Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 16

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Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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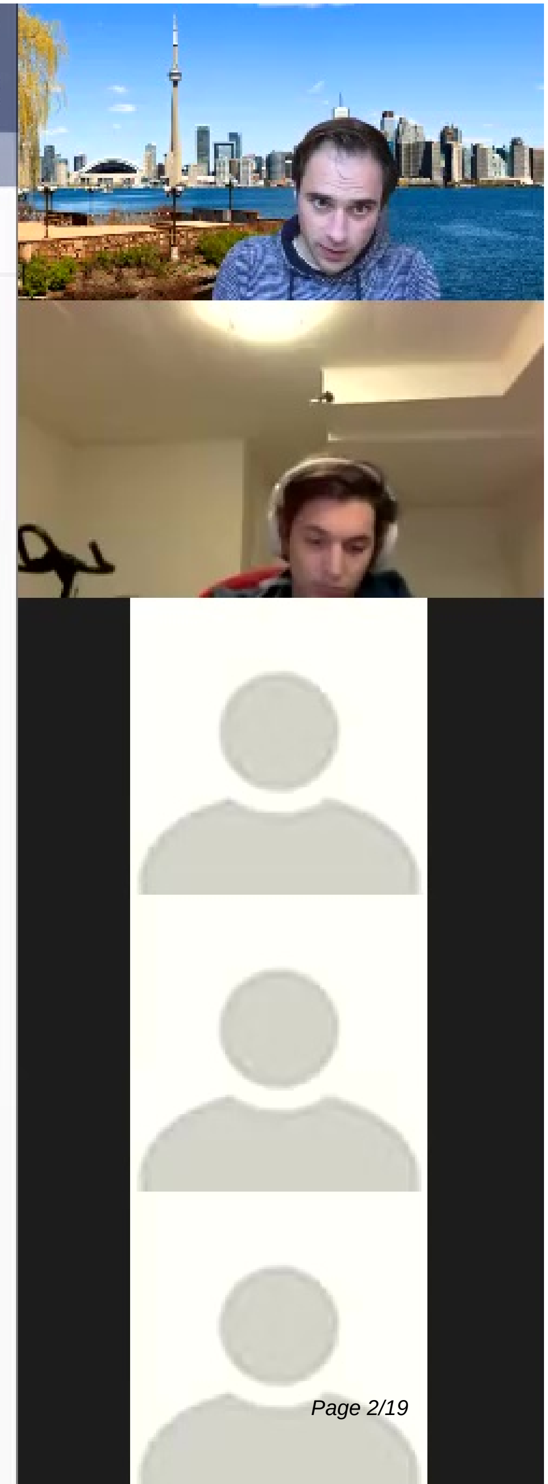
Chap. 6: Numerical schemes for hyperbolic systems of CLs

