Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 21

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Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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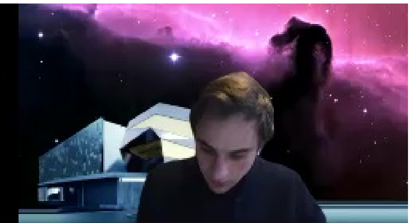


6.2.3 Wave Speed Estimates

Several papers:

Davis SIAM J. Sci. Stat. Comput. 9, 445 (1988)

Einfeldt SIAM J. Numer. Anal. 25, 294 (1988)

Einfeldt et al. J. Comp. Phys. 92, 273 (1991)

1) Davis: $s_L = u_L - c_L$, $s_R = u_R - c_R$

$$s_L = \min(u_L - c_L, u_R - c_R)$$

$$s_R = \max(u_L + c_L, u_R + c_R)$$

simple, but not recommended

2) Einfeldt (1991):

(HLLE) $s_L = \tilde{u} - \tilde{d}$, $s_R = \tilde{u} + \tilde{d}$

$$\tilde{u} \equiv \bar{u} = (\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R) / (\sqrt{\rho_L} + \sqrt{\rho_R})$$

simple, but not recommended

2) Einfeldt (1991):

(HLLE) $\quad s_L = \tilde{u} - \tilde{d}, \quad s_R = \tilde{u} + \tilde{d}$

$$\tilde{u} \equiv \bar{u} = \left(\sqrt{\rho_L}\, u_L + \sqrt{\rho_R}\, u_R\right) / \left(\sqrt{\rho_L} + \sqrt{\rho_R}\right)$$

$$\tilde{d}^2 \equiv \frac{\sqrt{\rho_L}\, c_L^2 + \sqrt{\rho_R}\, c_R^2}{\sqrt{\rho_L} + \sqrt{\rho_R}} + \eta_2 \left(u_R - u_L\right)^2$$

$$\eta_2 \equiv \frac{1}{2} \frac{\sqrt{\rho_L}\,\sqrt{\rho_R}}{\left(\sqrt{\rho_L} + \sqrt{\rho_R}\right)^2}$$

→ "positively conservative Riemann solvers"

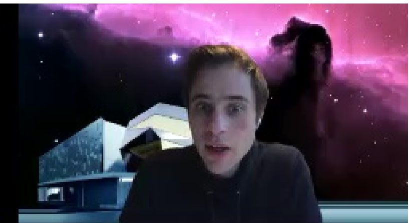
(ρ, e remain > 0 during evolution)

Chapter 8: Extensions to Balance laws and multi-D

8.1 Source terms

Consider a hyperbolic system of the type

$$(\mathrm{I}) \qquad u_t + f(u)_x = S(u)$$



Chapter 8: Extensions to Balance laws and multi-D

8.1 Source terms

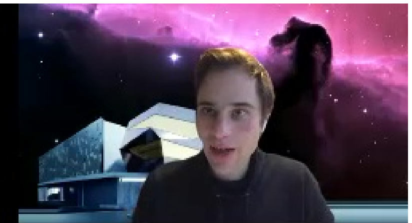
Consider a hyperbolic system of the type

(I) $\quad u_t + f(u)_x = S(u)$

$\uparrow$ geometric and/or physical source terms

Two approaches:

1) <u>unsplit methods</u>: direct discretization of (I) $\rightarrow$ numerical schemes can/



Two approaches:

1) unsplit methods: direct discretization of (I) → numerical schemes can/must be directly tailored to the specific equation

→ less general

2) split / "operator split" / "fractional-step" methods:

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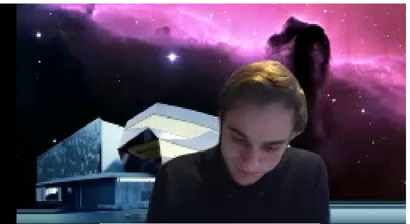
Split method into two subproblems

$$(\mathrm{II}) \begin{cases} u_t + f(u)_x = 0 \\ \frac{d}{dt} u = S(u) \end{cases}$$

that can be solved independently in an alternating manner

→ can use optimal existing schemes for each subproblem

→ can be easily applied



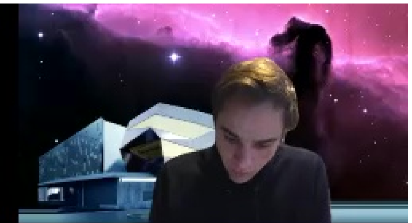
$$(\mathrm{II}) \begin{cases} u_t + f(u)_x = 0 \\ \frac{d}{dt} u = S(u) \end{cases}$$

that can be solved independently in an alternating manner

$\rightarrow$ can use optimal existing schemes for each subproblem

$\rightarrow$ can be easily applied to a large class and more complicated problem

$\leadsto$ wide use in hydrodynamics codes!



