Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 18

Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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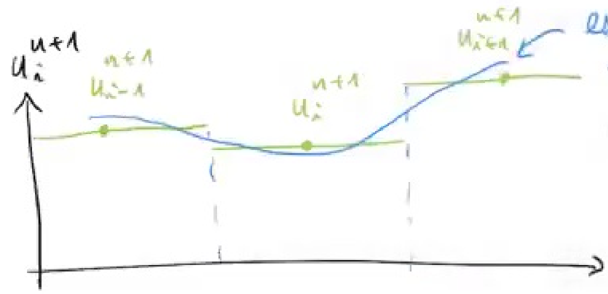
Graphical summary:

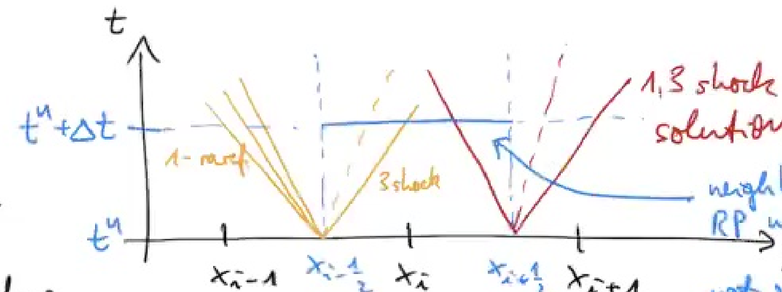
③ Average solution over cells



exact solution $v^n(x, t^{n+1})$ of local Riemann problems

② Evolve

RPs at cell interface



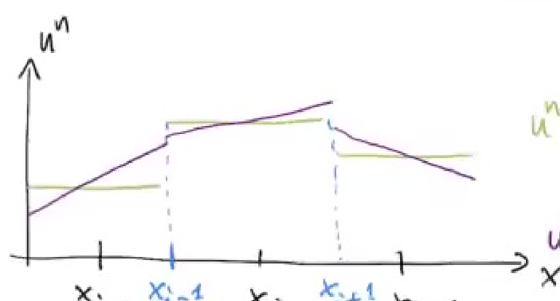
1,3 shock solution

neighboring RP must not intersect

$\Delta t < \frac{\Delta x}{2} \max\{\lambda_k\}$

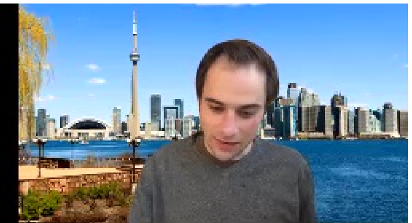
① $t = t^n$

Reconstruct from



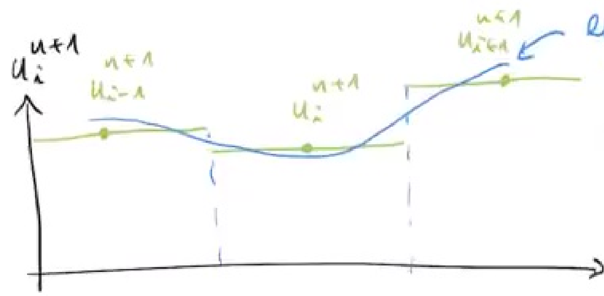
$u^n(x, t^n)$ piecewise const.

$u^n(x, t^n)$ piecewise



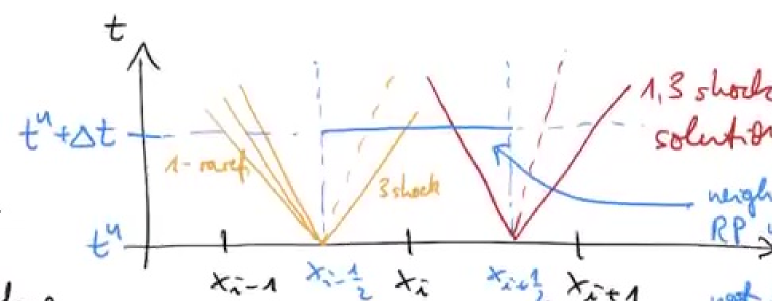
Graphical summary:

