Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 20

Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

Date: November 24, 2020 - 3:30 PM

URL: http://pirsa.org/20110015

(I)

$$= x_R u_R - x_L u_L + T\,(f_L - f_R)$$

$f_R = f(u_R)$, $f_L = f(u_L)$

Also:

(II)
$$\int_{x_L}^{x_R} u(x,T)\,dx = \int_{x_L}^{TS_L} \underbrace{u(x,T)}_{\substack{\text{const}\\ u_L}}\,dx + \int_{TS_L}^{TS_R} u(x,T)\,dx + \int_{TS_R}^{x_R} \underbrace{u(x,T)}_{\substack{\text{const}\\ u_R}}\,dx$$

$$= (TS_L - x_L)\,u_L + \int_{TS_L}^{TS_R} u(x,T)\,dx + (x_R - TS_R)\,u_R$$

(I), (II) $\Rightarrow$ average of u_* in $*$-region

$$\boxed{u^{HLL} = \frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} u(x,T)\,dx = \frac{S_R u_R - S_L u_L + f_L - f_R}{S_R - S_L}}$$

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Remarks: 1) this is true for any wave

(I), (III) $\Rightarrow$ average of u_* in $*$-region

$$\boxed{u^{HLL} = \frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} u(x,T)\,dx = \frac{S_R u_R - S_L u_L + f_L - f_R}{S_R - S_L}}$$

Remarks: 1) this is true for any wave structure and number of waves in the Riemann problem

2)

Riemann problem

2) need to know the fastest wave speeds S_L, S_R

Useful exact flux expressions at $x=0$:

Integrate (*) over $\begin{cases} V_- = [x_L, 0] \times [0, T] \\ V_+ = [0, x_R] \times [0, T] \end{cases}$

$$\underbrace{\int_{x_L}^{0} u(x,T)\,dx}_{\int_{x_L}^{TS_L} \underbrace{u(x,T)}_{u_L}\,dx + \int_{TS_L}^{0} u(x,T)\,dx} - \underbrace{\int_{x_L}^{0} \underbrace{u(x,0)}_{u_L}\,dx}_{-x_L u_L} = -\int_0^T \Big[f(u(0,t)) - \underbrace{f(u(x_L,t))}_{f_L} \Big] dt$$

$$\Leftrightarrow \int_{TS_L}^{0} u(x,T)\,dx = -TS_L u_L + T(f_L - f_{0L})$$

$$\Leftrightarrow f_{0L} = f_L - S_L u_L - \frac{1}{T} \int^{0} u(x,T)\,dx$$

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$$\int_{x_L}^{TS_L} \underbrace{u(x,T)}_{u_L} dx + \int_{TS_L}^{0} u(x,T)\,dx \qquad -x_L u_L \qquad f_L$$

$$\iff \int_{TS_L}^{0} u(x,T)\,dx = -TS_L u_L + T(f_L - f_{0L})$$

$$\iff f_{0L} = f_L - S_L u_L - \underbrace{\frac{1}{T}\int_{TS_L}^{0} u(x,T)\,dx}$$

$S_L \times$ average in $\cancel{f}$ – region where $x<0$

Similarly :

$S_L \times$ average in
$\mathcal{L}$ – region where $x<0$

Similarly for V_+:

$$f_{0R} = f_R - S_R u_R + \frac{1}{T}\int_0^{TS_R} u(x,T)\,dx$$

Note: requirement of $f_{0L} = f_{0R} \Leftrightarrow (\mathrm{I})$

HLL flux estimate at $x=0$:

2-wave approximation $\rightsquigarrow$ can replace averages in f_{0L}, f_{0R} by u^{HLL}:

$$f^{HLL} = f_{0L} = f_L + S_L\,(u^{HLL} - u_L)$$

$$f^{HLL} = f_{0R} = f_R + S_R\,(u^{HLL} - u_R)$$

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by u^{HLL}:

$$f^{HLL} = f_{0L} = f_L + S_L\left(u^{HLL} - u_L\right)$$

$$f^{HLL} = f_{0R} = f_R + S_R\left(u^{HLL} - u_R\right)$$

($\Leftrightarrow$ Rankine-Hugoniot relations across R, L waves!)

