Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 17

Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

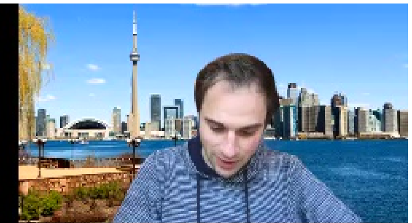
Date: November 12, 2020 - 3:30 PM

URL: http://pirsa.org/20110012

Def (Conservative scheme): Let $f \in C^1(\mathbb{R}^m; \mathbb{R}^m)$ and $g \in C^{0,1}(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$ be consistent with the system of CLs (I), i.e., $g(v,v) = f(v) \ \forall v \in \mathbb{R}^m$.
Assume we have a numerical grid $\mathcal{G} = \{(i\Delta x, n\Delta t) \mid i \in \mathbb{Z}, n \in \mathbb{N}\}$ and discretized initial data $u_i^0 \in \mathbb{R}^m$.
A scheme of the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\right] \Leftarrow$$

is said to be in conservation form with numerical flux g.



Lecture_17

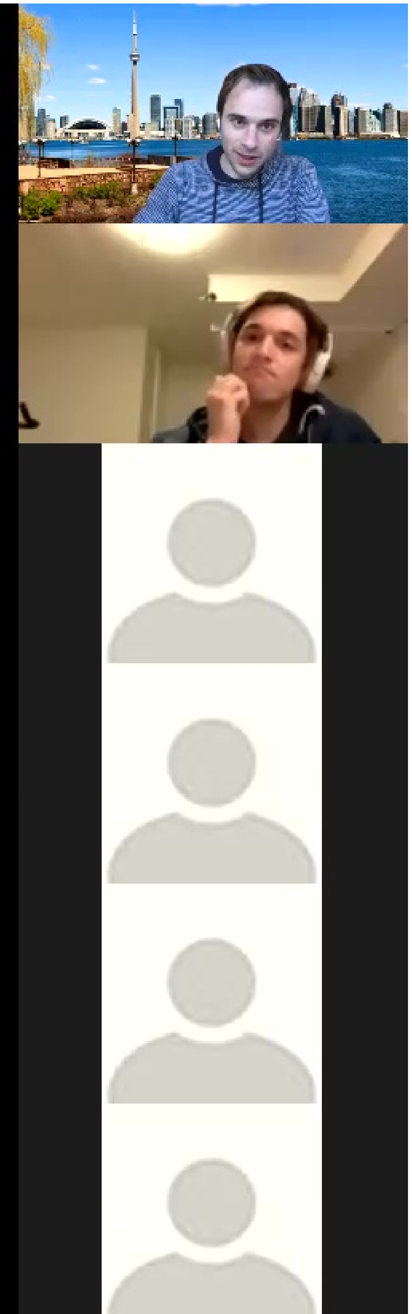
$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\right] \Leftarrow$$

is said to be in **conservation form** with **numerical flux g**.

Examples: Some straightforward choices for the numerical flux (scalar CLs): remember $g(v,v) = f(v)$

	g	$\frac{1}{\Delta x}\left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right)$
(i)	$g(u,v) = f(u)$	$\frac{f(u_i) - f(u_{i-1})}{\Delta x}$
(ii)	$g(u,v) = f(v)$	$\frac{f(u_{i+1}) - f(u_i)}{\Delta x}$
(iii)	$g(u,v) = \frac{1}{2}\left(f(u) + f(v)\right)$	$\frac{f(u_{i+1}) - f(u_{i-1})}{2\Delta x}$

$\leadsto$



9:41 AM Tue Jan 9 37%

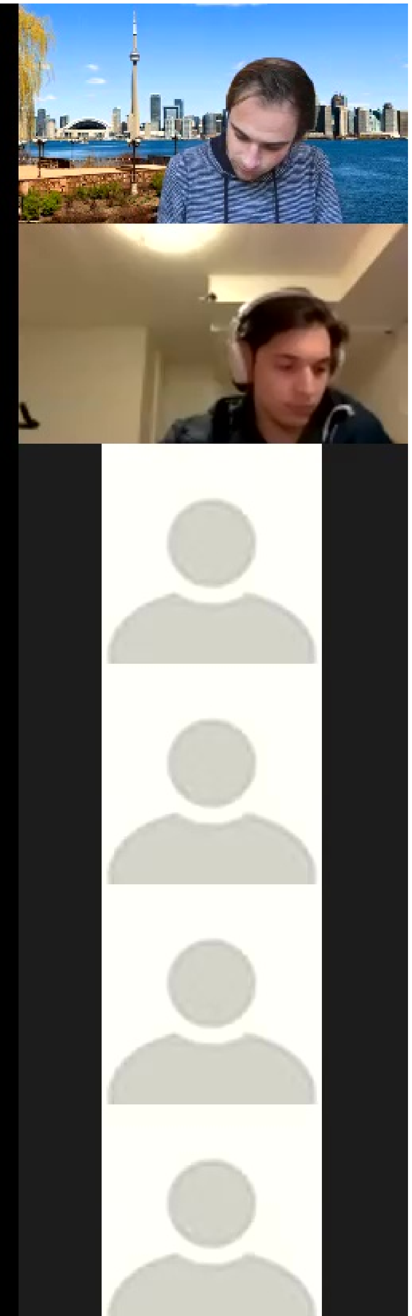
Lecture_17

Lecture_17 | Lecture_16 | Chap_6_NumericalSchem... | Leveque - Finite Volume M...

(iii) $g(u,v) = \frac{1}{2}\left(f(u) + f(v)\right)$ | $\dfrac{f(u_{i+1}) - f(u_{i-1})}{2\Delta x}$

$\rightsquigarrow$ can interpret conservative schemes as backward, forward, centred difference schemes with u_i^n approximating volume averages

- depending on f' (i) or (ii) can be unstable (violate the CFL condition)
  also (iii) is generally unstable
- Lax Friedrichs method: can (iii)
  a stable conservative sch



9:41 AM Tue Jan 9 38%

Lecture_17

Lecture_17 | Lecture_16 | Chap_6_NumericalSchem... | Leveque - Finite Volume M...

- <u>Lax Friedrichs method</u>: can (iii)

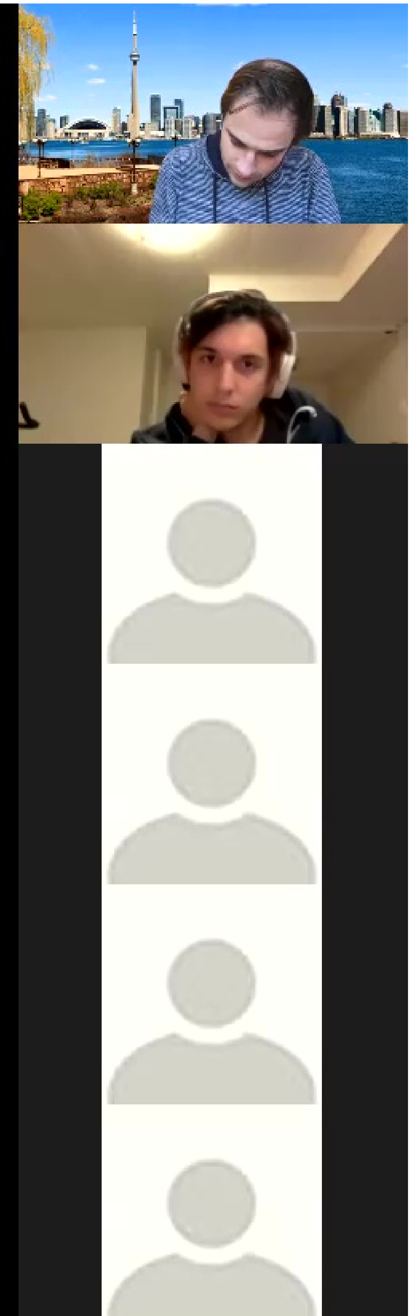
a stable conservative scheme

$$u_i^n \to \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right)$$

$$\rightsquigarrow u_i^{n+1} = \underbrace{\frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right)} - \frac{\Delta t}{2\Delta x}\left[f(u_{i+1}^n) - f(u_{i-1}^n)\right]$$

$$= u_i^n + \underbrace{\frac{1}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}_{\simeq\, \Delta x\, u_{xx}}$$

$\rightsquigarrow$ numerical



9:41 AM Tue Jan 9

Lecture_17

Lecture_17 | Lecture_16 | Chap_6_NumericalSchem... | Leveque - Finite Volume M...