③ **Average** solution over cells



exact solution $v^n(x, t^{n+1})$ of local Riemann problems

② **Evolve** RPs at cell interface



1,3 shock solution

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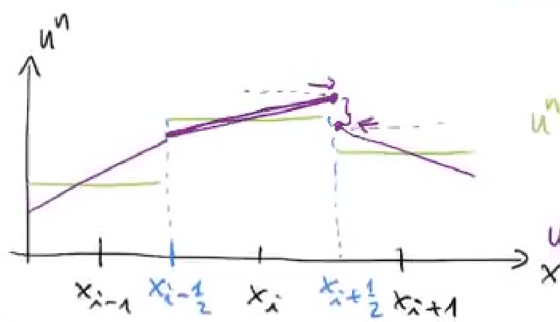
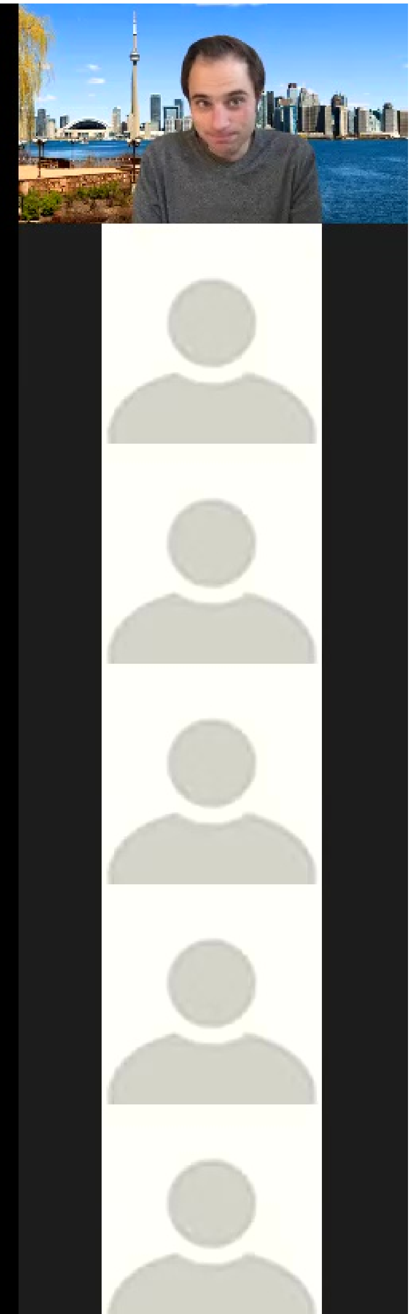


$u^n(x, t^n)$ piecewise const.

$u^n(x, t^n)$ piecewise linear



<u>Proposition:</u> The Godunov method can be written in conservative form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\right]$$

with intercell numerical flux

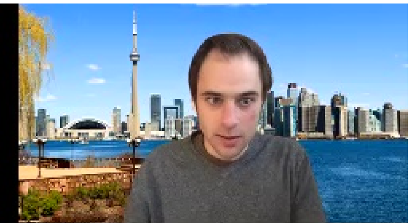
$$g(\underbrace{u_i^n, u_{i+1}^n}) \equiv f\left(v^n\left(x_{i+\frac{1}{2}}, t^n\right)\right)$$

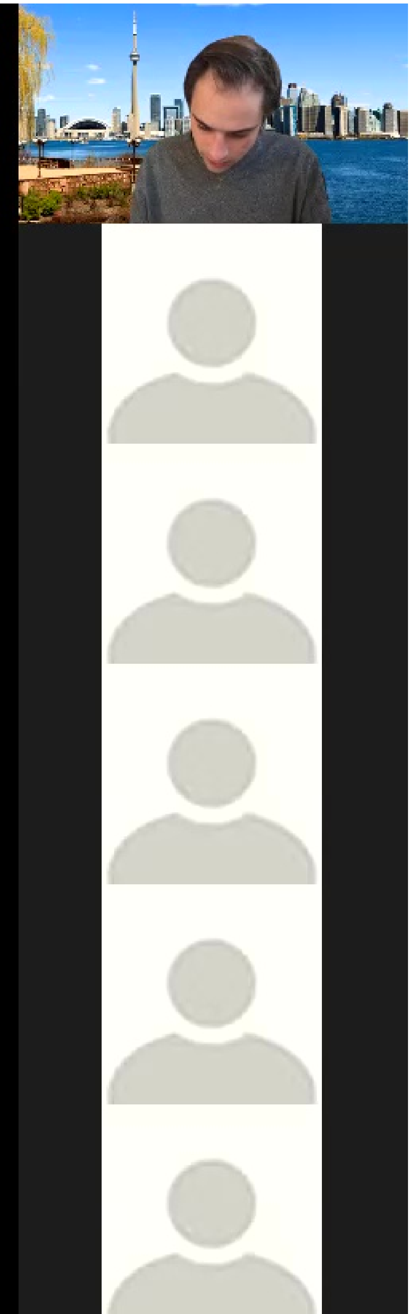
if the timestep Δt satisfies the condition $\Delta t < \frac{\Delta x}{2} \max\{|\lambda_k| \mid k \in \{1, \dots, m\}\}$

where λ_k are the eigenvalues of $Df(u)$.

The Godunov method is consistent and consistent with the entropy condition.

<u>Proof:</u>





where λ_k are the eigenvalues of $Df(u)$.

The Godunov method is consistent and consistent with the entropy condition.