$$f^{HLL} = \frac{S_R f_R - S_L f_R + S_L S_R (u_R - u_L)}{S_R - S_L}$$

$\rightsquigarrow$ set Godunov intercell flux to:

$$g^{HLL}_{i+\frac{1}{2}} \equiv f^{HLL} = \begin{cases} f_L, & 0 \leq S_L \\ \dfrac{S_R f_L - S_L f_R + S_L S_R (u_R - u_L)}{S_R - S_L}, & S_L \leq 0 \leq S_R \\ f_R, & S_R \leq 0 \end{cases}$$

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$$f^{HLL} = \frac{s_R f_R - s_L f_R + s_L s_R (u_R - u_L)}{s_R - s_L}$$

and set Godunov intercell flux to:

$$\boxed{g^{HLL}_{i+\frac{1}{2}} \equiv f^{HLL} = \begin{cases} f_L, & 0 \le s_L \\ \dfrac{s_R f_L - s_L f_R + s_L s_R (u_R - u_L)}{s_R - s_L}, & s_L \le 0 \le s_R \\ f_R, & s_R \le 0 \end{cases}}$$

where $f_L = f(u_i^n)$, $f_R = f(u_{i+1}^n)$.

Remarks: 1) Harten, Lax & van Leer (1983) showed that the Godunov scheme is

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<u>Remarks</u>: 1) Harten, Lax & van Leer (1983) showed that the Godunov scheme using the HLL expression for the intercell flux $g_{i+\frac{1}{2}}$

- converges to a weak solution of the system of conservation laws (if convergent)
- <u>converges to the entropy solution</u>

(if convergent)

- converges to the entropy solution

2) Disadvantage: cannot resolve any intermediate waves by construction

$\rightarrow$ diffusive, "numerical smearing"

$\rightarrow$ can be unacceptable for some problems

(especially for stationary intermediate waves wrt. to grid)

6.2.2 The HLLC solver

Idea: add intermediate wave to eliminate deficiencies of HLL(E) solver, at least for

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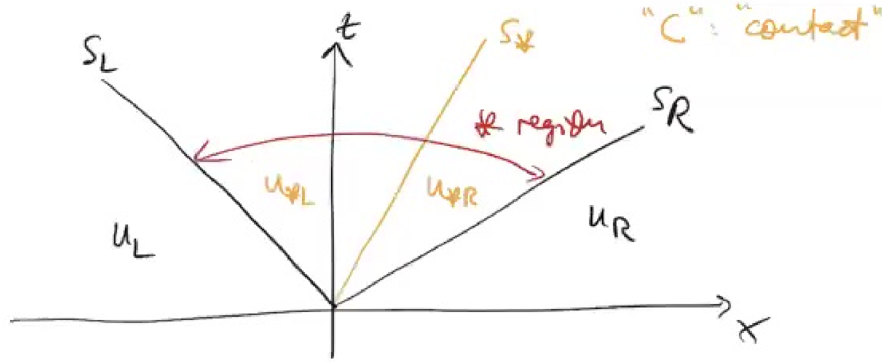
Idea: add intermediate wave to eliminate deficiencies of HLL(E) solver, at least for equations

Toro, Spruce & Speares, Shock Waves 4, 25 (1994)



Define averages in analogy to u^{HLL} before:

u_*

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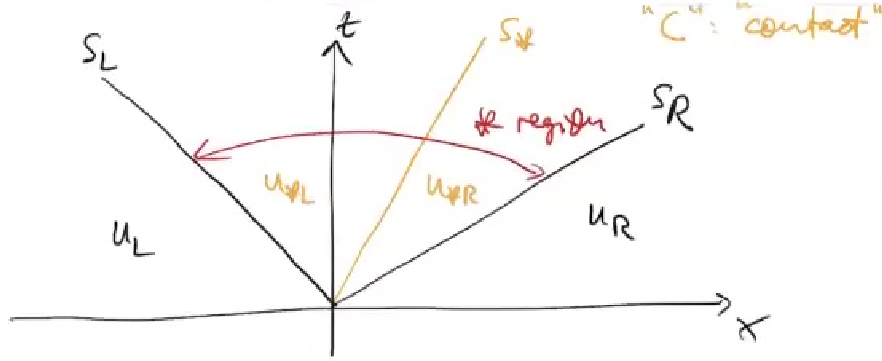

