Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 19

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Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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6.1 Roe's Riemann solver

<u>Idea:</u> replace actual Riemann problem by an approximate linearized problem defined locally at each cell

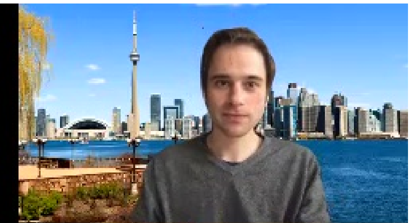
<u>Def:</u> (Riemann solver of Roe):

Replace $u_t + f(u)_x = 0$

$$u(x,0) = \begin{cases} u_l = u_{i-1}^n & x < 0 \\ u_r = u_i^n & x > 0 \end{cases}$$

by the linear problem

$$(*) \qquad w_t + A_{lr} w_x = 0$$
$$w(x,0) = \begin{cases} w_l & x < 0 \\ w_r & x \geq 0 \end{cases}$$



where $A_{\ell r} = A(u_\ell, u_r) \in \mathrm{Mat}(\mathbb{R}^{m\times m})$ satisfies the conditions

(i) $f(v) - f(w) = A(v,w)\,(v-w)$

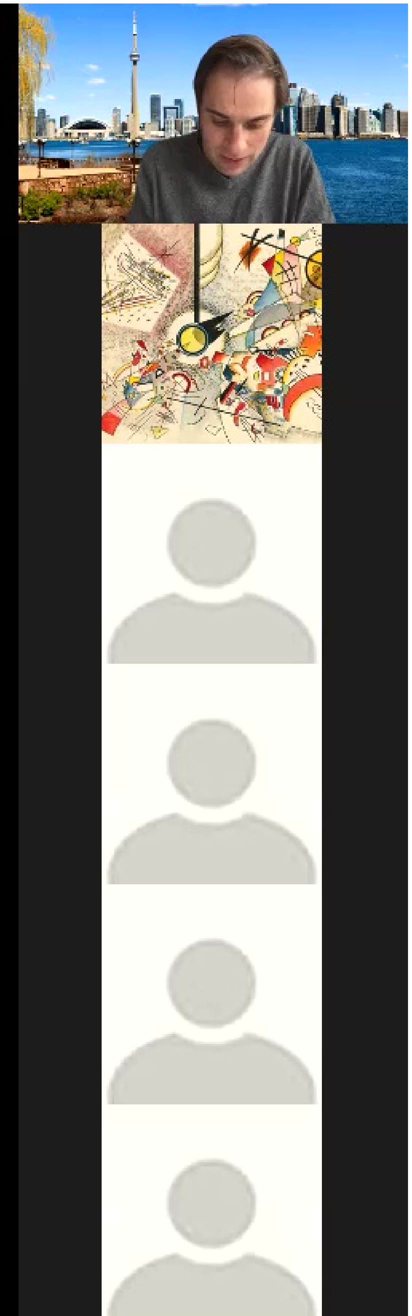
(conservation across discontinuities)

(ii) $\|A(v,w) - Df(v)\| \to 0$ as $\|w-v\| \to 0$

(consistency with the exact Jacobian)

(iii) $A(v,w)$ has only real eigenvalues and a complete set of eigenvectors

The Roe scheme is defined by replacing the exact solution to the local RP in Godunov's method by the exact solution to $(*)$.