Proof: (1) $t = t^n$. v^n is (locally in time) the exact solution

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Big[\underbrace{v^n(x,t^{n+1})}_{\stackrel{def}{=} u_i^{n+1}\cdot\Delta x} - \underbrace{v^n(x,t^n)}_{= u_i^n \Delta x}\Big]\,dx = -\int_{t^n}^{t^{n+1}} \Big[f(v^n(x_{i+\frac{1}{2}},t)) - f(v^n(x_{i-\frac{1}{2}},t))\Big]\,dt$$

Note: $v^n(x_{i\pm\frac{1}{2}}, t)$ is constant on

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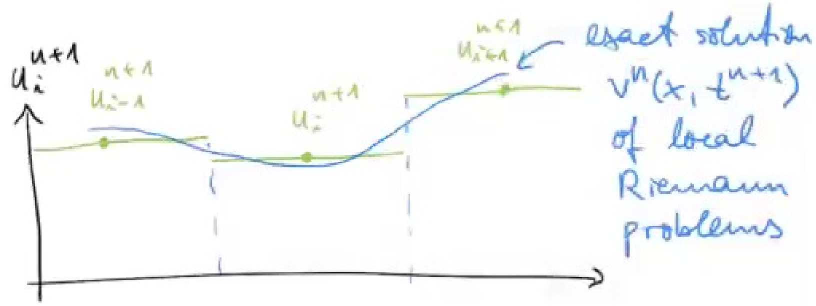
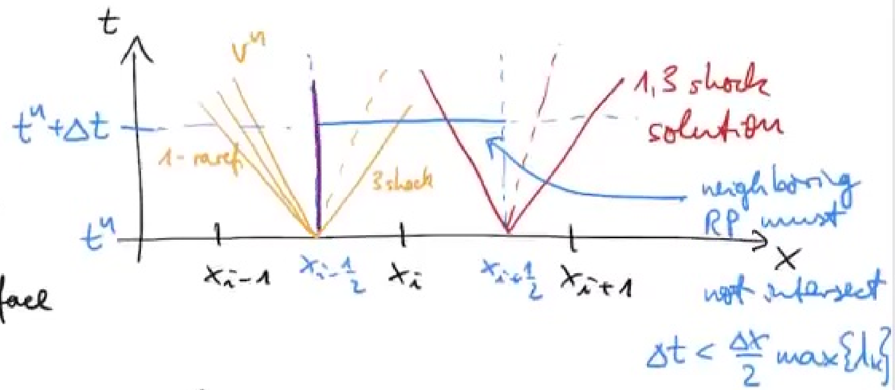
Graphical summary:

③ Average solution over cells

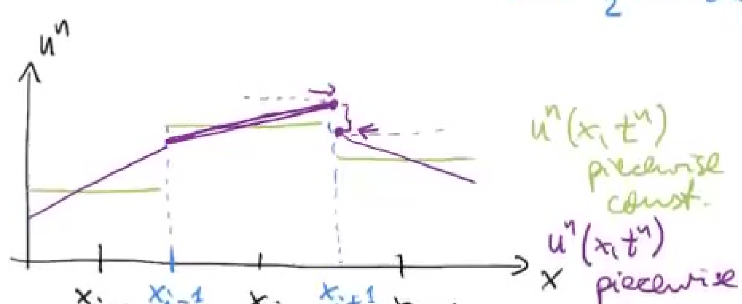


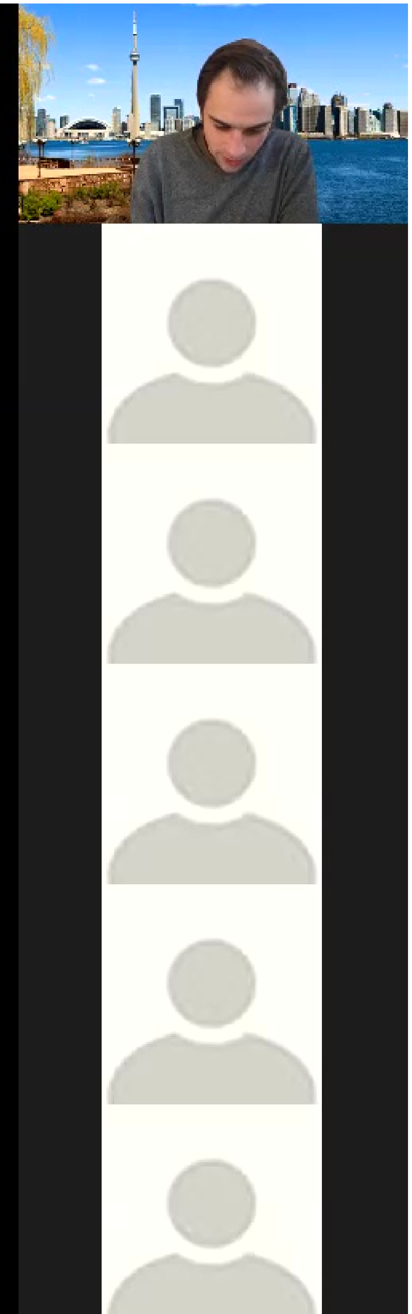
② Evolve RPs at cell interface



① $t = t^n$

Reconstruct from





the exact solution

$$(*)\quad \int_{x_{i-\frac12}}^{x_{i+\frac12}} \Big[\underbrace{v^n(x,t^{n+1})}_{\overset{def}{=} u_i^{n+1}\cdot\Delta x} - \underbrace{v^n(x,t^n)}_{= u_i^n \Delta x}\Big]dx = -\int_{t^n}^{t^{n+1}} \Big[f(v^n(x_{i+\frac12},t)) - f(v^n(x_{i-\frac12},t))\Big]dt$$