Idea:

$u \to u+1$

$$\left.\begin{aligned} &u_t + f(u)_x = 0 \\ \text{IC: } &u(x, t^n) = u^n(x) \end{aligned}\right\} \Rightarrow \bar{u}^{n+1}(x)$$

intermediate advection update

$$\left.\begin{aligned} &\frac{d}{dt} u = S(u) \\ \text{IC: } &\bar{u}^{n+1}(x) \end{aligned}\right\} \Rightarrow u^{n+1}(x)$$

final update

8.1.1 Accuracy of split methods

Consider linear

8.1.1 Accuracy of split methods

Consider linear system and write (I) or (II) as

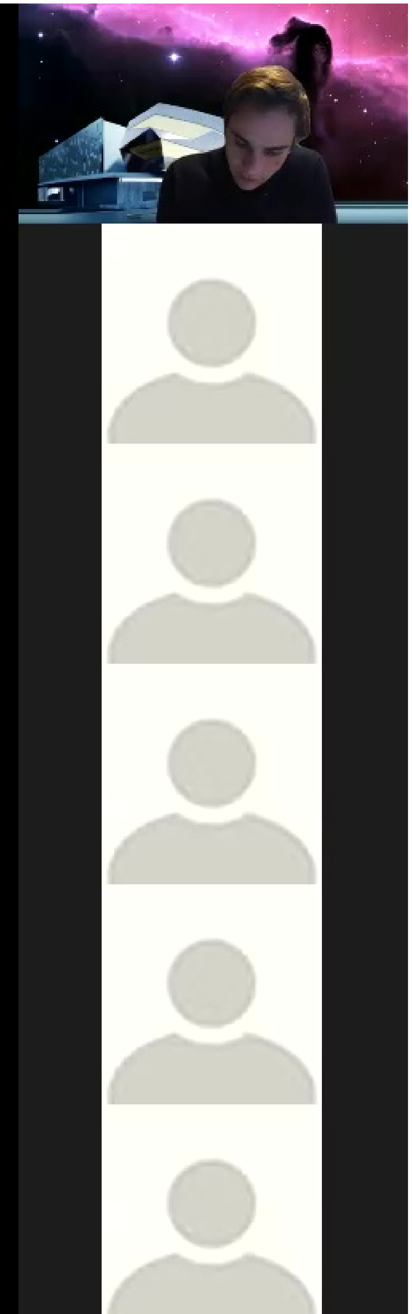
$$u_t = (A + S)\,u \qquad \longleftarrow$$

and assume A & S do not depend on time t.

Example: $A = A\,\partial_x, \quad S = S(x)$

Note: $\partial_t^j u = (A+S)^j u$

Theorem: The spl



8.1.1 Accuracy of split methods

Consider linear system and write (I) or (II) as

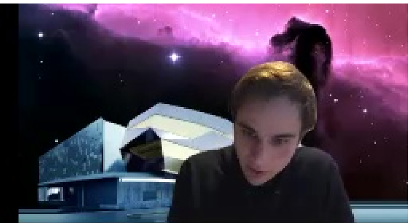
$$u_t = (\mathcal{A} + \mathcal{S})\, u$$

and assume $\mathcal{A}$ & $\mathcal{S}$ do not depend on time t.

Example: $\mathcal{A} = A\,\partial_x, \quad \mathcal{S} = S(x)$

<u>Note:</u> $\partial_t^j u = (\mathcal{A} + \mathcal{S})^j u$

Theorem: The splitting error will be $\mathcal{O}(\Delta t^2)$ in general and vanish if $\mathcal{A}$ and $\mathcal{S}$ commute, i.e. if $\mathcal{A}\mathcal{S} = \mathcal{S}\mathcal{A}$.



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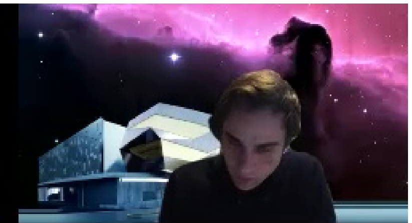

A and S commute, i.e. if $AS = SA$.