<u>Define</u> averages in analogy to u^{HLL} before:

$$u_{*L} \equiv \frac{1}{T(S_* - S_L)} \int_{TS_L}^{TS_*} u(x,T)\, dx$$

$$u_{*R} \equiv \frac{1}{T(S_R - S_*)} \int_{TS_*}^{TS_R} u(x,T)\, dx$$

$$\text{(I)} \Rightarrow u^{HLL} = \frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} u(x,T)\, dx$$

$$= \frac{S_* - S_L}{S_R - S_L} u_{*L} + \frac{S_R - S_*}{S_R - S_L}$$

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<u>HLLC approach:</u> replace exact solution to local RP by

$$u(x,t) = \begin{cases} u_L & \frac{x}{t} \le S_L \\ u_{*L} & S_L \le \frac{x}{t} \le S_* \\ u_{*R} & S_* \le \frac{x}{t} \le S_R \\ u_R & S_R \le \frac{x}{t} \end{cases}$$

using the intercell Godunov flux

$$g^{HLL}_{i+\frac{1}{2}} = \begin{cases} f_L & \frac{x}{t} \le S_L \\ f_{*L} & S_L \le \frac{x}{t} \le S_* \\ f_{*R} & S_* \le \frac{x}{t} \le S_R \\ f_R & S_R \le \frac{x}{t} \end{cases}$$

with f_{*L}, f_{*R}, u_{*L}, u_{*R} still to be determined from the Rankine-Hugoniot relation across the three waves:

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using the intercell Godunov flux

$$
g^{HLL}_{i+\frac{1}{2}} = \begin{cases} f_L & \frac{x}{t} \le S_L \\ f_{*L} & S_L \le \frac{x}{t} \le S_* \\ f_{*R} & S_* \le \frac{x}{t} \le S_R \\ f_R & S_R \le \frac{x}{t} \end{cases}
$$

with $f_{*L}, f_{*R}, u_{*L}, u_{*R}$ still to be determined from the Rankine-Hugoniot relation across the three waves:

(i) $f_{*L} = f_L + S_L(u_{*L} - u_L)$

(ii) $f_{*R} = f_{*L} + S_*(u_{*R} - u_{*L})$

(iii) $f_{*R} = f_R + S_R(u_{*R} - u_R)$

Idea: 1) specify / estimate $S_* = S_*(S_L, S_R, u_L, u_R$

$\rightarrow$ HLLC problem is reduced to the HLL problem

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with $f_{*L}, f_{*R}, u_{*L}, u_{*R}$ still to be determined from the Rankine–Hugoniot relation across the three waves:

(i) $f_{*L} = f_L + s_L(u_{*L} - u_L)$

(ii) $f_{*R} = f_{*L} + s_*(u_{*R} - u_{*L})$

(iii) $f_{*R} = f_R + s_R(u_{*R} - u_R)$

Idea: 1) specify / estimate $s_* = s_*(s_L, s_R, u_L, u_R)$

$\rightarrow$ HLLC problem is reduced to the HLL problem of estimating s_L, s_R.

2) determine u_*

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estimating s_L, s_R.

2) determine u_{*L}, u_{*R} and then use above expressions to find required fluxes f_{*L}, f_{*R}.

Need additional conditions:

(iv) $p_{*L} = p_{*R} = p_*$

(v) $S_* = v_* = \lambda_2$

(i) & (iii) using the fact that

$f^i(\vec{u}) = v_i \vec{u} + p \vec{D}_i$ for the Euler

equs., where $\vec{D}_i = [0, e_i, v_i]$, $e_i = (1,0,0)$

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(iv) $p_{*L} = p_{*R} = p_*$

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(i) & (iii) using the fact that

$$f^i(\vec{u}) = v_i \vec{u} + p \vec{D}_i \quad \text{for the Euler}$$

equs., where $\vec{D}_i = [0, e_i, v_i]$, $e_i = (\underset{\uparrow \atop i}{1}, 0, 0)$