Note: $v^n(x_{i\pm\frac12},t)$ is constant on $[t^n,t^{n+1}]$

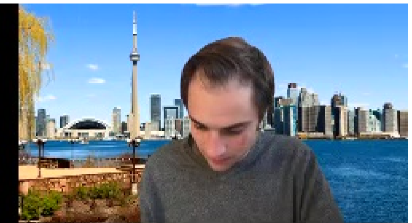
(solution of RP on ray $\frac{x}{t}=0$ through $x_{i\pm\frac12}$)

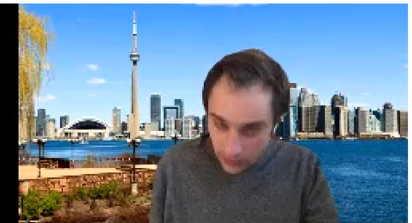
$$v^n(x_{i\pm\frac12},t^n) \overset{def}{=} w_i^n(x_{i\pm\frac12},t^n)$$

$$\Rightarrow \int_{t^n}^{t^{n+1}} f(v^n(x_{i\pm\frac12},t))\,dt = \Delta t\; f(v^n(x_{i\pm\frac12},t))$$

Therefore:

$(*)$





$$\Rightarrow \int_{t^n} f\left(v^n(x_{i\pm\frac{1}{2}},t)\right)dt = \Delta t \, f\left(v^n(x_{i\pm\frac{1}{2}},t^n)\right)$$

Therefore:

$$(*) \iff u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[f\left(v^n(x_{i+\frac{1}{2}},t)\right) - f\left(v^n(x_{i-\frac{1}{2}},t)\right)\right]$$

Define numerical flux

$$g(u_i^n, u_{i+1}^n) \equiv f\left(v^n(x_{i+\frac{1}{2}},t^n)\right)$$

$$= u_i^n - \frac{\Delta t}{\Delta x}\left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\right]$$

$\Rightarrow$ scheme is in conservative form

Also: • if $u_i^n = u_{i+1}^n \equiv \tilde{u}$, then $v^n(x_{i+\frac{1}{2}},t^n) = \tilde{u}$

$\Rightarrow g(\hat{u},\tilde{u}) = f(\tilde{u}) \Rightarrow g$ is consistent

• g lipshitz continuous if f is lipshitz continuous

② entropy condition:

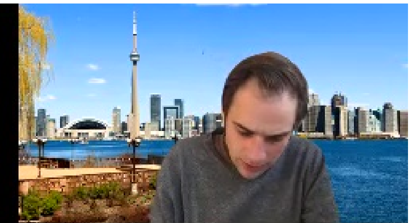
$v^n(x,t)$ as exact solution to the RP satisfies the entropy condition:

let (Φ, Ψ) be an entropy pair

$$\left(\leadsto \Phi(v^n)_t + \Psi(v^n)_x \le 0\right)$$

$$\Rightarrow \int_{x_{i-\frac12}}^{x_{i+\frac12}} \Phi(v^n(x,t^{n+1}))\,dx - \int_{x_{i-\frac12}}^{x_{i+\frac12}} \Phi(v^n(x,t^n))\,dx$$

$$(**) \qquad + \int_{t^n}^{t^{n+1}} \Psi(v^n(x_{i+\frac12},t))\,dt - \int_{t^n}^{t^{n+1}} \Psi(v^n(x_{i-\frac12},t))\,dt \le 0$$



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(**)

$$+\int_{t^n}^{t^{n+1}} \Psi\left(v^n(x_{i+\frac12},t)\right)dt - \int_{t^n}^{t^{n+1}} \Psi\left(v^n(x_{i-\frac12},t)\right)dt \le 0$$

- Since $v^n(x,t^n)$ is constant on $[x_{i-\frac12}, x_{i+\frac12}]$

$$\int_{x_{i-\frac12}}^{x_{i+1}} \Phi\left(v^n(x,t^n)\right)dx = \Delta x\, \Phi\left(v^n(x,t^n)\right)$$

$$= \Delta x\, \Phi\left(\frac{1}{\Delta x}\int_{x_{i-\frac12}}^{x_{i+\frac12}} v^n(x,t^n)\,dx\right)$$

