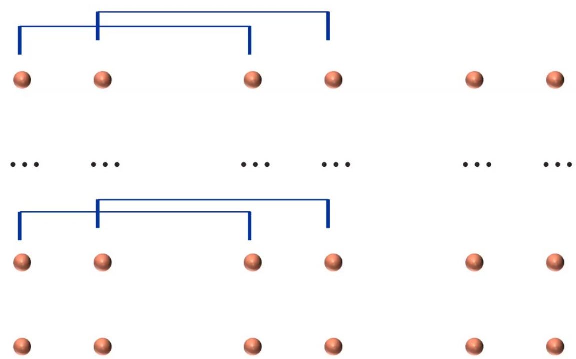


It is not a „tensor network"

Choi state: $R = 1 + k_N \sum_{s_1,\ldots,s_N=0,1} c_{s_1,\ldots,s_N} \left[\otimes_{n=1}^{N} \left(\sigma_n^x \otimes \sigma_{n'}^x\right)^{s_n} \left(\sigma_n^z \otimes \sigma_{n'}^z\right)^{1-s_n} \right]$

It is an LPQC but not tnQC nor dQC

AREA LAW

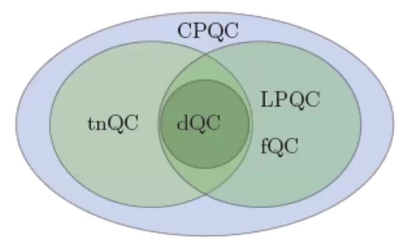
Definition: A sequence of QC obeys an area law if for all A, the state obtained by applying the channel to any product state fulfills

$$I(A:A^c) \leq c\,|\partial A|$$

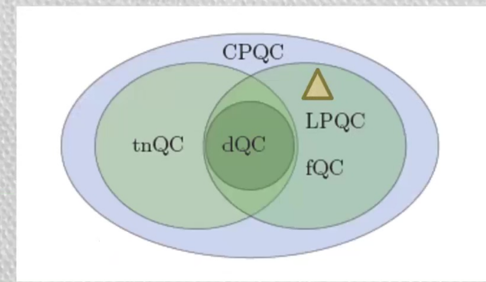
I is the quantum mutual information: $I(A:A^c) = S_A + S_{A^c} - S_{A,A^c}$

Results:

- LPQC and tnQC obey an area law
- CPQC do not obey an area law in general



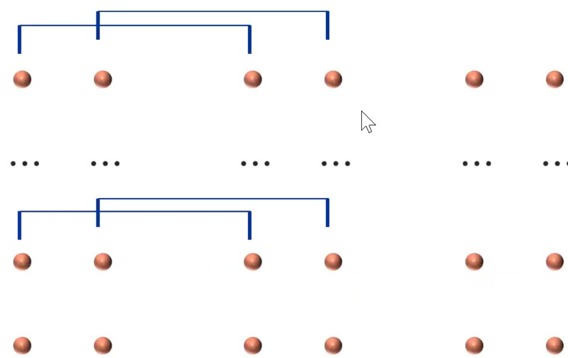
EXAMPLES



Example 3: Consider a state $\mathrm{E} = \otimes \mathrm{E}_{n,n+e}$

in any spatial dimension

$$\mathrm{E}_{n,m}(\rho) = \frac{1}{2}\left[\rho + \left(\sigma_n^z \otimes \sigma_m^z\right)\rho\left(\sigma_n^z \otimes \sigma_m^z\right)\right]$$



Classification of QCA on 1D:

- Index theorem:

  Gross, Nesme, Vogts, Werner, Comm. Math. Phys. **310**, 419 (2012)

- MPU:

  JIC, Perez-Garcia, Schuch, Verstraete, J. Stat. Mech. 083105 (2017)

- Symmetries:

  Gong, Sünderhauf, Schuch, JIC, PRL **124**, 100402 (2017)

- Fermions:

  Po, Fidkowski, Vishwanath, Pot-ter, PRB **96**, 245116 (2017)

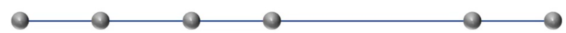
- fMPU:

  Piroli, Turzillo, Shukla, JIC, arXiv:2007.11905

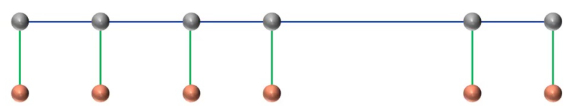
CONSTRUCTION

- Take an LME state (eg stabilizer)

  Kruszynska, Kraus, PRA **79**, 052304 (2009)

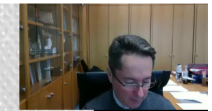


- Locally entangle it to ancillas locally, so that it is maximally entangled



Choi state

  - The state is a Choi state that inherits the properties of the original state
  - The corresponding unitary as well
  - Define unitaries with topological order
  - They can be deterministically applied given the Choi state
  - Extension of LOCC to geometries



CONCLUSIONS

- Extended QCA to channels
- Two different definitions: CPQC and LPQC
- Connected QCA and LPQC with TN
- Proven area laws
- Classification of QCA and other PEPU