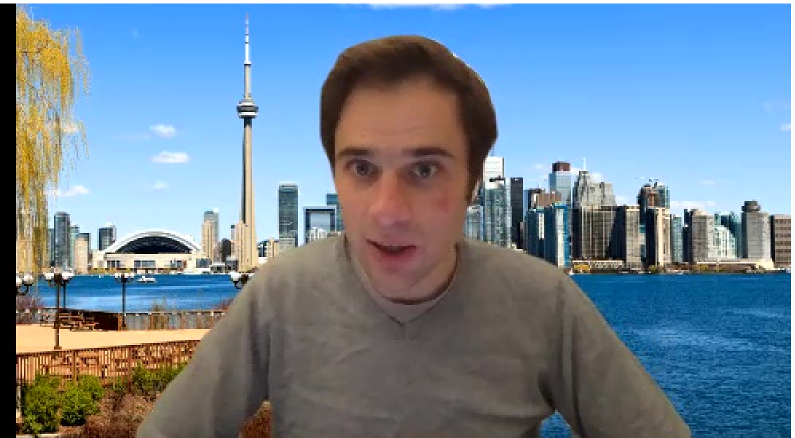
and p_{*L}, p_{*R} as functions of L, R, S_*

$$p_{*L} = p_L + \rho_L (S_L - u_L)(S_* - u_L)$$

$$p_{*R} = p_R + \rho_R (S_R - u_R)(S_* - u_R)$$

(iv)
and $S_* = S_*(S_L, S_R)$

$$= \frac{p_R - p_L + \rho_L v_L (S_L - v_L) - \rho_R v_R (S_R - v_R)}{\rho_L (S_L - v_L) - \rho_R (S_R - v_R)}$$



equs., where $\vec{D}_i = [0, e_i, v_i]$, $e_i = (1, 0, 0)$
$\uparrow$
i

and p_{*L}, p_{*R} as functions of L, R, S_*

$$p_{*L} = p_L + \rho_L(S_L - u_L)(S_* - u_L)$$

$$p_{*R} = p_R + \rho_R(S_R - u_R)(S_* - u_R)$$

(vi) (iv) and $S_* = S_*(S_L, S_R)$

$$= \frac{p_R - p_L + \rho_L v_L(S_L - v_L) - \rho_R v_R(S_R - v_R)}{\rho_L(S_L - v_L) - \rho_R(S_R - v_R)}$$

→ reduced to HLL problem of specifying the wave speeds S_L, S_R.

Using (i)-(vi), several possible choices for $u_{*L}, u_{*R} \Rightarrow f_{*L}, f_{*R}$

(Toro Sec. 10.4.2)

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(Toro Sec. 10.4.2)

For example: (i) & (iii) & p_{*L}, p_{*R} as above, find:

$$f_{*L,R} = f_{L,R} + S_{L,R}\left(U_{*L,R} - U_{L,R}\right)$$

$$U_{*L,R} = \rho_{L,R}\left(\frac{S_{L,R}-u_{L,R}}{S_{L,R}-S_*}\right)\begin{pmatrix} 1 \\ S_* \\ \frac{E_{L,R}}{\rho_{L,R}} + (S_* - u_{L,R})\left[S_* + \frac{p_{L,R}}{\rho_{L,R}(S_{L,R}-u_{L,R})}\right] \end{pmatrix}$$

Remark: The HLL & HLLC Riemann solver discussed so far are independent of the equation of state.

For example: (i) & (iii) & p_{*L}, p_{*R} as above, find:

$$f_{*L,R} = f_{L,R} + S_{L,R}\left(U_{*L,R} - U_{L,R}\right)$$

$$U_{*L,R} = \rho_{L,R}\left(\frac{S_{L,R} - u_{L,R}}{S_{L,R} - S_*}\right)\begin{pmatrix} 1 \\ S_* \\ \frac{E_{L,R}}{\rho_{L,R}} + (S_* - u_{L,R})\left[S_* + \frac{p_{L,R}}{\rho_{L,R}(S_{L,R} - u_{L,R})}\right] \end{pmatrix}$$

<u>Remark</u>: The HLL & HLLC Riemann solver discussed so far are independent of the equation of state. The EOS only enters through the wave speed S_L, S_R, which we still need to specify.