- <u>Lax Friedrichs method</u>: can (iii)

a stable conservative scheme

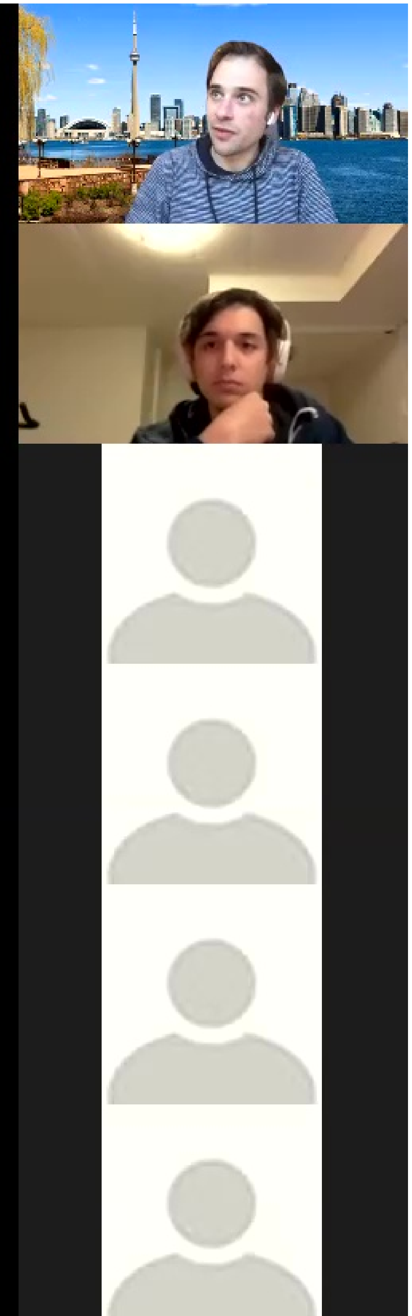
$$u_i^n \to \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right)$$

$$\rightsquigarrow u_i^{n+1} = \underbrace{\frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right)}_{} - \frac{\Delta t}{2\Delta x}\left[f(u_{i+1}^n) - f(u_{i-1}^n)\right]$$

$$= u_i^n + \underbrace{\frac{1}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}_{\simeq \Delta x \, u_{xx}}$$

$\to 0$ as $\Delta x \to 0$

$\rightsquigarrow$ numerical dissipation / damping



$$\rightsquigarrow u_i^{n+1} = \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left[f(u_{i+1}^n) - f(u_{i-1}^n)\right]$$

$$= u_i^n + \frac{1}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

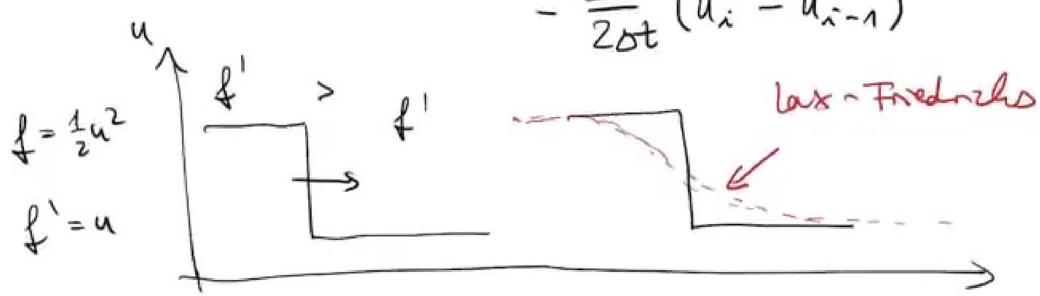
$$\simeq \Delta x \, u_{xx}$$

$\to 0$ as $\Delta x \to 0$

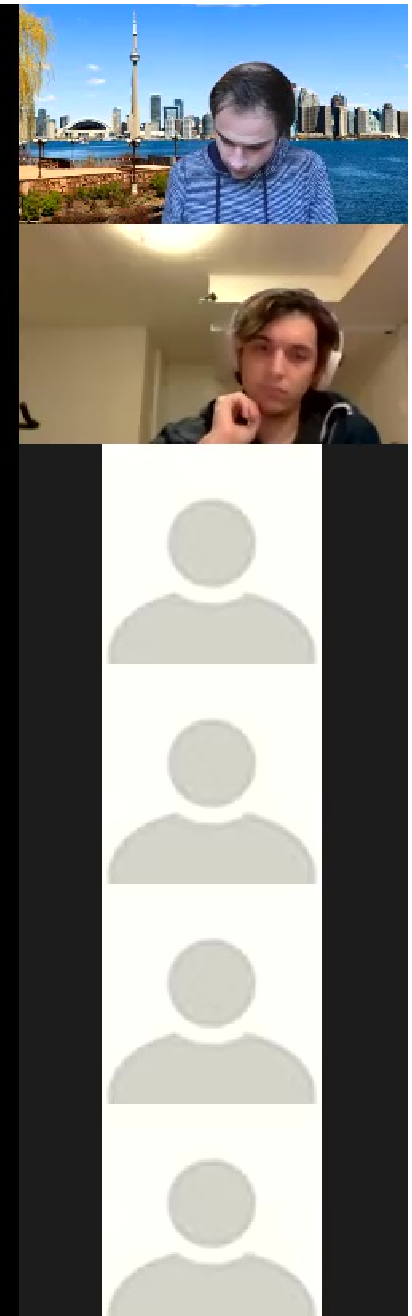
$\rightsquigarrow$ numerical dissipation / damping

write in conservative form:

$$g(u_{i-1}^n, u_i^n) = \frac{1}{2}\left[f(u_{i+1}^n) + f(u_{i-1}^n)\right] - \frac{\Delta x}{2\Delta t}\left(u_i^n - u_{i-1}^n\right)$$



Remark: Importance of Conservation form

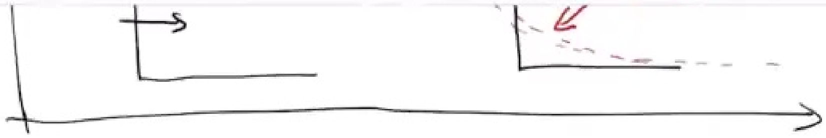


9:41 AM Tue Jan 9

Lecture_17

Lecture_17 | Lecture_16 | Chap_6_NumericalSchem... | Leveque - Finite Volume M

$f' = u$



Remark: Importance of Conservation form

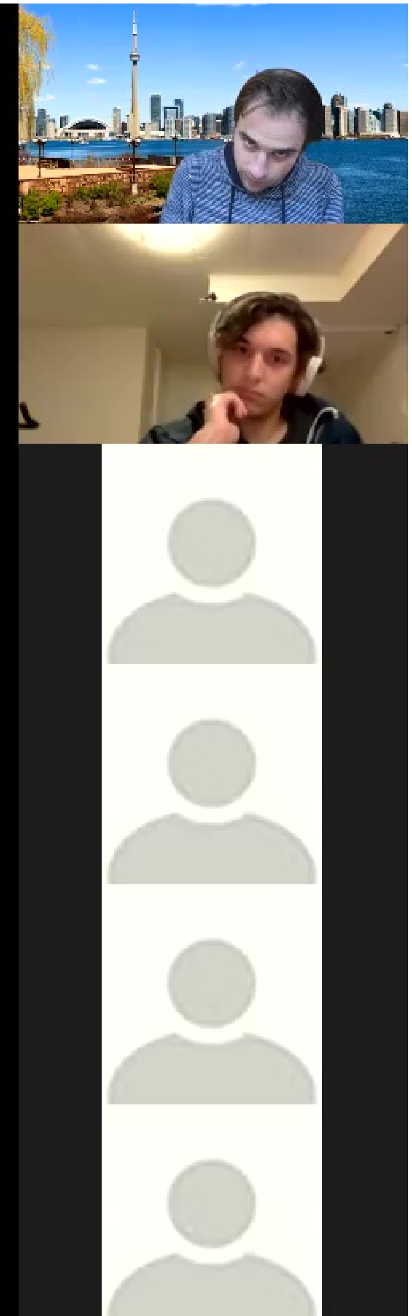
Consider Burgers' equation, assume $u>0$.