Motivation:

- linearized problem justified at most cell interfaces

$$\|u^n_{i+1} - u^n_i\| \sim \mathcal{O}(\Delta x)$$

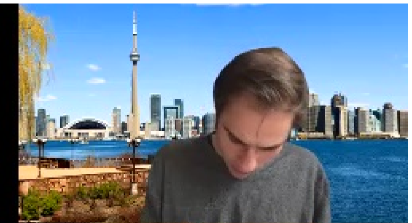
$$Df(u_{i+1}) \approx Df(u_i)$$

$$u_t + f(u)_x = 0 \rightsquigarrow u_t + \overbrace{Df(\bar{u})}^{A} u_x = 0$$

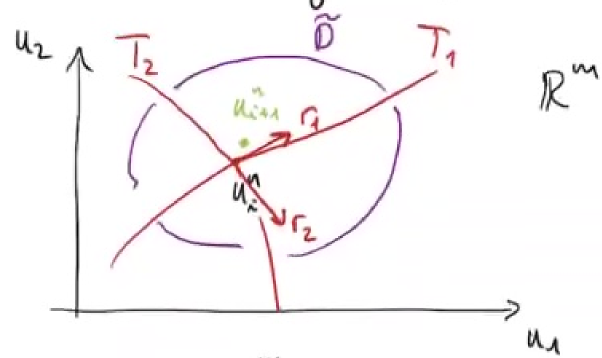
const. matrix

$\bar{u}$: some average state $\bar{u} = \bar{u}(u_{i+1}, u_i)$

State space: integral & shock sets need to connect u^n_{i+1}, u^n_i are nearly straight lines



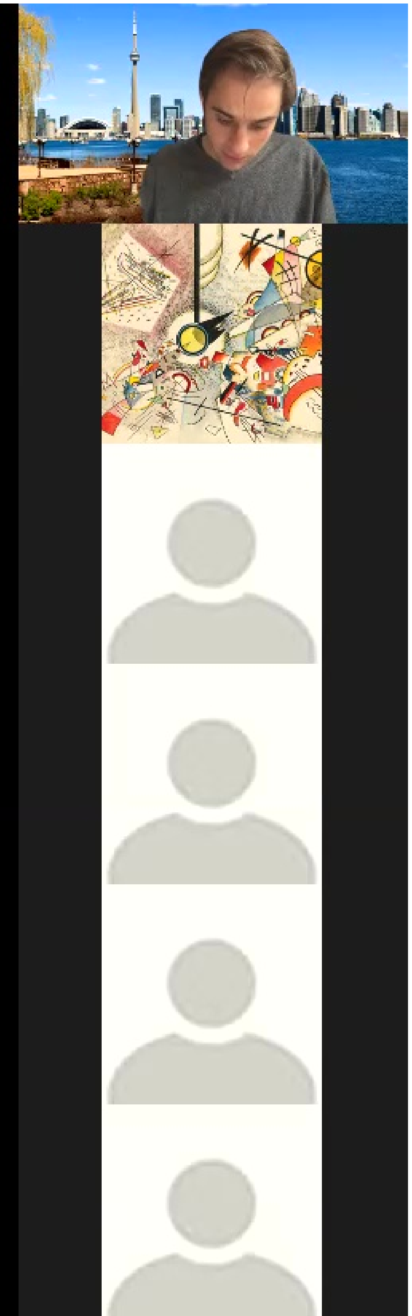
State space: integral & shock sets need to connect u_{i+1}^n, u_i^n are nearly straight lines

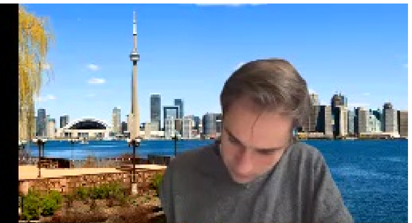


$$u_{i+1}^n - u_i^n = \sum_{k=1}^{m} \alpha_k r_k$$

r_k eigenvectors of $Df(u_i^n)$

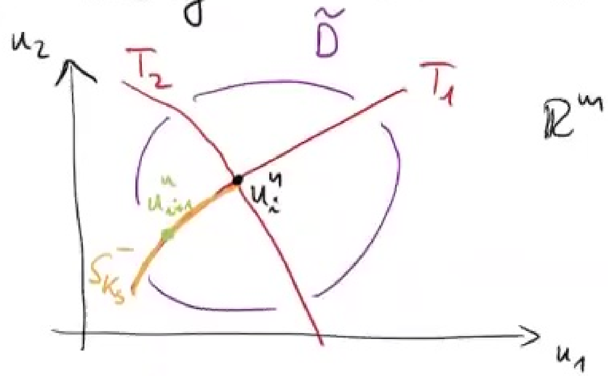
(infinitesimal Riemann problem)
linearized





linearized

- near shocks: u_{i+1}^n, u_i^n may be separated far in state space at least in one direction $k=k_s$ along a shock set S_k^-