<u>Proof</u>: First Taylor expand actual (exact) solution to (I):

$$u(x,\Delta t) = u(x,0) + \Delta t\,(A+S)\,u(x,0) + \tfrac{1}{2}(\Delta t)^2 (A+S)^2 u(x,0) + \dots$$

$$= \left(\mathbb{1} + \Delta t(A+S) + \tfrac{1}{2}\Delta t^2 (A+S)^2 + \dots\right) u(x,0)$$

$$= \sum_{k=0}^{\infty} \frac{(\Delta t)^k}{k!} (A+S)^k\, u(x,0)$$

$$= e^{\Delta t (A+S)}\; u(x,0)$$

Split method:

$$\bar{u}(x,\Delta t) = e^{\Delta t A}\, u(x,0)$$

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Split method:

$$\bar{u}(x, \Delta t) = e^{\Delta t A} u(x, 0)$$

$$u^S(x, \Delta t) = e^{\Delta t S} \bar{u}(x, \Delta t)$$

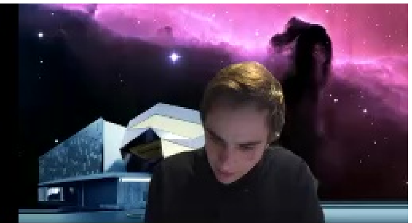
$$= e^{\Delta t S} e^{\Delta t A} u(x, 0) \leftarrow$$

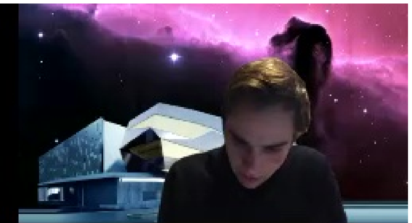
$$u(x, \Delta t) - u^S(x, \Delta t) = \left(e^{\Delta t (A+S)} - \underline{e^{\Delta t S} e^{\Delta t A}}\right) u(x, 0)$$

$$u^S(x, \Delta t) = \left(\mathbb{1} + \Delta t S + \tfrac{1}{2}(\Delta t)^2 S^2 + \dots\right)$$

$$\times \left(\mathbb{1} + \Delta t A + \tfrac{1}{2}(\Delta t)^2 A^2 + \dots\right)$$

$$u($$





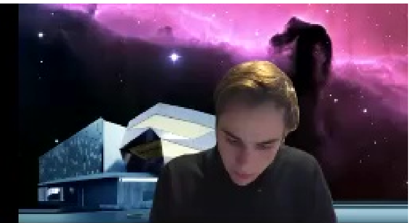
$$u^S(x,\Delta t) = \left(\mathbb{1} + \Delta t\, S + \tfrac{1}{2}(\Delta t)^2 S^2 + \dots\right)$$
$$\times \left(\mathbb{1} + \Delta t\, A + \tfrac{1}{2}(\Delta t)^2 A^2 + \dots\right)$$
$$u(x,0)$$

$$= \left(\mathbb{1} + \Delta t\,(A+S) + \tfrac{1}{2}(\Delta t)^2 (A^2 + 2SA + S^2) + \dots\right) u(x,0)$$

$\Longrightarrow$ Splitting error:

$$\Delta E^S \equiv u(x,\Delta t) - u^S(x,\Delta t)$$

$$= \tfrac{1}{2}\Delta$$



$$u^S(x,\Delta t) = \left(\mathbb{1} + \Delta t\, S + \tfrac{1}{2}(\Delta t)^2 S^2 + \dots\right) \times \left(\mathbb{1} + \Delta t\, A + \tfrac{1}{2}(\Delta t)^2 A^2 + \dots\right) u(x,0)$$

$$= \left(\mathbb{1} + \Delta t\,(A+S) + \tfrac{1}{2}(\Delta t)^2 (A^2 + 2SA + S^2) + \dots\right) u(x,0)$$

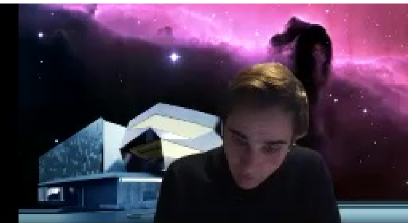
$\Longrightarrow$ Splitting error:

$$\boxed{\begin{aligned} \Delta\varepsilon^S &\equiv u(x,\Delta t) - u^S(x,\Delta t) \\ &= \tfrac{1}{2}(\Delta t)^2 \underbrace{(AS - SA)}_{\neq 0 \text{ in general}} u(x,0) + \mathcal{O}(\Delta t^3) \end{aligned}}$$

$\square$

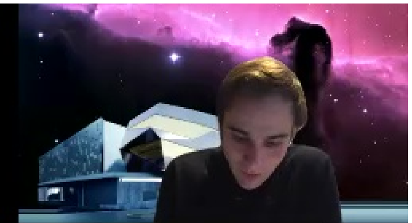
Remark: If

$$\boxed{= \frac{1}{2}(\Delta t)^2 \underbrace{(AS - SA)}_{\neq 0 \text{ in general}} u(x,0) + O(\Delta t^3)}$$



Remark: 1) If A & S commute split methods can be "exact".