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$

Conservative upwind:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right)$$

Quasi-linear (non-conserva



Conservative upwind:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right)$$

Quasi-linear (non-conservative): $u_t + u u_x = 0$

'upwind':

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n)$$

$$= u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right)$$

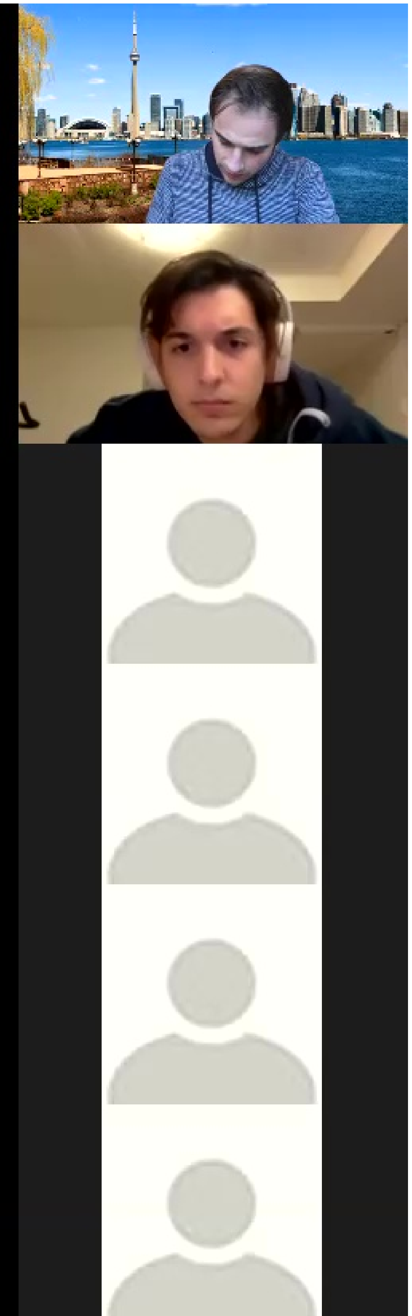
$$+ \frac{1}{2}\Delta t \Delta x \left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right)^2$$

$$\simeq \frac{1}{2}\Delta x (u_x)^2$$

$\uparrow$

$\Delta x, \Delta t \to 0$

does not vanish for shocks (u_x is not bounded)



Conservative upwind:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right)$$

Quasi-linear (non-conservative): $u_t + u u_x = 0$

'upwind': $$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n)$$

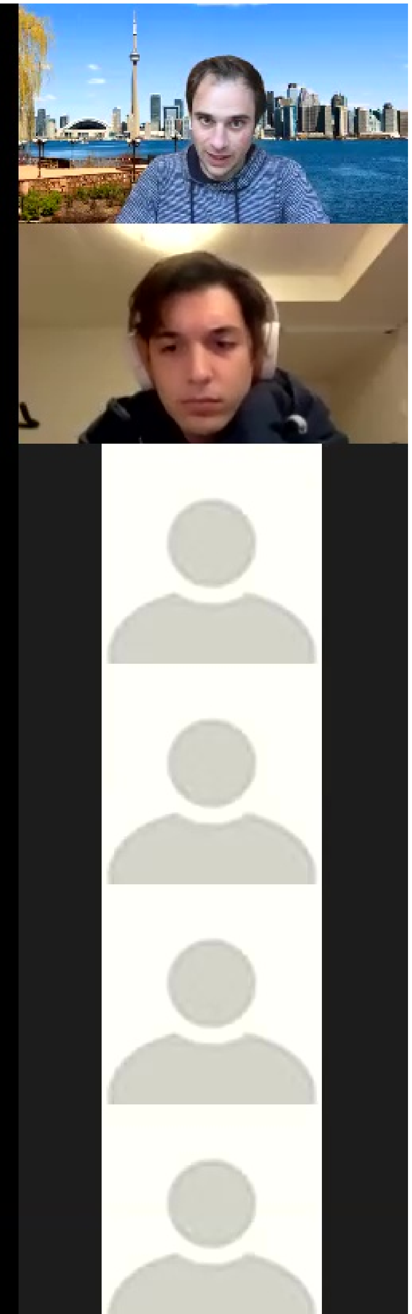
$$= u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right)$$

$$+ \frac{1}{2}\Delta t \Delta x \left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right)^2$$

$$\simeq \frac{1}{2}\Delta x \,(u_x)^2$$

↑ $\Delta x, \Delta t \to 0$

does not vanish for shocks (u_x is not bounded)



of the Rankine–Hugoniot conditions (see Section 11.8) that govern the form and speed of shock waves. It thus makes sense that a conservative method based on the integral form might be more successful than other methods based on the differential equation. In fact, we will see that the use of conservative finite volume methods is essential in computing weak solutions to conservation laws. Nonconservative methods can fail, as illustrated below. With conservative methods, one has the satisfaction of knowing that if the method converges to some limiting function as the grid is refined, then this function is a weak solution. This is further explained and proved in Section 12.10 in the form of the *Lax–Wendroff theorem*.

In Section 12.11 we will see that similar ideas can be used to show that the limiting function also satisfies the entropy condition, provided the numerical method satisfies a natural discrete version of the entropy condition.

Consider Burgers' equation $u_t + \frac{1}{2}(u^2)_x = 0$, for example. If $u > 0$ everywhere, then the conservative upwind method (Godunov's method) takes the form

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(U_i^n)^2 - \frac{1}{2}(U_{i-1}^n)^2\right). \qquad (12.24)$$

On the other hand, using the quasilinear form $u_t + uu_x = 0$, we could derive the nonconservative upwind method

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n). \qquad (12.25)$$

On smooth solutions, both of these methods are first-order accurate, and they give comparable results. When the solution contains a shock wave, the method (12.25) fails to converge to a weak solution of the conservation law. This is illustrated in Figure 12.5. The conservative method (12.24) gives a slightly smeared approximation to the shock, but it is smeared about the correct location. We can easily see that it must be, since the method has the discrete conservation property (4.8). The nonconservative method (12.25), on the other hand, gives the results shown in Figure 12.5(b). These clearly do not satisfy (4.8), and as the grid is

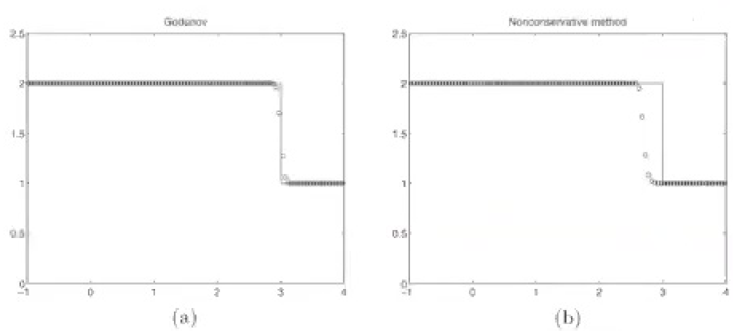
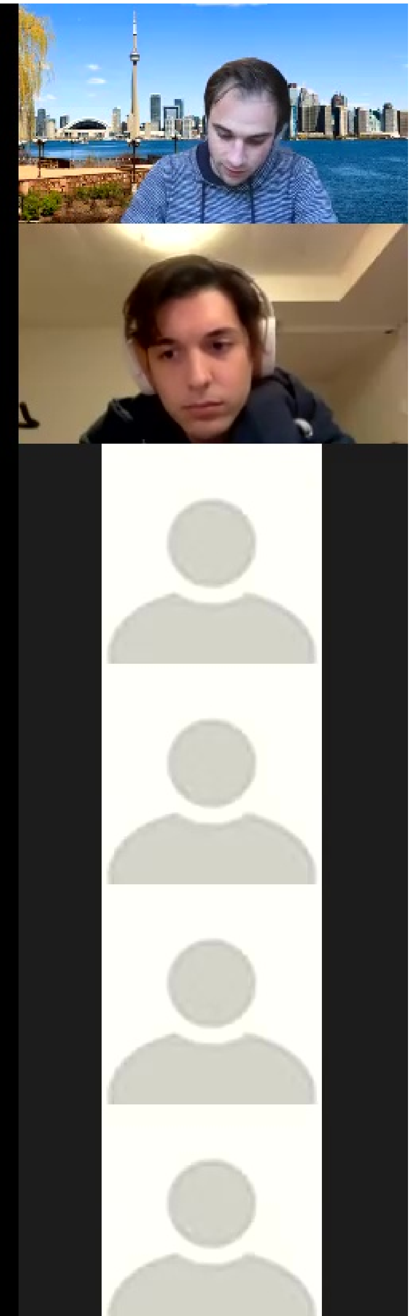