$\longrightarrow$ need to make sure that the corresponding single shock wave $w = u_{i+1}^n - u_i^n$ is an eigenvector of $A(u_i^n, u_{i+1}^n)$, so

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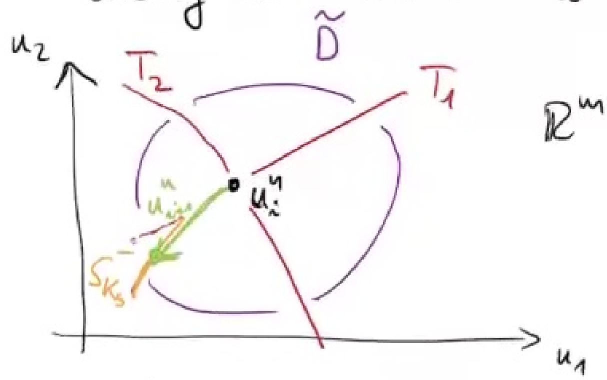
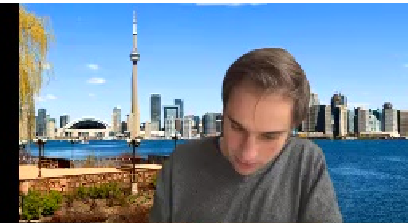
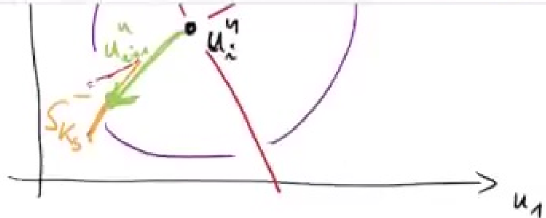


→ need to make sure that the corresponding single shock wave $w = u_{i+1}^n - u_i^n$ is an eigenvector of $A(u_i^n, u_{i+1}^n)$, so that the linearized problem "captures" the shock





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RH jump conditions: $f(v) - f(w) = s(v-w)$

Require $A(v,w)(v-w) \overset{!}{=} s(v-w) = f(v) - f(w)$

$\Rightarrow$ requirement (i)

<u>Construction of $A(v,w)$:</u>

Construction of $A(v,w)$:

- Consider the line integral along the path

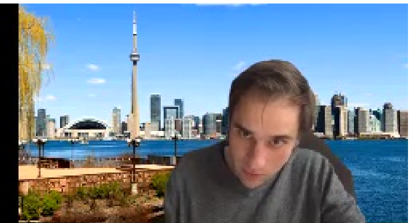
$$u(\xi) = u_i^n + (u_{i+1}^n - u_i^n)\,\xi, \quad 0 \le \xi \le 1$$

$$f(u_{i+1}) - f(u_i) = \int_0^1 \frac{d f(u(\xi))}{d\xi}\, d\xi$$

$$= \int_0^1 Df(u(\xi))\, u'(\xi)\, d\xi$$

$$= (u_{i+1}^n - u_i^n) \underbrace{\int_0^1 Df(u(\xi))\, d\xi}_{= A(u_i, u_{i+1})}$$

$\rightarrow$ satisfy



$$= (u^n_{i+1} - u^n_i) \underbrace{\int_0^1 Df(u(\xi))\,d\xi}_{= A(u_i,\, u_{i+1})}$$