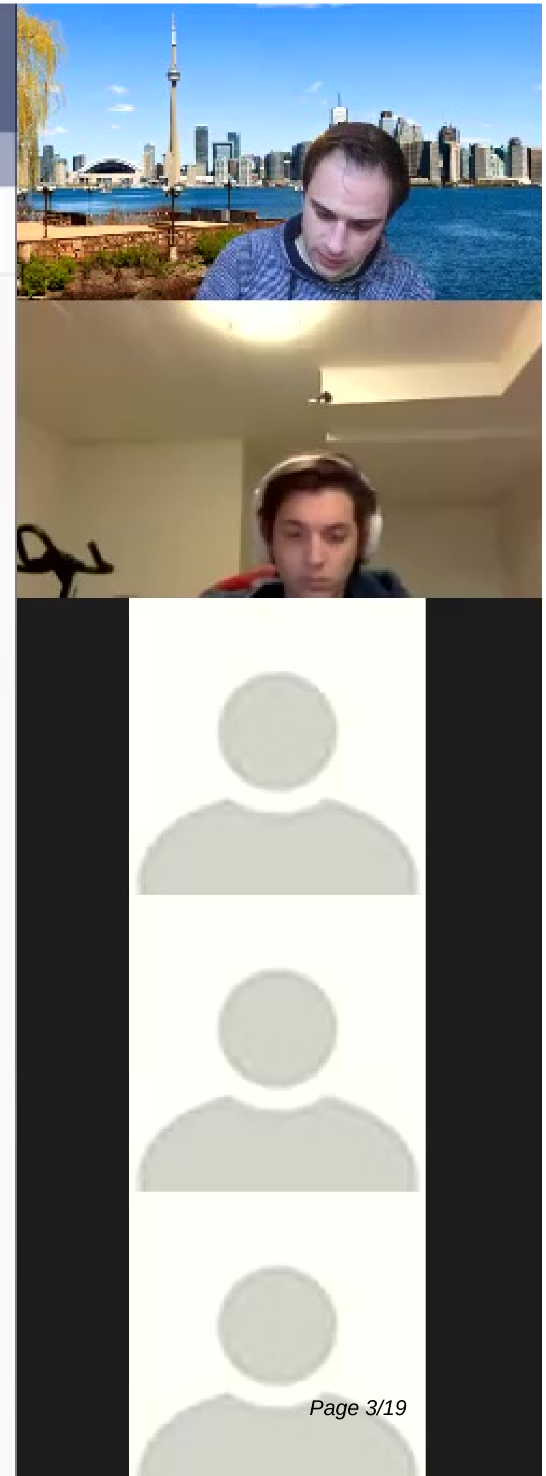
Consider hyperbolic system

$$\text{(I)} \qquad u_t + f(u)_x = 0 \qquad \text{in } \mathbb{R} \times (0,\infty)$$

$$\text{(II)} \qquad u(x,0) = u^0(x) \qquad \text{in } \mathbb{R} \times \{0\}$$

6.1 Conservative schemes

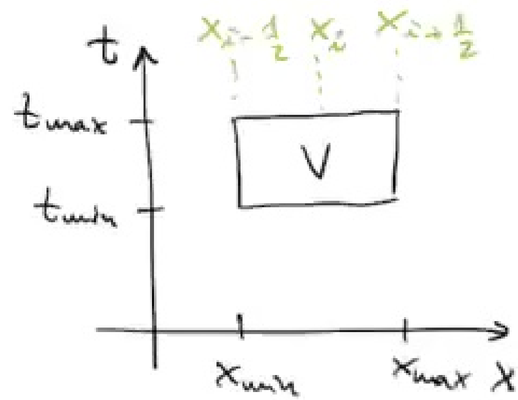




6.1 Conservative schemes

Motivation:

Integrate (I) over control volume V:



$$\iint \left[u_t + f(u)_x\right] dt\, dx$$

$$= \int \left[u(x, t_{max}) - u(x, t_{min})\right] dx$$

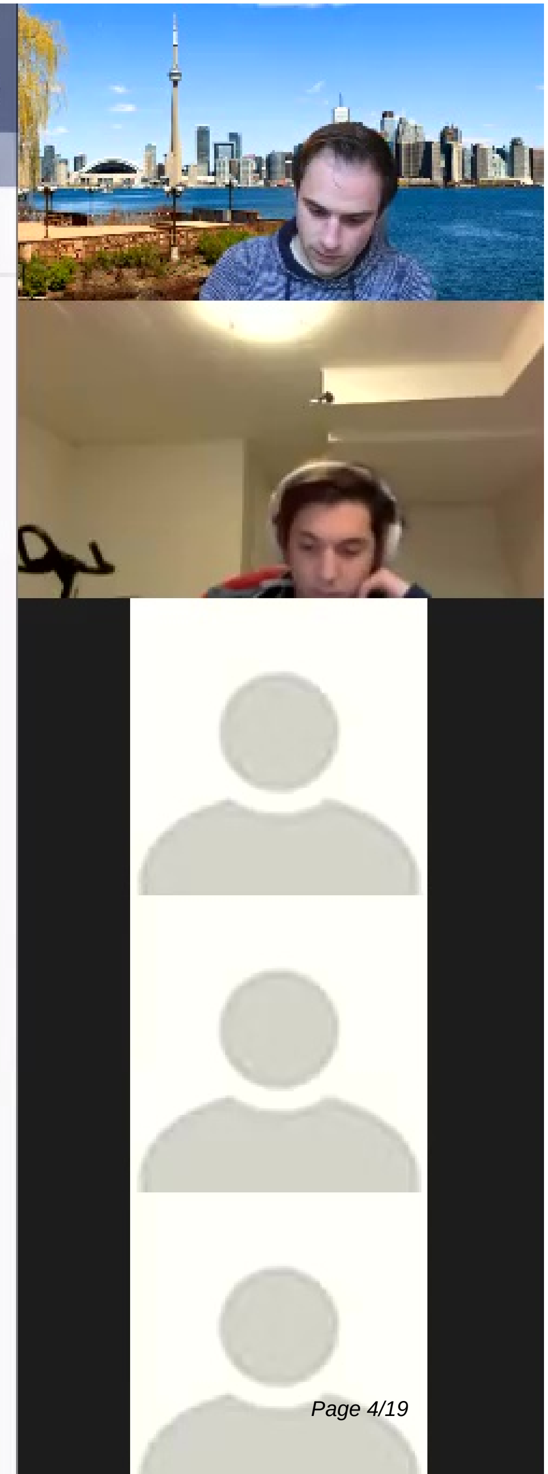
$$(\ast)\quad \underbrace{\int_{x_{i-\frac12}}^{x_{i+\frac12}} \left[u(x,t^{n+1}) - u(x,t^n)\right] dx}_{\substack{\text{change in } u \\ \text{in the volume} \\ [x_{i-\frac12},\, x_{i+\frac12}] \text{ during} \\ [t^n,\, t^{n+1}]}} + \underbrace{\int_{t^n}^{t^{n+1}} \left[f(u(x_{i+\frac12},t)) - f(u(x_{i-\frac12},t))\right] dt}_{\substack{\text{flow difference} \\ \text{through cell} \\ \text{interfaces during} \\ [t^n,\, t^{n+1}]}} = 0$$