Fig. 12.5. True and computed solutions to a Riemann problem for Burgers' equation with data $u_l = 2$, $u_r = 1$, shown at time $t = 2$: (a) using the conservative method (12.24), (b) using the nonconservative method (12.25). [claw/book/chap12/nonconservative]



'upwind':

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n \left(u_i^n - u_{i-1}^n\right)$$

$$= u_i^n - \frac{\Delta t}{\Delta x} \left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right) + \frac{1}{2}\Delta t \Delta x \left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right)^2$$

$$\simeq \frac{1}{2}\Delta x \,(u_x)^2$$

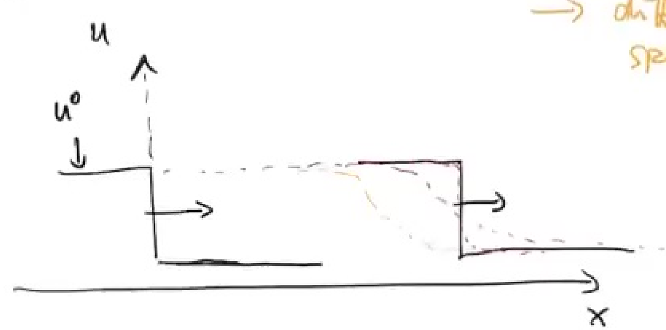
$\uparrow$

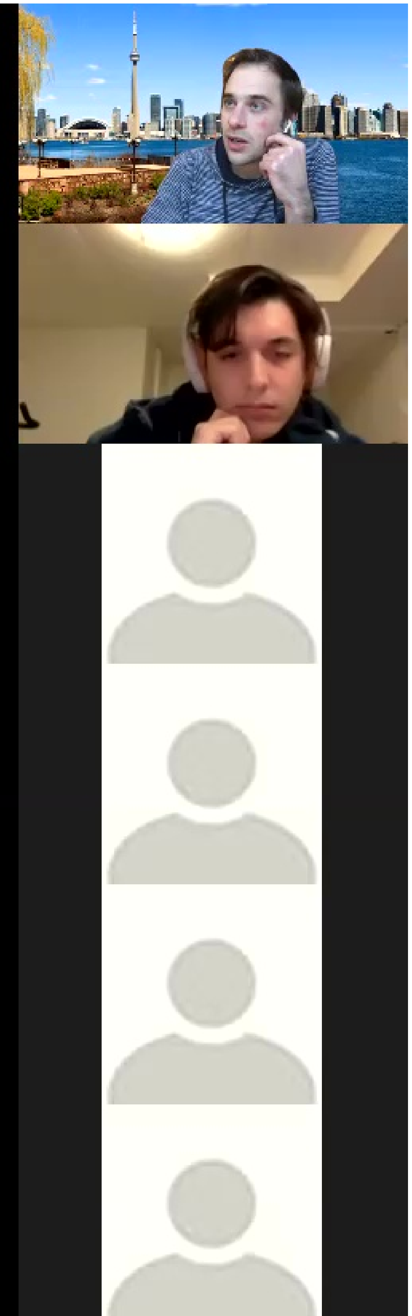
$\Delta x, \Delta t \to 0$

does not vanish for shocks (u_x is not bounded)

→ different shock speed

$f' > 0$





some limiting function as the grid is refined, then this function is a weak solution. This is further explained and proved in Section 12.10 in the form of the *Lax–Wendroff theorem*.

In Section 12.11 we will see that similar ideas can be used to show that the limiting function also satisfies the entropy condition, provided the numerical method satisfies a natural discrete version of the entropy condition.

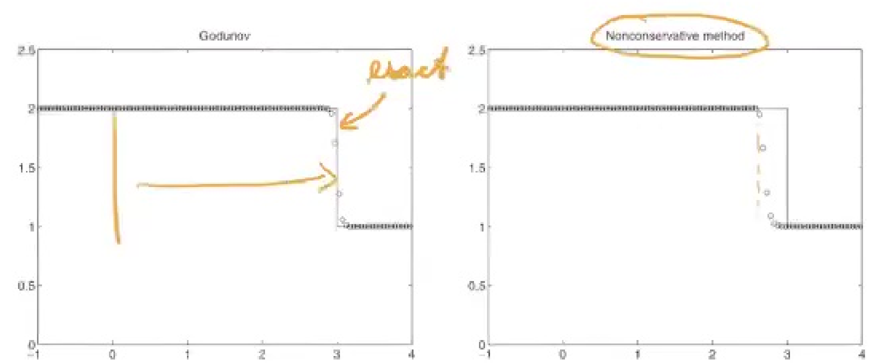
Consider Burgers' equation $u_t + \frac{1}{2}(u^2)_x = 0$, for example. If $u > 0$ everywhere, then the conservative upwind method (Godunov's method) takes the form

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(U_i^n)^2 - \frac{1}{2}(U_{i-1}^n)^2\right). \tag{12.24}$$

On the other hand, using the quasilinear form $u_t + uu_x = 0$, we could derive the nonconservative upwind method

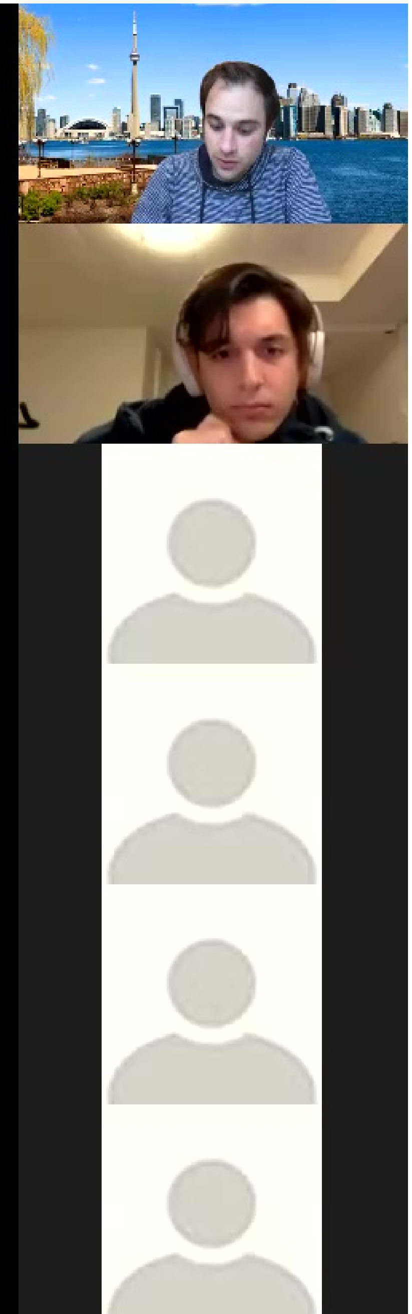
$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n). \tag{12.25}$$

On smooth solutions, both of these methods are first-order accurate, and they give comparable results. When the solution contains a shock wave, the method (12.25) fails to converge to a weak solution of the conservation law. This is illustrated in Figure 12.5. The conservative method (12.24) gives a slightly smeared approximation to the shock, but it is smeared about the correct location. We can easily see that it must be, since the method has the discrete conservation property (4.8). The nonconservative method (12.25), on the other hand, gives the results shown in Figure 12.5(b). These clearly do not satisfy (4.8), and as the grid is



(a) (b)

Fig. 12.5. True and computed solutions to a Riemann problem for Burgers' equation with data $u_l = 2$, $u_r = 1$, shown at time $t = 2$: (a) using the conservative method (12.24), (b) using the nonconservative method (12.25). [claw/book/chap12/nonconservative]



some limiting function as the grid is refined, then this function is a weak solution. This is further explained and proved in Section 12.10 in the form of the *Lax–Wendroff theorem*.