2) Even if the subproblems are solved exactly or to some high order $O((\Delta t)^2)$, the split error introduces $O(\Delta t) \simeq \frac{T}{\Delta t} O((\Delta t)^2)$ after $\frac{T}{\Delta t}$ time steps to reach some specif



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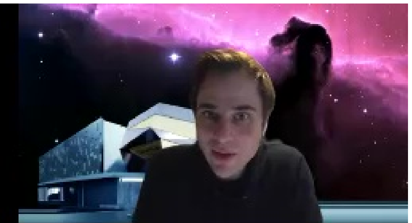
3) Thro

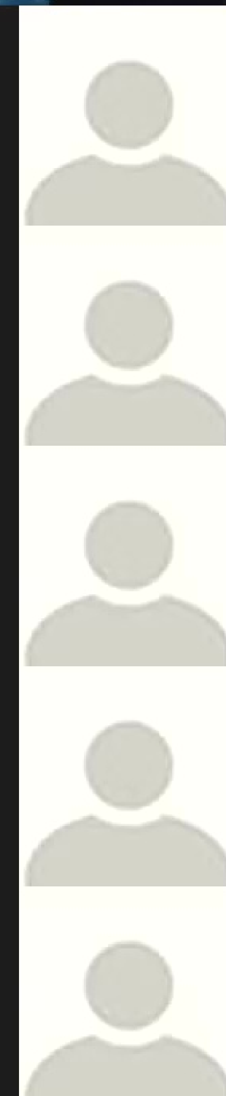
2) Even if the subproblems are solved exactly or to some high order $O((\Delta t)^2)$, the split error introduces $O(\Delta t) \simeq \frac{T}{\Delta t}\, O((\Delta t)^2)$ after $\frac{T}{\Delta t}$ time steps to reach some specified time T.

3) Through a slight modification, a higher-order split scheme can be obtained

$\longrightarrow$ "Strang splitting"

Strang SIAM J. Num. Anal. 5, 506 (1968)





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5, 506 (1968)

Idea: update with

$$e^{\Delta t S} e^{\Delta t A} \longrightarrow e^{\frac{1}{2}\Delta t A} e^{\Delta t S} e^{\frac{1}{2}\Delta t A}$$

to cancel $O((\Delta t)^2)$ error term:

$$e^{\frac{1}{2}\Delta t A} e^{\Delta t S} e^{\frac{1}{2}\Delta t A} = \left(\mathbb{1} + \frac{1}{2}\Delta t A + \frac{1}{8}(\Delta t)^2 A^2 + \dots\right)$$

$$\times \left(\mathbb{1} + \Delta t S + \frac{1}{2}(\Delta t)^2 S^2 + \dots\right)$$

$$\times \left(\mathbb{1} + \frac{1}{2}\Delta t A + \frac{1}{8}(\Delta t)^2 A^2 + \dots\right)$$

$$= \mathbb{1} + \Delta t (A+S) + \frac{1}{2}(\Delta t)^2 \big(\overbrace{A^2 + AS + SA + S^2}^{(A+S)^2}\big) + O((\Delta t)^3)$$

$$\Rightarrow \Delta \varepsilon = O((\Delta t)^3)$$

$$\Rightarrow \Delta t = O((\Delta x)^2)$$

8.1.2 ODE solver and timestepping

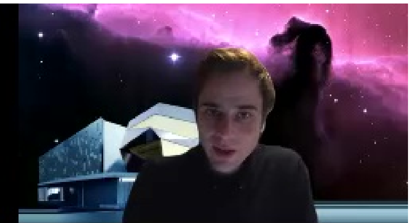
$$\frac{d}{dt} u = S(u)$$

first-order non-linear system

Note: 1) generally want to use

Note: 1) generally want to use a numerical scheme that is at least 2nd order accurate, in order to preserve accuracy of split method

2) Choice of numerical method depends on properties of S:



2) Choice of numerical method depends on properties of S:

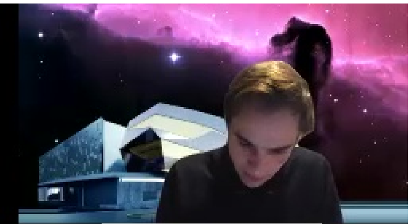
stability: the ODE system is called <u>asymptotically</u> stable if $|u_1(t) - u_2(t)| \xrightarrow{t\to\infty} 0$ for $u_1(0)$ and $u_2(0)$ sufficiently close.