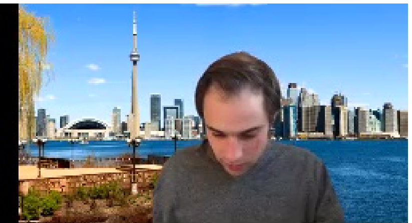
$$\overset{\text{def}}{=} \Delta x\, \Phi(u_i^n)$$

- Define

$$G(u_i^n, u_{i+1}^n) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \Psi\left(\underbrace{v^n(x_{i+\frac12},t)}\right)dt$$

$$= \Psi\left(v^n(x_{i+\frac12},t)\right)$$

$\uparrow$ v^n constant on $x_{i\pm\frac12}$ and $[t^n, t^{n+1}]$



v^n constant on $x_{i\pm\frac{1}{2}}$ and $[t^n, t^{n+1}]$

For $u_i^n = u_{i+1}^n \equiv \tilde{u}$

$$\Rightarrow \quad G(\hat{u}, \tilde{u}) = \Psi(\hat{u})$$

$\rightsquigarrow$ G is numerical entropy flux

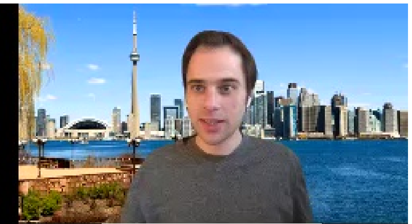
Then:

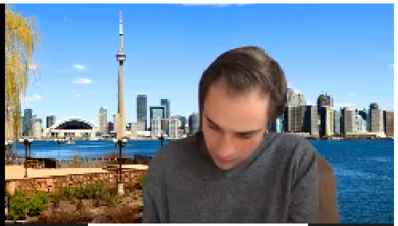
$$\Phi(u_i^{n+1}) = \Phi\left(\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v^n(x, t^{n+1})\, dx\right)$$

$$\overset{\Phi \text{ convex}}{\leq} \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \Phi(v^n(x, t^{n+1}))\, dx$$

$$\overset{(**)}{\leq} \Phi(u_i^n) - \frac{\Delta t}{\Delta x}\Big[G(u_i^n, u_{i+1}^n) - G(u_{i-1}^n, u_i^n)\Big]$$

$\Rightarrow$ scheme is consistent with the (discrete) entropy condition $\square$





Weak solutions under coordinate transformations

let $u = u(x,t)$ be a classical solution (C^1) solution to a system of CLs

$$u_t + f(u)_x = 0 \qquad (i)$$

Consider: $u \xrightarrow{h^{-1}} \tilde{u} = h^{-1}(u)$ change of conserved variables

$\rightsquigarrow$ then $\tilde{u}$ is a C^1 solution of

$$\tilde{u}_t + B(\tilde{u})\tilde{u}_x = 0,$$

$$B(v) = [Dh(v)]^{-1} Df(h(v))\, Dh(v) \qquad v \in \mathbb{R}^m$$

assume there exists a function g with

$$Dg(v) = B(v) \qquad \forall\, v \in$$

solution to a system of CLs

$$u_t + f(u)_x = 0 \qquad (i)$$

Consider: $\quad u \overset{h^{-1}}{\longmapsto} \tilde{u} = h^{-1}(u)$ change of conserved variables