In Section 12.11 we will see that similar ideas can be used to show that the limiting function also satisfies the entropy condition, provided the numerical method satisfies a natural discrete version of the entropy condition.

Consider Burgers' equation $u_t + \frac{1}{2}(u^2)_x = 0$, for example. If $u > 0$ everywhere, then the conservative upwind method (Godunov's method) takes the form

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(U_i^n)^2 - \frac{1}{2}(U_{i-1}^n)^2\right). \tag{12.24}$$

On the other hand, using the quasilinear form $u_t + uu_x = 0$, we could derive the nonconservative upwind method

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}U_i^n(U_i^n - U_{i-1}^n). \tag{12.25}$$

On smooth solutions, both of these methods are first-order accurate, and they give comparable results. When the solution contains a shock wave, the method (12.25) fails to converge to a weak solution of the conservation law. This is illustrated in Figure 12.5. The conservative method (12.24) gives a slightly smeared approximation to the shock, but it is smeared about the correct location. We can easily see that it must be, since the method has the discrete conservation property (4.8). The nonconservative method (12.25), on the other hand, gives the results shown in Figure 12.5(b). These clearly do not satisfy (4.8), and as the grid is

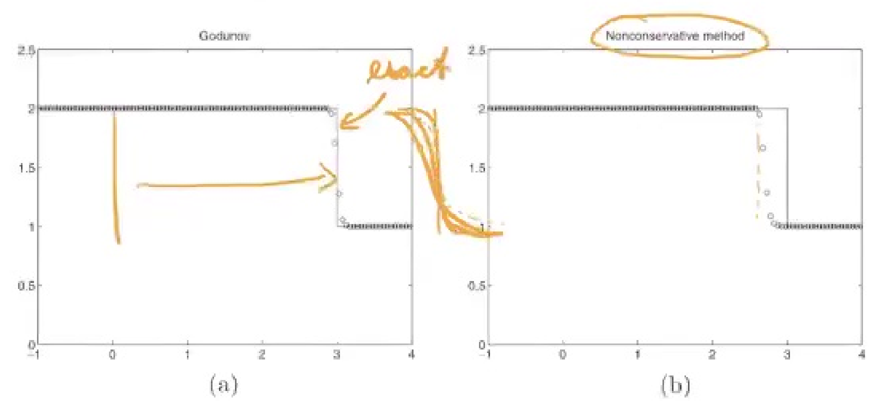
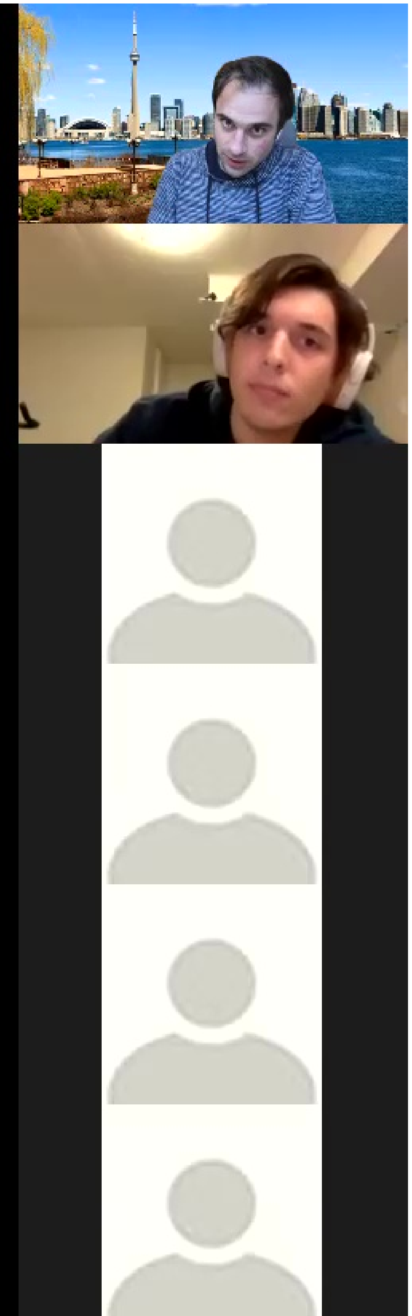


Fig. 12.5. True and computed solutions to a Riemann problem for Burgers' equation with data $u_l = 2$, $u_r = 1$, shown at time $t = 2$: (a) using the conservative method (12.24), (b) using the nonconservative method (12.25). [claw/book/chap12/nonconservative]

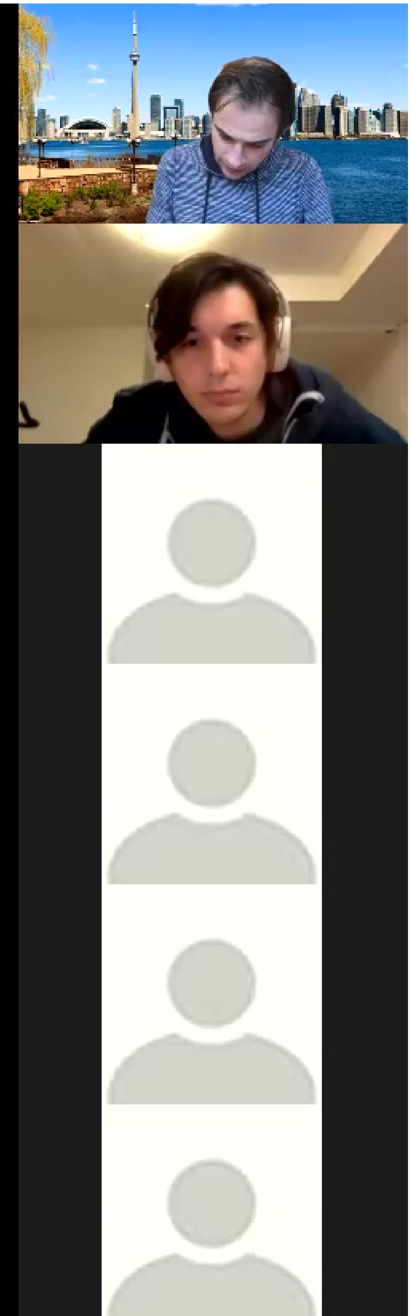


Def (Discrete entropy condition):

Let (Φ, Ψ) be an entropy pair for the system of CLs $u_t + f(u)_x = 0$ and let $G \in C^{0,1}(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$ such that $G(v,v) = \Psi(v)$ $\forall\, v \in \mathbb{R}^m$. Let u_i^n be the solution of a numerical scheme in conservation form. Then u_i^n is said to satisfy a <u>discrete entropy condition</u> iff:

$$\Phi(u_i^{n+1}) - \Phi(u_i^n) \leq -\theta \frac{\Delta t}{\Delta x}\Big[G(u_i^n, u_{i+1}^n) - G(u_{i-1}^n, u_i^n)\Big] - (1-\theta)\frac{\Delta t}{\Delta x}\Big[G(u_i^{n+1}, u_{i+1}^{n+1})$$