This is the case for

$$\mathrm{Re}(\lambda_k) < 0 \quad \forall\, k = 1, \dots, m$$

where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of the Jacobian

$$J(u) = \frac{\partial S}{\partial u} = \begin{pmatrix} \frac{\partial S_1}{\partial u_1} & \cdots & \frac{\partial S_1}{\partial u_m} \\ \vdots & & \\ \frac{\partial S_m}{\partial u_1} & \cdots & \frac{\partial S_m}{\partial u_m} \end{pmatrix}$$



stiffness: disparate time scales

$\longrightarrow$ rapid <u>and</u> slow variations

<u>Def</u>: (i) $\mathrm{Re}(\lambda_k) < 0 \quad \forall\, k = 1, \dots, m$

(ii) $\lambda_{max} \equiv \max_k |\mathrm{Re}(\lambda_k)|$

$\gg \lambda_{min} \equiv \min_k |\mathrm{Re}(\lambda_k)|$

<u>Stiffness ratio</u>: $R_{stiff} \equiv \dfrac{\lambda_{max}}{\lambda_{min}}$

$\gtrsim 20$ already be problematic for explicit schemes

be problematic for explicit schemes

Example: $u'(t) = \lambda u(t)$

Euler method: $u^{n+1} = u^n + \Delta t\, S(t^n, u^n)$

(explicit)

Trapezoidal method: $u^{n+1} = u^n + \frac{1}{2}\Delta t \left[S(t^n, u^n) + S(t^{n+1}, u^{n+1}) \right]$

(implicit)

$\leadsto$ Euler: $u^{n+1} = (1 + \lambda \Delta t)\, u^n$

(implicit)

$\rightsquigarrow$ Euler : $u^{n+1} = (1+\lambda \Delta t)\, u^n$

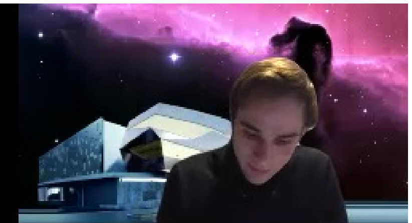
stability requires $J = \frac{\partial S}{\partial u} = \lambda \leq 0$

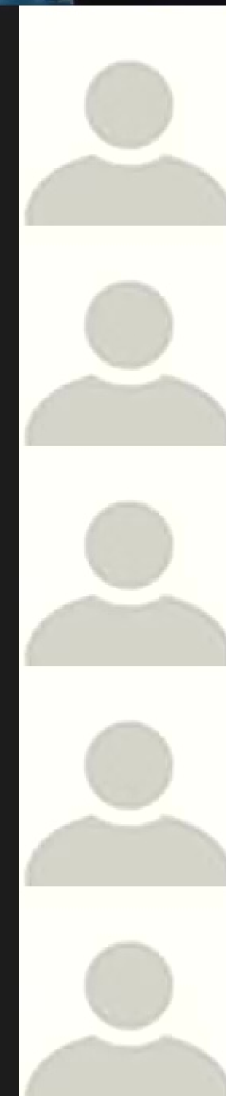
and solution remains bounded

$$(1+\lambda \Delta t) \leq 1$$

$$\Rightarrow \quad \Delta t \leq \frac{2}{|\lambda|}$$

$\rightsquigarrow$ for stiff ODE (large $|\lambda|$), Δt becomes impractically small





$$\Rightarrow \quad \Delta t \leq \frac{}{|\lambda|}$$

$\rightsquigarrow$ for stiff ODE (large $|\lambda|$), Δt becomes impractically small

Trapezoidal:

$$u^{n+1} = \underbrace{\frac{(1 + \frac{1}{2}\Delta t \lambda)}{1 - \frac{1}{2}\Delta t \lambda}}_{\leq 1 \text{ for } \lambda \leq 0} u^n$$

$\Rightarrow$ unconditionally stable for any Δt whenever the ODE

itself asymptotically stable