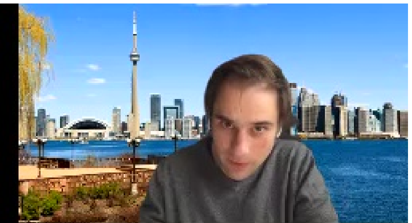
$\rightarrow$ satisfy (i) & (ii), but (iii) is not necessarily guaranteed

- Roe (1981): introduce coordinate transform. $z(u)$

path: $z(\xi) = z_i + (z_{i+1} - z_i)\xi$, $z_i \equiv z(u_i)$

$$\rightsquigarrow f(u_{i+1}) - f(u_i) = \int_0^1 \frac{d f(z(\xi))}{d\xi}\, d\xi$$

$$=$$



$$\rightsquigarrow \quad f(u_{i+1}) - f(u_i) = \int_0^1 \frac{d f(z(\xi))}{d\xi}\, d\xi$$

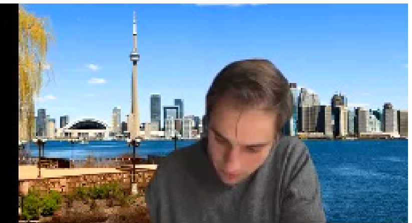
$$= (z_{i+1} - z_i) \underbrace{\int_0^1 Df_z(z(\xi))\, d\xi}_{\equiv\, C(u_i, u_{i+1})}$$

↑ w.r.t. to z

$$u_{i+1} - u_i = \int_0^1 \frac{d u(z(\xi))}{d\xi}\, d\xi$$

$$= \int_0^1 \frac{d u(z(\xi))}{dz}\, z'(\xi)\, d\xi$$

$$= (z_{i+1} - z_i) \underbrace{\int_0^1 \frac{d u(z(\xi))}{dz}\, d\xi}_{\equiv\, B}$$



$$\equiv C(u_i, u_{i+1})$$

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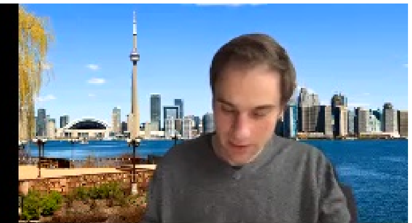
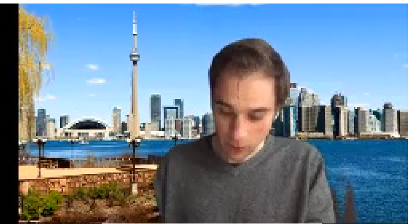
$$= (z_{i+1} - z_i) \underbrace{\int_0^1 \frac{d u(z(\xi))}{dz}\, d\xi}_{\equiv B(u_i, u_{i+1})}$$

$$\rightsquigarrow \quad A(u_i, u_{i+1}) \equiv C(u_i, u_{i+1})\, B(u_i, u_{i+1})$$





$$= (z_{i+1} - z_i) \underbrace{\int_0^1 \frac{dw(z(\xi))}{dz}\, d\xi}_{\equiv B(u_i, u_{i+1})}$$

$\leadsto$ $A(u_i, u_{i+1}) \equiv C(u_i, u_{i+1})\, B(u_i, u_{i+1})$

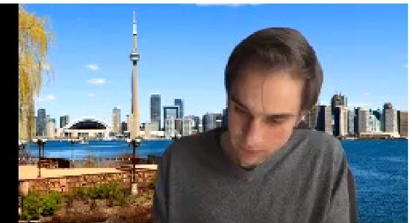
Harten, Lax, van Leer (1983): this procedure guarantees (iii) if the system has a entropy function Φ