$-$



9:41 AM Tue Jan 9

Lecture_17

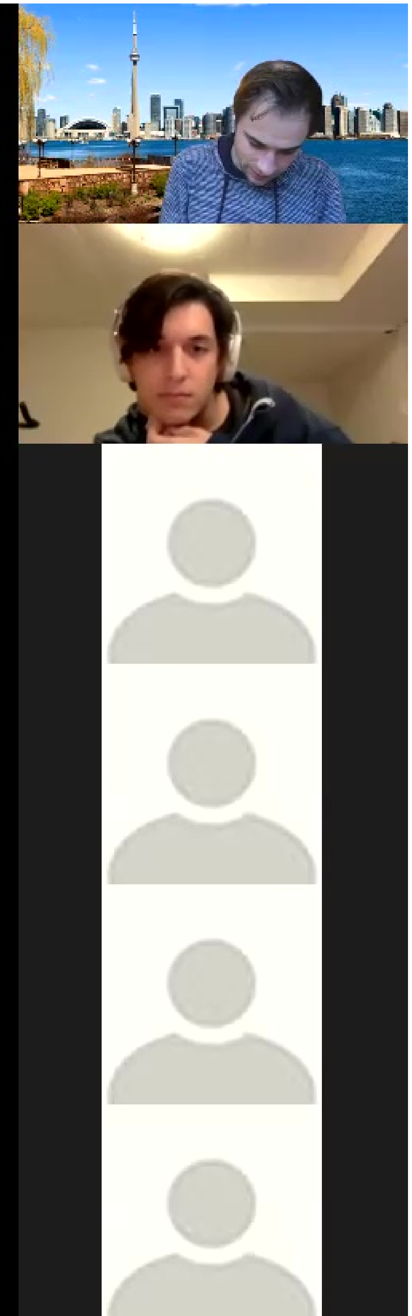
Lecture_17 | Lecture_16 | Chap_6_NumericalSchem... | Leveque - Finite Volume M...

numerical scheme in conservation form. Then u_i^n is said to satisfy a <u>discrete entropy condition</u> if:

$$\Phi(u_i^{n+1}) - \Phi(u_i^n) \leq -\theta \frac{\Delta t}{\Delta x}\left[G(u_i^n, u_{i+1}^n) - G(u_{i-1}^n, u_i^n)\right]$$

$$(\ast) \qquad - (1-\theta)\frac{\Delta t}{\Delta x}\left[G(u_i^{n+1}, u_{i+1}^{n+1}) - G(u_{i-1}^{n+1}, u_i^{n+1})\right]$$

<u>G is called numerical entropy flux</u>. The scheme is said to be consistent with the entropy condition if $(\ast)$ holds uniformly for $\Delta t, \Delta x \to 0$ for any Φ and G.

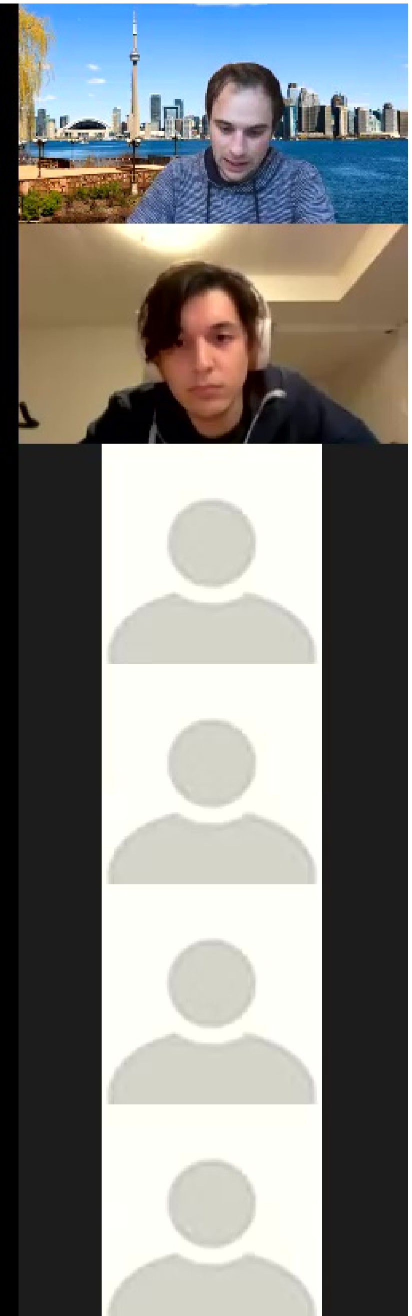


(*)

$$- (1-\theta)\frac{\Delta t}{\Delta x}\Big[G\big(u_i^{n+1}, u_{i+1}^{n+1}\big) - G\big(u_{i-1}^{n+1}, u_i^{n+1}\big)\Big]$$

<u>G is called numerical entropy flux.</u> The scheme is said to be consistent with the entropy condition if (*) holds uniformly for $\Delta t, \Delta x \to 0$ for any Φ and G.

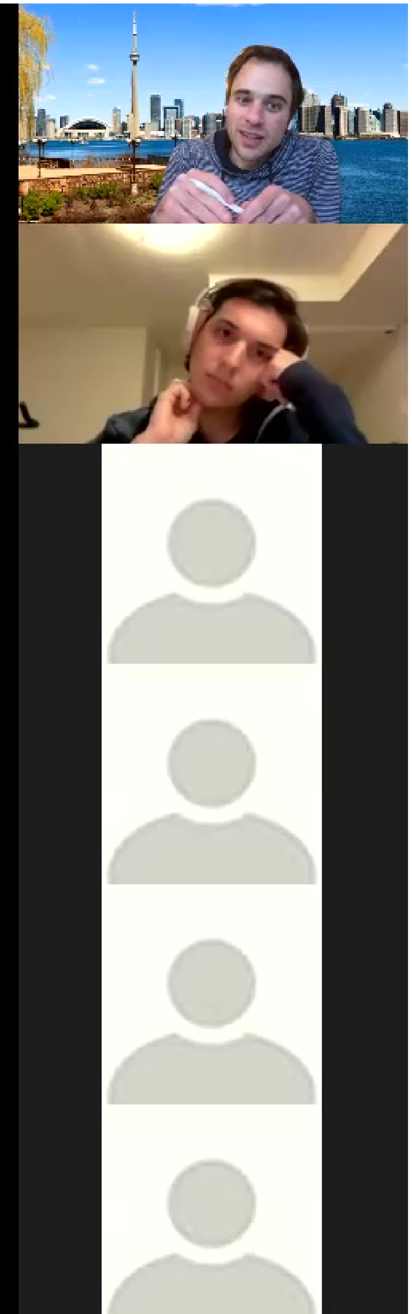
<u>Theorem</u> (Convergence to entropy solution): Consider situation of Lax-Wendroff theorem and assume that the scheme is consistent with the entropy condition. Then the scheme converges to a weak solution that satisfies the entropy condition.



Lecture_17

6.3 Godunov's method for nonlinear systems of CLs

Motivation:



9:41 AM Tue Jan 9

Lecture_17

Lecture_17 | Lecture_16 | Chap_6_NumericalSchem... | Leveque - Finite Volume M...

nonlinear systems of CLs

Motivation: find conservative scheme that does not try to eliminate discontinuities but rather exploits them

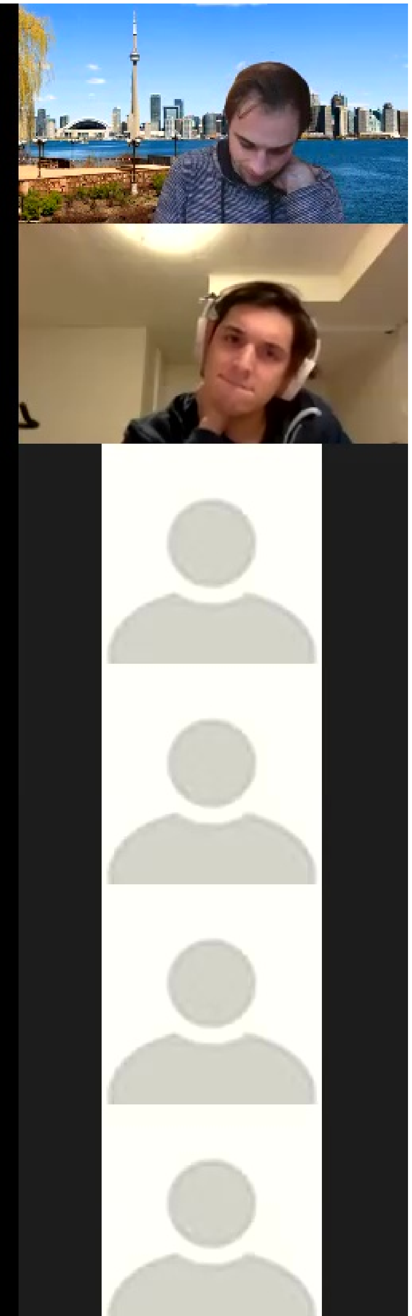
Idea: REA: reconstruct – evolve – average

1. Reconstruct a piecewise polynomial function $u^n(x,t^n)$ from given cell-averages u_i^n

   → simplest case piece-wise constant funct

   $$u^n(x,t^n) \equiv u_i^n \,, \; x_{i-\frac{1}{2}} < x \le x_{i+\frac{1}{2}}$$

2.