$\leadsto$ then $\tilde{u}$ is a C^1 solution of

$$\tilde{u}_t + B(\tilde{u})\tilde{u}_x = 0,$$

$$(*) \qquad B(v) = [Dh(v)]^{-1} Df(h(v))\, Dh(v) \quad v \in \mathbb{R}^m$$

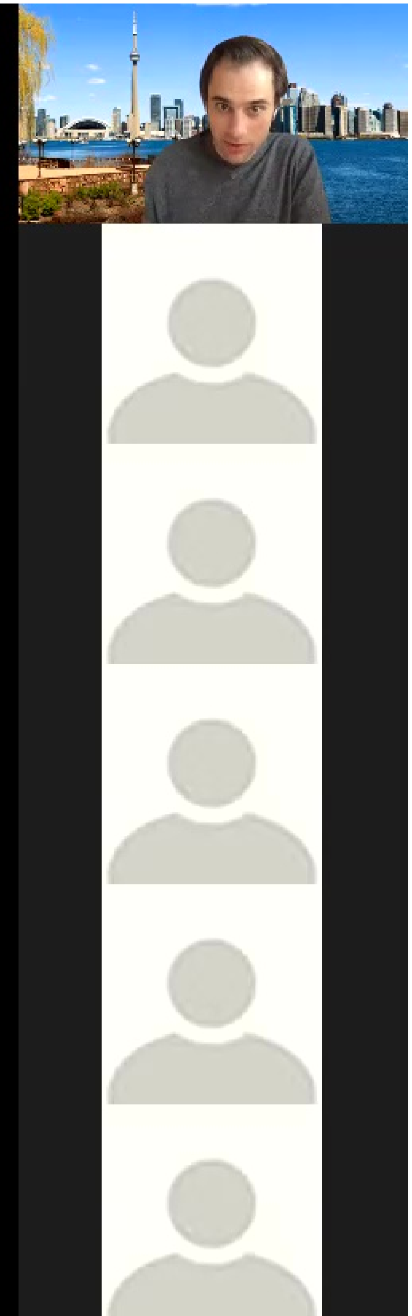
assume there exists a function g with

$$Dg(v) = B(v) \quad \forall\, v \in \mathbb{R}^m$$

$$\leadsto (*) \Leftrightarrow \tilde{u}_t + g(\tilde{u})_x = 0 \qquad (ii)$$

Note: • $\tilde{u}$ C^1 solution of (ii) iff $u = h(\tilde{u})$ is a classical solution to (i)

• this equivalence of solutions



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$u = u(\tilde{u})$ is a classical solution to (i)

- this equivalence of solutions breaks down for <u>weak solutions</u>

<u>Example:</u> Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0 \quad (\mathrm{I})$$

Consider: $\tilde{u} \equiv u^3 \rightsquigarrow g(\tilde{u}) = \frac{3}{4}\tilde{u}^{4/3}$

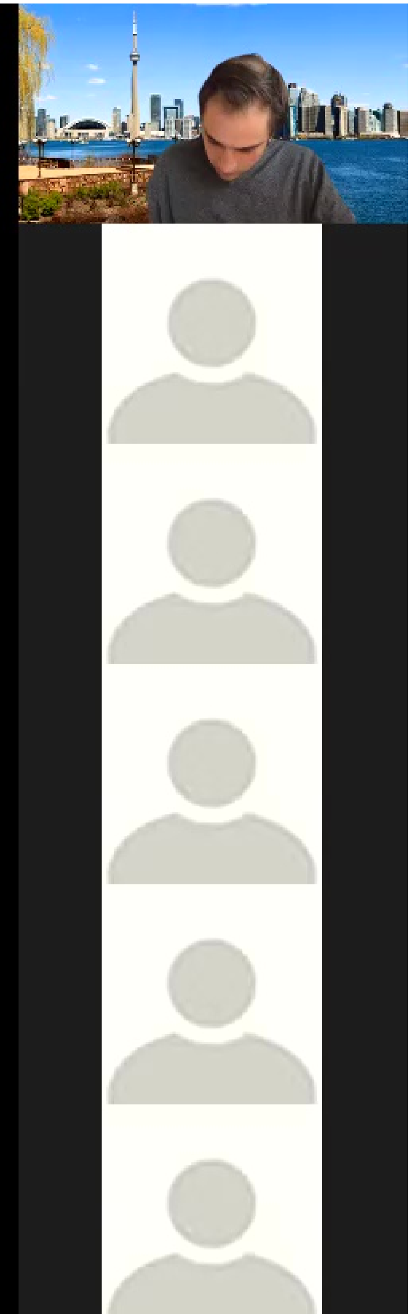
$$\left[\begin{aligned} &u_t + u u_x = 0 \\ &3u^2 u_t + 3u^3 u_x = 0 \\ &(u^3)_t + \left(\tfrac{3}{4}u^4\right)_x = 0 \end{aligned}\right.$$

$\rightsquigarrow \tilde{u}_t + g(\tilde{u})_x = 0 \rightsquigarrow$ conservation law for u^3

Consider Riemann problem with

$u_l > u_r$

$\rightsquigarrow$ unique weak solution is a sh



Example: Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0 \quad (I)$$

Consider: $\tilde{u} \equiv u^3$ $\rightsquigarrow$ $g(\tilde{u}) = \frac{3}{4}\tilde{u}^{4/3}$

$$u_t + u u_x = 0$$

$$3u^2 u_t + 3u^3 u_x = 0$$

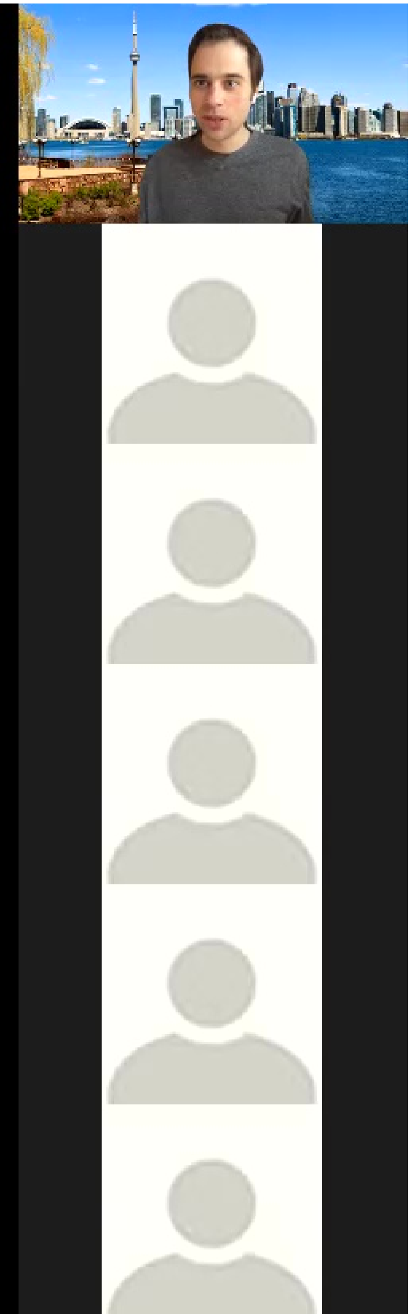
$$(u^3)_t + \left(\tfrac{3}{4}u^4\right)_x = 0$$