$\leadsto$ choose $z(u) = \nabla \Phi(u)$ wrt u

$\leadsto$ $A = A_{HLL}$ is symmetric

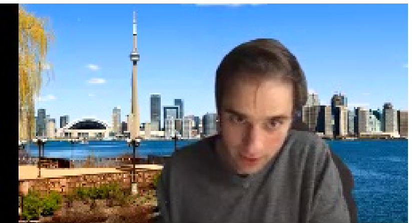
$\to$ real eigenvalues

Proposition: Assume that there exists a matrix $A(v,w)$ as in the previous



Proposition: Assume that there exists a matrix $A(v,w)$ as in the previous definition. Then the Roe scheme as defined above can be written in conservative form

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left[g(w_i^n, w_{i+1}^n) - g(w_{i-1}^n, w_i^n)\right)$$



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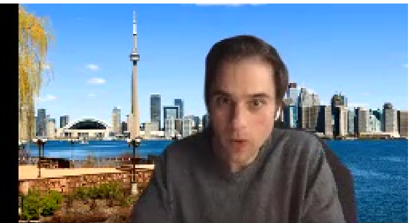
where the numerical flux g is given by

$$g(v,w) = \frac{1}{2}\left[f(v) - f(w) \right] - \frac{1}{2}\sum_{k=1}^{m} |\lambda_k| \alpha_k r_k$$

with (λ_k, r_k) being the eigenvalues and eigenvectors of $A(v,w)$. The coefficients ("wave strengths") α_k are determined

$$w - v = \sum_{k=1}^{m} \alpha_k r_k$$

<u>Proof:</u> Toro Sec. 11.1



$$w - v = \sum_{k=1} \alpha_k r_k$$

<u>Proof:</u> Toro Sec. 11.1

<u>Remarks:</u> 1) Note that the exact solution to the linearized RP can be easily written down:

$$w(x,t) = \begin{cases} u_\ell & \frac{x}{t} < \lambda_1 \\ u_k & \lambda_k \leq \frac{x}{t} < \lambda_{k+1}, \quad k = \{1, \dots, m-1\} \\ u_r & \lambda_m \leq \frac{x}{t} \end{cases}$$

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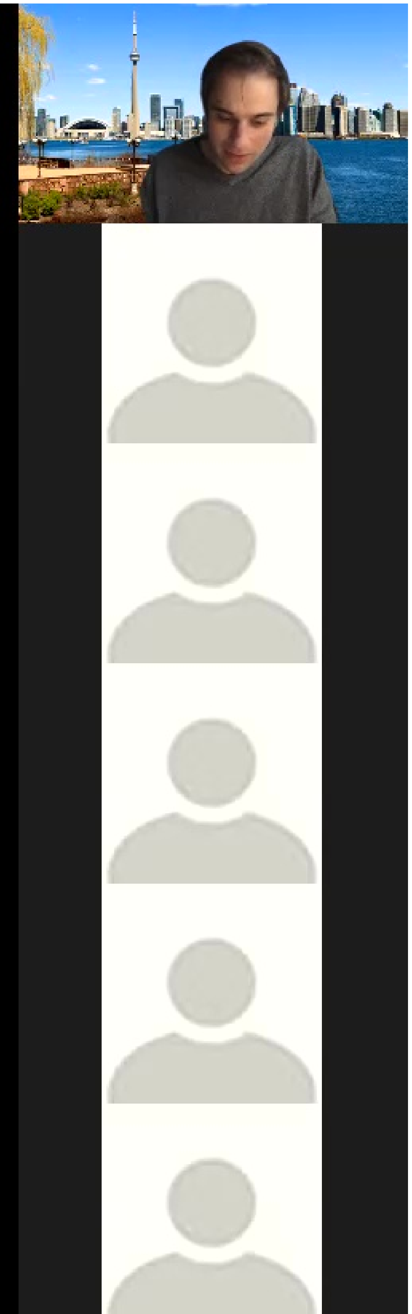
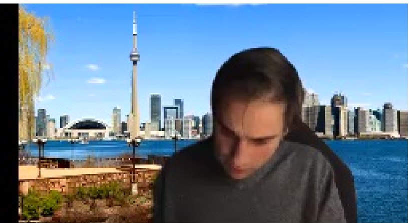
where $u_k \equiv u_\ell + \sum_{j=1}^{k} \alpha_j r_j$, and α_j are given by