→ simplest case piece-wise constant funct

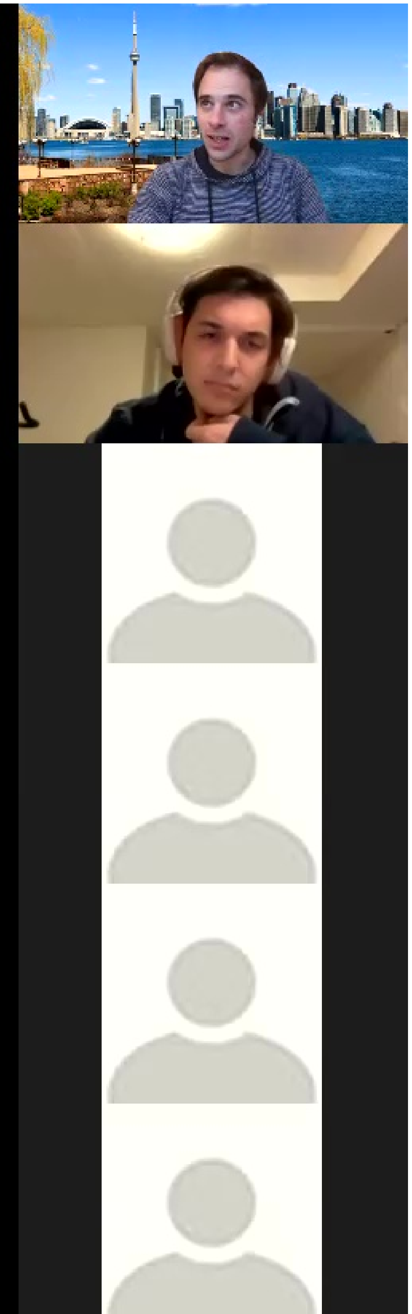
$$u^n(x,t^n) \equiv u_i^n \,, \quad x_{i-\frac{1}{2}} < x \leq x_{i+\frac{1}{2}}$$

2. Evolve the system of CLs exactly (or approximately) with this initial data to obtain $\underline{u^{n+1}(x,t^{n+1})}$ at $t^n + \Delta t = t^{n+1}$ (series of Riemann problems)

3. Average function over grid cell to obtain new cell averges

$$u_i^{n+1} \equiv \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^{n+1}(x,t^{n+1})\, dx$$

and start over with step 1.



9:41 AM Tue Jan 9

Lecture_17

Lecture_17 | Lecture_16 | Chap_6_NumericalSchem... | Leveque - Finite Volume M...

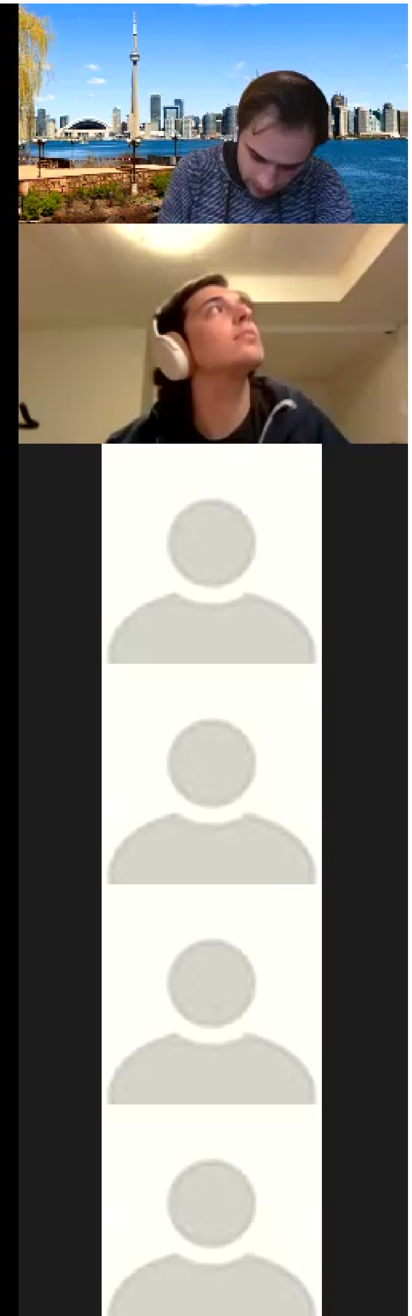
at $t^n + \Delta t = t^{n+1}$ (series of Riemann problems)

3. Average function over grid cell to obtain new cell averages

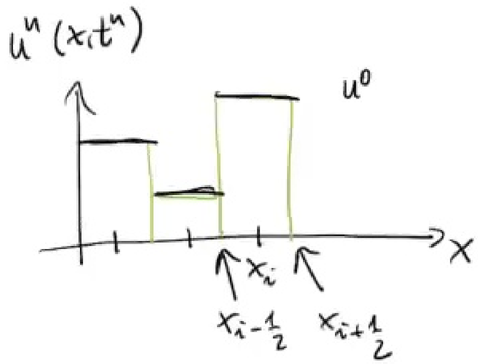
$$u_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^{n+1}(x, t^{n+1})\, dx$$

and start over with step 1.

In detail: spatial grid of finite volumes: $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$



In detail: spatial grid of finite volumes: $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$



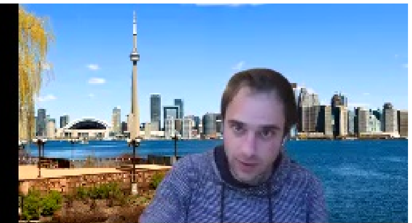
Assume $u^0(x) \in L^1(\mathbb{R})$ given and define

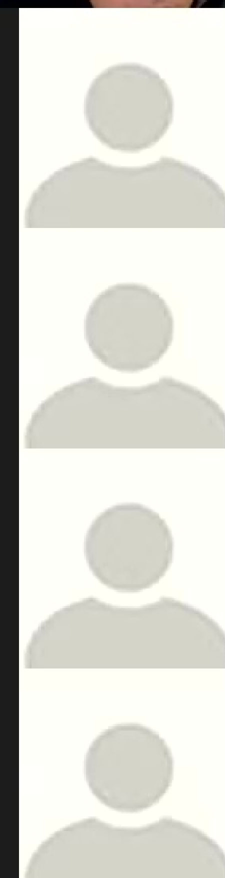
$$u_i^0 \equiv \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^0(x)\,dx$$

$\rightsquigarrow$ u_i^n is known

- Reconstruct piecewise constant function

$$u^n(x,t^n) \equiv u_i^n \quad \text{for} \quad x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$