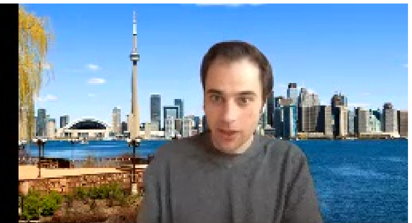
$\rightsquigarrow$ $\tilde{u}_t + g(\tilde{u})_x = 0$ $\rightsquigarrow$ conservation law for u^3

Consider Riemann problem with

$u_l > u_r$

$\rightsquigarrow$ unique weak solution is a shock





$$u_t + u u_x = 0$$

$$3u^2 u_t + 3u^3 u_x = 0$$

$$(u^3)_t + \left(\tfrac{3}{4} u^4\right)_x = 0$$

$\leadsto \quad \hat{u}_t + g(\hat{u})_x = 0 \quad \leadsto$ conservation law for u^3

Consider Riemann problem with

$u_\ell > u_r$

$\leadsto$ unique weak solution is a shock with speed

$$S_I = \frac{\frac{1}{2}u_\ell^2 - \frac{1}{2}u_r^2}{u_\ell - u_r} = \frac{1}{2}(u_\ell + u_r)$$

$$S_{II} = \frac{\frac{3}{4}u_\ell^4 - \frac{3}{4}u_r^4}{u_\ell^3 - u_r^3} = \frac{3}{4}\,\frac{u_\ell^4 - u_r^4}{u_\ell^3 - u_r^3} \neq S_I$$

if $u_\ell \neq u_r$

Consider: $\tilde{u} = u^3 \rightsquigarrow g(\tilde{u}) = \frac{3}{4}\tilde{u}^{4/3}$

$$\begin{cases} u_t + u u_x = 0 \\ 3u^2 u_t + 3u^3 u_x = 0 \\ (u^3)_t + \left(\frac{3}{4}u^4\right)_x = 0 \end{cases}$$

$\rightsquigarrow \tilde{u}_t + g(\tilde{u})_x = 0 \rightsquigarrow$ conservation law for u^3

Consider Riemann problem with

$u_l > u_r$

$\rightsquigarrow$ unique weak solution is a shock with speed

$$S_I = \frac{\frac{1}{2}u_l^2 - \frac{1}{2}u_r^2}{u_l - u_r} = \frac{1}{2}(u_l + u_r)$$

$$S_{II} = \frac{\frac{3}{4}u_l^4 - \frac{3}{4}u_r^4}{u_l^3 - u_r^3} = \frac{3}{4}\frac{u_l^4 - u_r^4}{u_l^3 - u_r^3} \neq S_I$$

if $u_l \neq u_r$

$\Rightarrow$ different weak solutions

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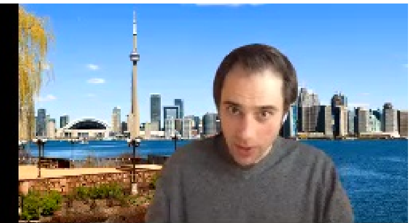
Chapter 7: Approximate Riemann solvers

<u>Idea:</u>

- Godunov-type methods are computationally expensive as one must solve a Riemann problem at every cell interface at each time step
- But: all we need is the local solution to the RP at the cell interfaces $v^n(x_{i+\frac{1}{2}}, t^n)$ which are constant on $[t^n, t^{n+1}]$
- replace "exact" Riemann solver by an "approximate" (cheaper)