$$u_r - u_\ell = \sum_{j=1}^{m} \alpha_j r_j$$





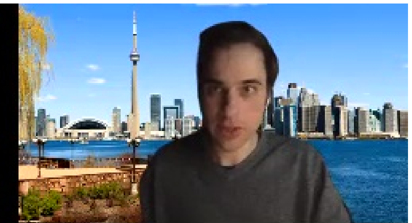
$j=1$

2) In general, the Roe scheme does not approximate the entropy solution

→ additional entropy "fixes" required (see Toro Sec. 11.4, Leveque 15.3)

3) Construction of $A(v,w)$ is complicated and expensive

→ Roe-Pike method (Toro Sec. 11.3) offer approach to compute all quantities needed for flux computation etc. <u>without</u> actually constructing $A(v,w)$



Roe's scheme for the Euler equations:

Consider $u_t + f(u)_x = 0$

Roe (1981): $z = \frac{u}{\sqrt{\rho}}$ $\rightsquigarrow$ find that

$$A(u_\ell, u_r) = Df(\bar{u}), \quad \text{where}$$

wrt. to u

$$\bar{u} = \begin{pmatrix} \bar{\rho} \\ \bar{\rho}\bar{u} \\ \bar{e} \end{pmatrix} \quad \text{and}$$

$$\bar{\rho} = \sqrt{\rho_r \rho_\ell}$$

$$\bar{u} = \frac{\sqrt{\rho_\ell}\, u_\ell + \sqrt{\rho_r}\, u_r}{\sqrt{\rho_\ell} + \sqrt{\rho_r}}$$

$$\bar{H} = \frac{\sqrt{\rho_\ell}\, H_\ell + \sqrt{\rho_r}\, H_r}{\sqrt{\rho_\ell} + \sqrt{\rho_r}},$$

"Roe mean values"

$H = \frac{e+p}{\rho}$ specific enthalpy

then $A(u_\ell, u_r)$ satisfies (i) – (iii).

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Lecture_19

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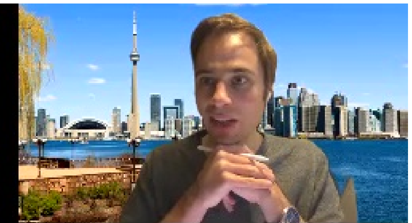

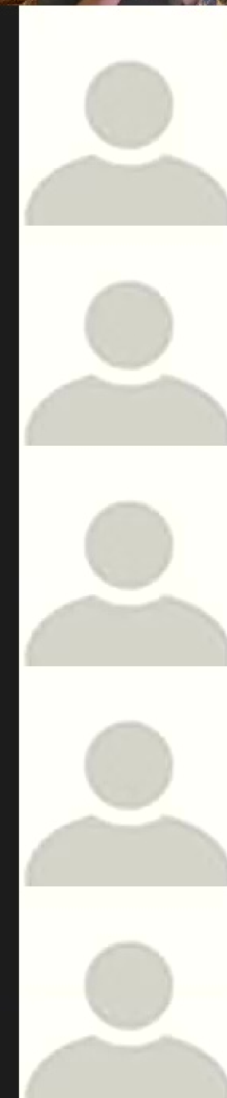

then $A(u_l, u_r)$ satisfies (i) – (iii).

Remark: The Roe-scheme does not guarantee positive pressure and densities

$\rightsquigarrow$ cases in which $\rho < 0$, $p < 0$

disadvantages for problems where low densities are expected

→ additional fixes (Einfeldt et al. 1991)

→ Einfeldt show that for certain Riemann problems there is no linearization that preserves positivity of ρ, e, p