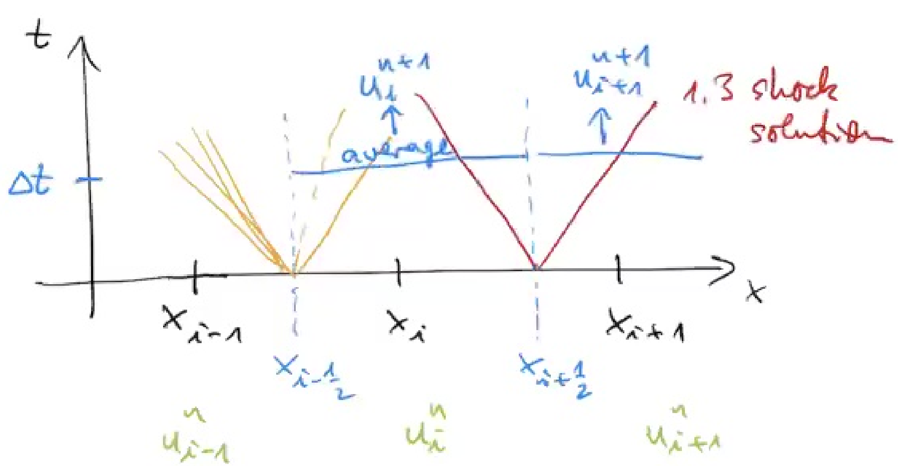
• Reconstruct piecewise constant function

$$u^n(x,t^n) \equiv u_i^n \quad \text{for} \quad x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$

• For each cell (x_i, x_{i+1}) solve the local Riemann problem

$$u_t + f(u)_x = 0$$

$$u(x, t=t^n) = \begin{cases} u_i^n, & x < x_{i+\frac{1}{2}} \\ u_{i+1}^n, & x > x_{i+\frac{1}{2}} \end{cases}$$



9:41 AM Tue Jan 9

Lecture_17

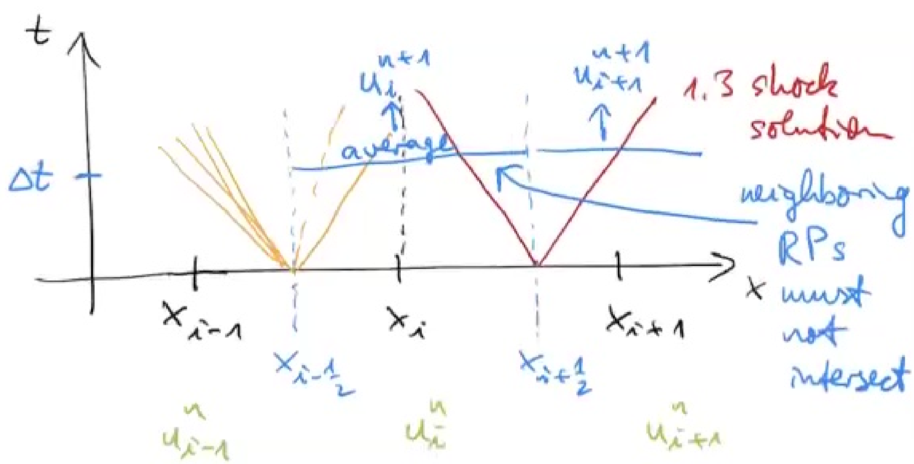
Lecture_17 | Lecture_16 | Chap_6_NumericalSchem... | Leveque - Finite Volume M...

$$u^n(x, t^n) \equiv u_i^n \quad \text{for} \quad x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$

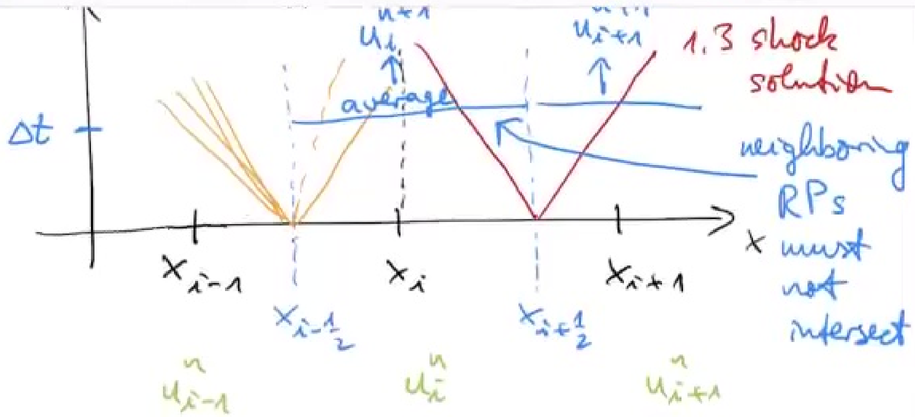
- For each cell (x_i, x_{i+1}) solve the local Riemann problem

$$u_t + f(u)_x = 0$$

$$u(x, t=t^n) = \begin{cases} u_i^n, & x < x_{i+\frac{1}{2}} \\ u_{i+1}^n, & x > x_{i+\frac{1}{2}} \end{cases}$$



→ obtain exact (or approximate solution w,



→ obtain exact (or approximate) solution $w_i^n(x,t)$

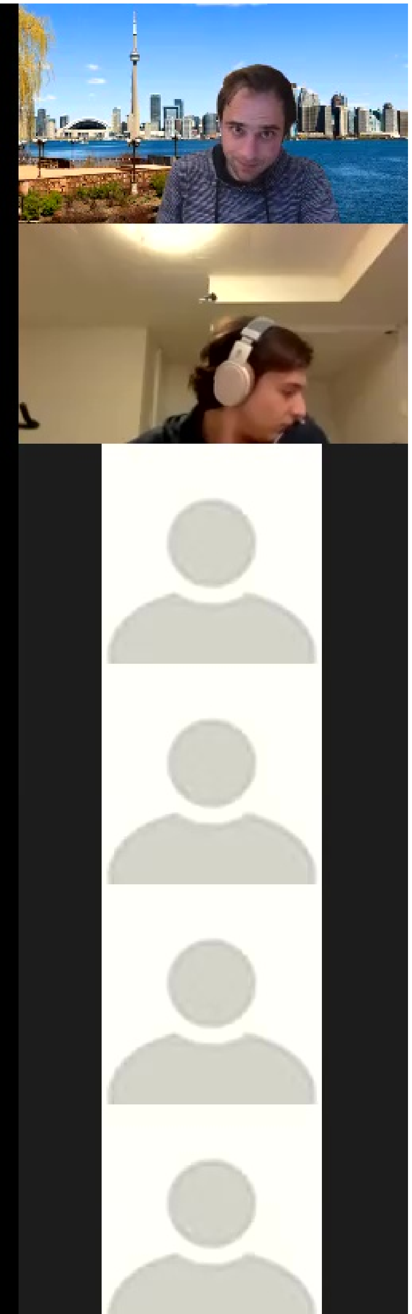
→ define global solution

$$v^n(x,t) \equiv w_i^n(x,t) \qquad t^n \le t \le t^{n+1}, \quad x_i \le x \le x_{i+\frac12}$$

must ensure that shocks with

$$\lambda_k(u_i^n) < s_{i+\frac12} < \lambda_k(u_{i+1}^n)$$

or rarefaction waves with speeds $\lambda_k(u_i^n)$



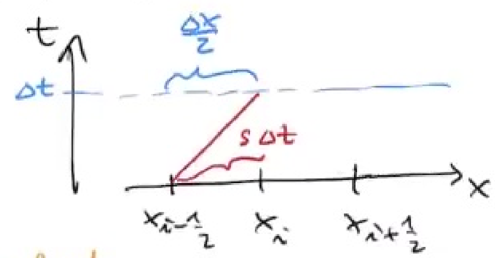
9:41 AM Tue Jan 9

Lecture_17

Lecture_17 | Lecture_16 | Chap_6_NumericalSchem... | Leveque - Finite Volume M...

$$\lambda_k(u_i^n) < s_{i+\frac{1}{2}} < \lambda_k(u_{i+1}^n)$$

or rarefaction waves with speeds $\lambda_k(u_i^n)$ do not intersect



maximum signal speed throughout the domain

$$\max\{|\lambda_k| \mid k \in \{1, \dots, m\}\}\, \Delta t \leq \frac{\Delta x}{2}$$

"CFL-like" condition

- Compute new cell averages

$$u_i^{n+1} \equiv \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v^n(x, t^{n+1})\, dx$$

$$= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w_{i-1}^n (x, t^{n+1})\, dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w_i^n (x, t^{n+1})\, dx$$