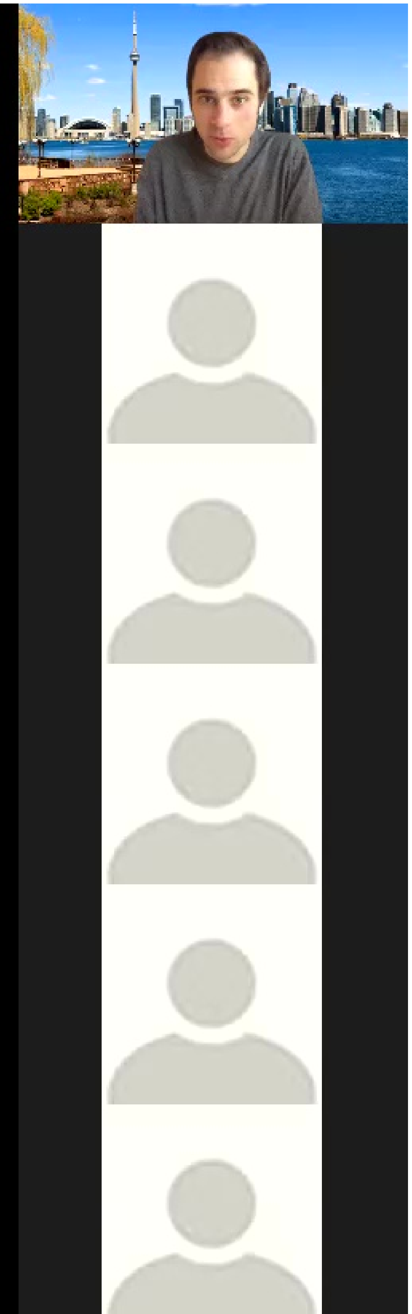


at every cell interface at each time step

- But: all we need is the local solution to the RP at the cell interfaces $v^n(x_{i\pm\frac{1}{2}}, t^n)$ which are constant on $[t^n, t^{n+1}]$

- replace "exact" Riemann solver by an "approximate" (cheaper) Riemann solver

$\longrightarrow$ defines class of Godunov-type methods



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by an "approximate" (cheaper)

Riemann solver

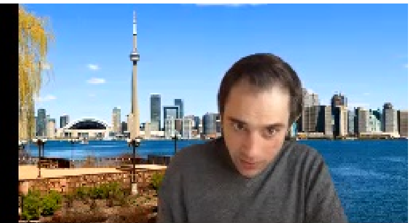
→ defines class of Godunov-type

methods

6.1 Roe's Riemann solver

<u>Idea</u>: replace actual Riemann problem by an approximate linearized problem defined locally at each cell

D



problem defined locally at each cell

<u>Def</u>: (Riemann solver of Roe):

Replace $u_t + f(u)_x = 0$

$$u(x,0) = \begin{cases} u_l = u^n_{i-1} & x < 0 \\ u_r = u^n_i & x > 0 \end{cases}$$

by the linear problem

$$(*) \quad w_t + A_{lr} w_x = 0$$

$$w(x,0) = \begin{cases} w_l & x < 0 \\ w_r & x \geq 0 \end{cases}$$

where $A_{lr} = A(u_l, u_r) \in Mat(\mathbb{R}^{m \times m})$

satisfies the conditions

$$u(x,0) = \begin{cases} u_l = u_{i-1}^n & x < 0 \\ u_r = u_i^n & x > 0 \end{cases}$$

by the linear problem

$$(*) \quad \begin{aligned} & w_t + A_{lr} w_x = 0 \\ & w(x,0) = \begin{cases} w_l & x < 0 \\ w_r & x \geq 0 \end{cases} \end{aligned}$$

where $A_{lr} = A(u_l, u_r) \in \mathrm{Mat}(\mathbb{R}^{m \times m})$ satisfies the conditions

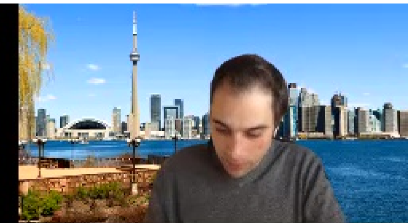
(i) $f(v) - f(w) = A(v,w)(v-w)$ $\leftarrow$

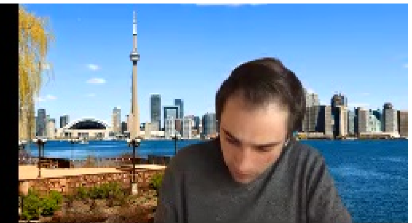
(conservation across discontinuities)

(ii) $\|A(v,w) - Df(v)\| \to 0$ as $\|w - v\| \to 0$

(consistency with the exact Jacobian)

(iii) $A(v,w)$ has only real eigenvalues





where $A_{\ell r} = A(u_\ell, u_r) \in \mathrm{Mat}(\mathbb{R} \quad)$

satisfies the conditions

(i) $f(v) - f(w) = A(v,w)(v-w)$ $\leftarrow$

(conservation across discontinuities)

(ii) $\|A(v,w) - Df(v)\| \to 0$ as $\|w-v\| \to 0$

(consistency with the exact Jacobian)

(iii) $A(v,w)$ has only real eigenvalues and a complete set of eigenvectors

The Roe scheme is defined by replacing the exact solution to the local RP in Godunov's method by the exact solution to $*$