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6.2 The HLL family of Riemann solvers

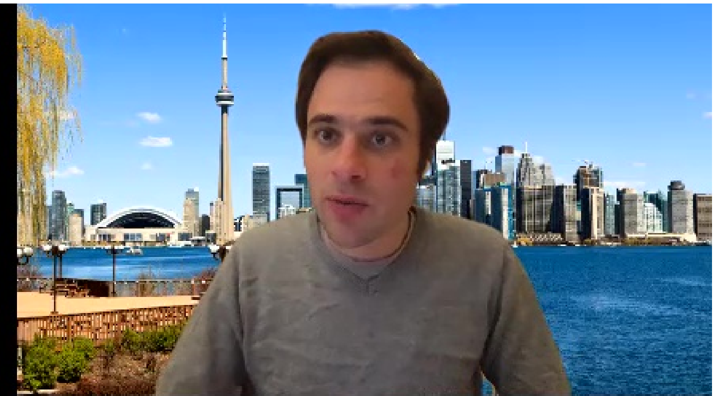
HLL: 2-wave approximation to the Riemann problem due to Harten, Lax, van Leer (1983)

$\rightarrow$ complete only for $m=2$ systems

HLLC: 3-wave approximation for the Euler equations (assumes 2-wave is a contact discontinuity) due to Toro et al. (1994)

"complete solver" for the Euler equs.

HLLE: HLL plus specific wave speed estimates by Einfeldt (1988)



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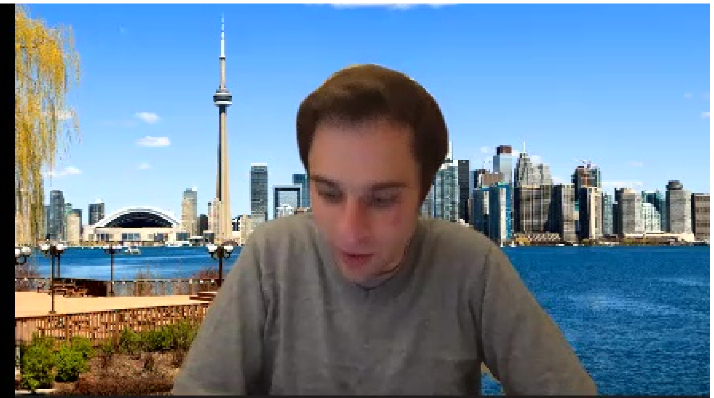

6.2.1 The HLL Riemann Solver

Consider the system

$$u_t + f(u)_x = 0, \qquad x \in [0,L]$$

ICs $\qquad u(x,0) = u^0(x)$

BCs $\qquad u(0,t) = u_l^B(t), \quad u(L,t) = u_r^B(t)$

Conservative scheme:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g_{i+\frac{1}{2}} - g_{i-\frac{1}{2}}\right]$$

Godunov original flux: $g_{i+\frac{1}{2}} = g(u_i^n, u_{i+1}^n)$

$$= f(\underbrace{v^n(x_{i+\frac{1}{2}}, t)}_{\text{exact solution to the RP at cell interface}})$$

$$= \text{"} f(v^n_{i+\frac{1}{2}}(0)) \text{"}$$

$$= f(\underbrace{v^n(x_{i+\frac{1}{2}}, t)})$$

exact solution to the RP at cell interface

$$\equiv \text{"} f(v^n_{i+\frac{1}{2}}(0)) \text{"}$$

where $v^n_{i+\frac{1}{2}}(0)$ is the exact solution of the local Riemann problem

(*) $\quad u_t + f(u)_x = 0$

$$u(x,0) = \begin{cases} u_L \equiv u_i^n, & x < 0 \ (x < x_{i+\frac{1}{2}}) \\ u_R \equiv u_{i+1}^n, & x \geq 0 \ (x \geq x_{i+\frac{1}{2}}) \end{cases}$$

Harten, Lax & van Leer (1983) SIAM Review 25(1):35

replace exact solution

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$$(*)\qquad u_t + f(u)_x = 0$$

$$u(x,0) = \begin{cases} u_L \equiv u_i^n, & x < 0 \;(x < x_{i+\frac{1}{2}}) \\ u_R \equiv u_{i+1}^n, & x \geq 0 \;(x \geq x_{i+\frac{1}{2}}) \end{cases}$$

Harten, Lax & van Leer (1983) SIAM Review 25(1): 35

replace exact solution by approximate solution

$$u(x,t) = \begin{cases} u_L, & \frac{x}{t} < S_L \\ u^{HLL}, & S_L \leq \frac{x}{t} \leq S_R \\ u_R, & \frac{x}{t} > S_R \end{cases}$$

S_L, S_R: fe

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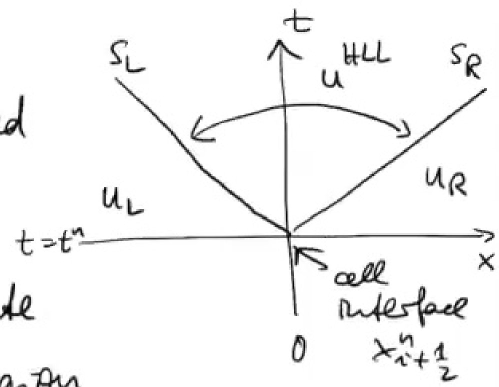
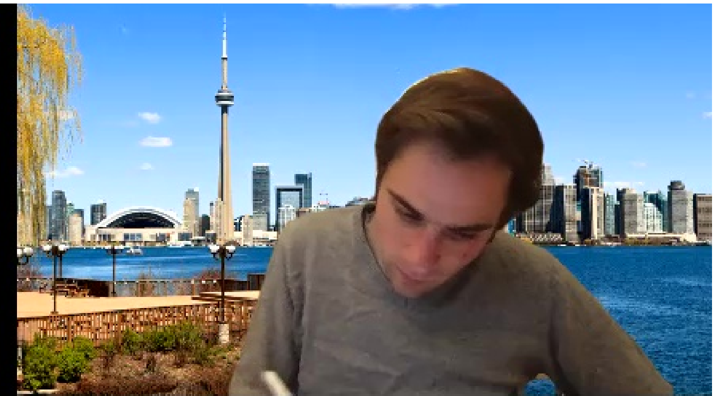
S_L, S_R: fastest wave speeds, assumed to be known

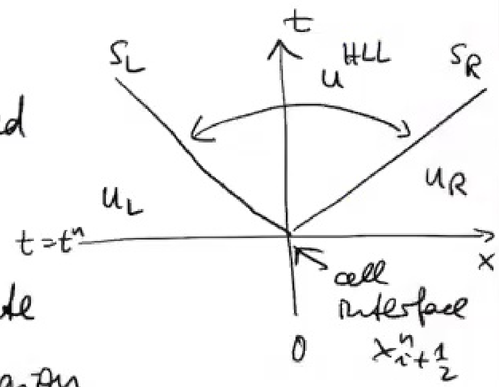


u^{HLL}: approximate state vector in region

Determine u^{HLL} using exact integral



S_L, S_R: fastest wave speeds, assumed to be known

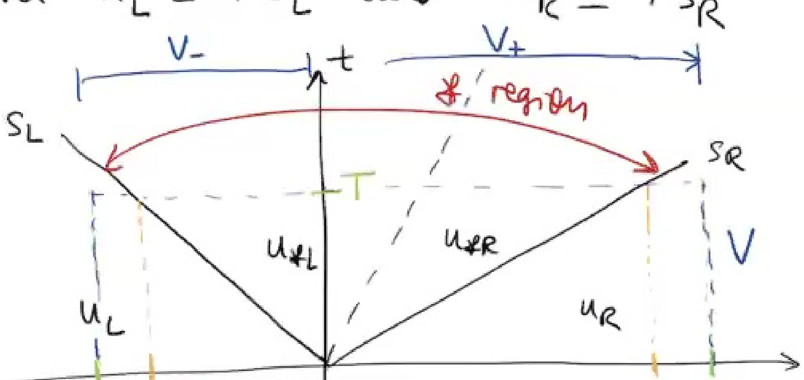


u^{HLL}: approximate state vector in * region

Determine u^{HLL} using exact integral relations:

Consider control volume $V = [x_L, x_R] \times [0, T]$

with $x_L \leq T S_L$ and $x_R \geq T S_R$



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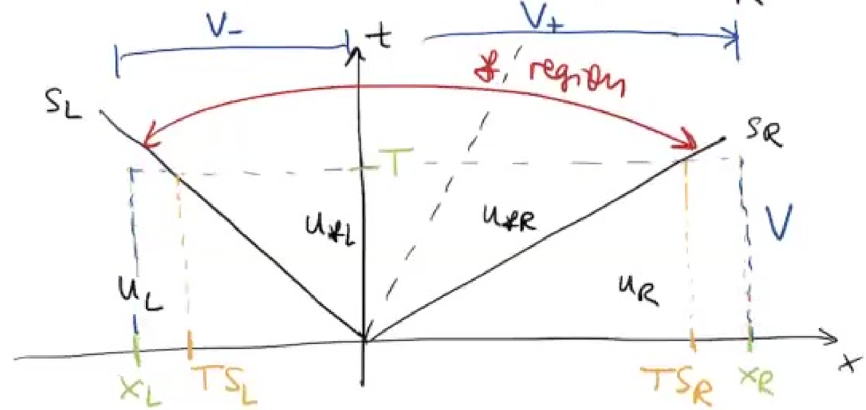
Determine u^{HLL} using exact integral relations:

Consider control volume $V = [x_L, x_R] \times [0,T]$

with $x_L \le TS_L$ and $x_R \ge TS_R$



Integrate (*) over V: $\displaystyle\int_{x_L}^{x_R}\int_0^T \left[u_t + f(u)_x\right] dt\,dx$

$$\rightsquigarrow \int_{x_L}^{x_L} u(x,T)\,dx = \int_{x_L}^{x_R} \underbrace{u(x,0)}_{\substack{\text{const}\\ \text{on } [x_L,0]\\ \text{and } [0,x_R]}}\,dx + \int_0^T f(\underbrace{u(x_L,t)}_{u_L})\,dt - \int_0^T f(u(x_R,t))\,dt$$

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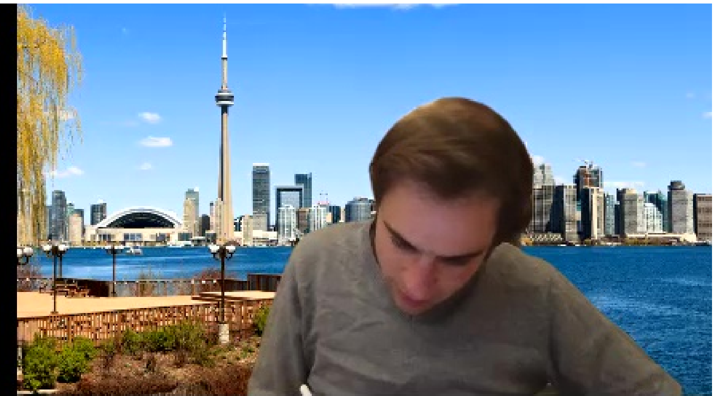

integrate (*) over ... $\int_{x_L}\int_0 [u_t + f(u)_x]\,dt\,dx$

$$\Rightarrow \int_{x_L}^{x_L} u(x,T)\,dx = \int_{x_L}^{x_R} \underbrace{u(x,0)}_{\substack{\text{const}\\ \text{on } [x_L,0] \\ \text{and } [0,x_R]}}\,dx + \int_0^T f(\underbrace{u(x_L,t)}_{u_L})\,dt - \int_0^T f(\underbrace{u(x_R,t)}_{u_R})\,dt$$

(I)

$$= x_R u_R - x_L u_L + T(f_L - f_R)$$

$f_R = f(u_R)$

$f_L = f(u_L)$

Also:

(II)
$$\int_{x_L}^{x_R} u(x,T)\,dx = \int_{x_L}^{TS_L} \underbrace{u(x,T)}_{\substack{\text{const}\\ u_L}}\,dx + \int_{TS_L}^{TS_R} u(x,T)\,dx + \int_{TS_R}^{x_R} \underbrace{u(x,T)}_{\substack{\text{const}\\ u_R}}\,dx$$

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(I)

$$= x_R u_R - x_L u_L + T(f_L - f_R)$$

$f_R = f(u_R)$

$f_L = f(u_L)$

Also:

(II)

$$\int_{x_L}^{x_R} u(x,T)\,dx = \int_{x_L}^{TS_L} \underbrace{u(x,T)}_{\text{const } u_L}\,dx + \int_{TS_L}^{TS_R} u(x,T)\,dx + \int_{TS_R}^{x_R} \underbrace{u(x,T)}_{\text{const } u_R}\,dx$$

$$= (TS_L - x_L)\,u_L + \int_{TS_L}^{TS_R} u(x,T)\,dx + (x_R - TS_R)\,u_R$$

(I), (III) $\Rightarrow$ average of u_* in $*$-region

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