Assume there is a function $g: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$

("numerical flux") such that:

average conserved variable

$$u_i^n = \frac{1}{\Delta x} \int_{x_{i-\frac12}}^{x_{i+\frac12}} u(x,t^n)\, dx$$



average flux through interface in Δt

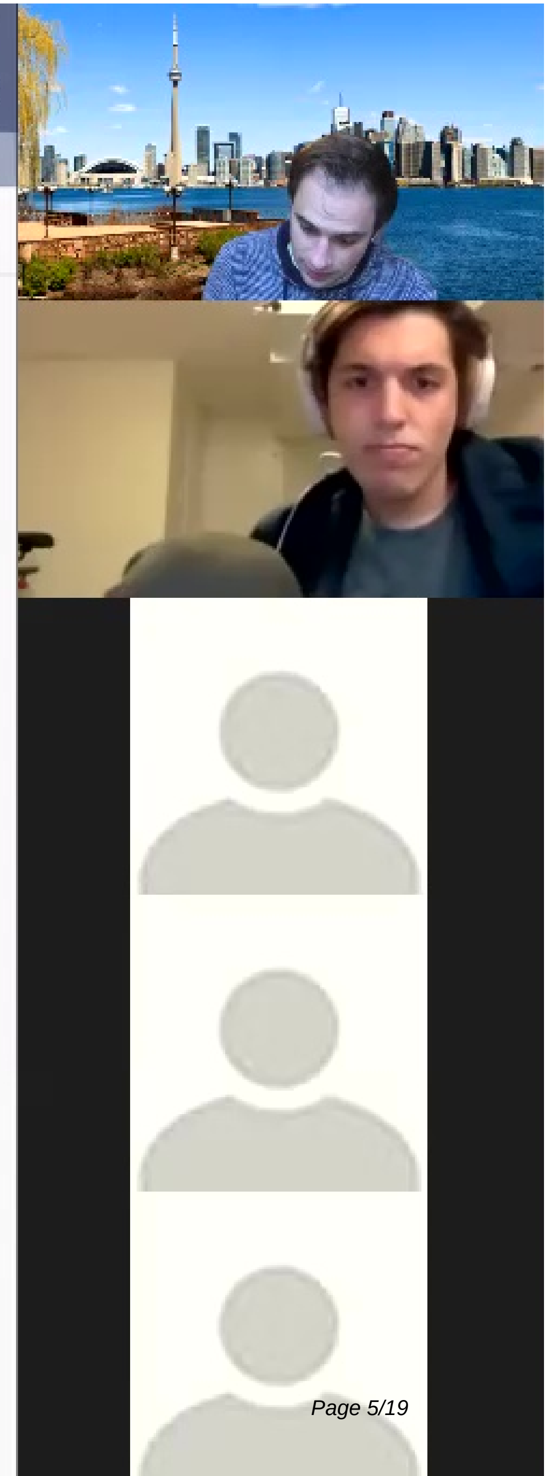
$$g^n_{i+\frac{1}{2}} \equiv g(\underbrace{u^n_i, u^n_{i+1}}) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}},t))\,dt$$

$$g^n_{i-\frac{1}{2}} \equiv g(u^n_{i-1}, u^n_i) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i-\frac{1}{2}},t))\,dt$$

$\rightsquigarrow$ (*) $$u^{n+1}_i - u^n_i = -\frac{\Delta t}{\Delta x}\left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right)$$

<u>Consistency</u>: • We need to ensure that g is consistent with physical flux:

$$g(v,v) = f(v) \quad \forall\, v \in \mathbb{R}^m$$



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Consistency: • We need to ensure that g is consistent with physical flux:

$$g(v,v) = f(v) \quad \forall\, v \in \mathbb{R}^m$$

This is because:

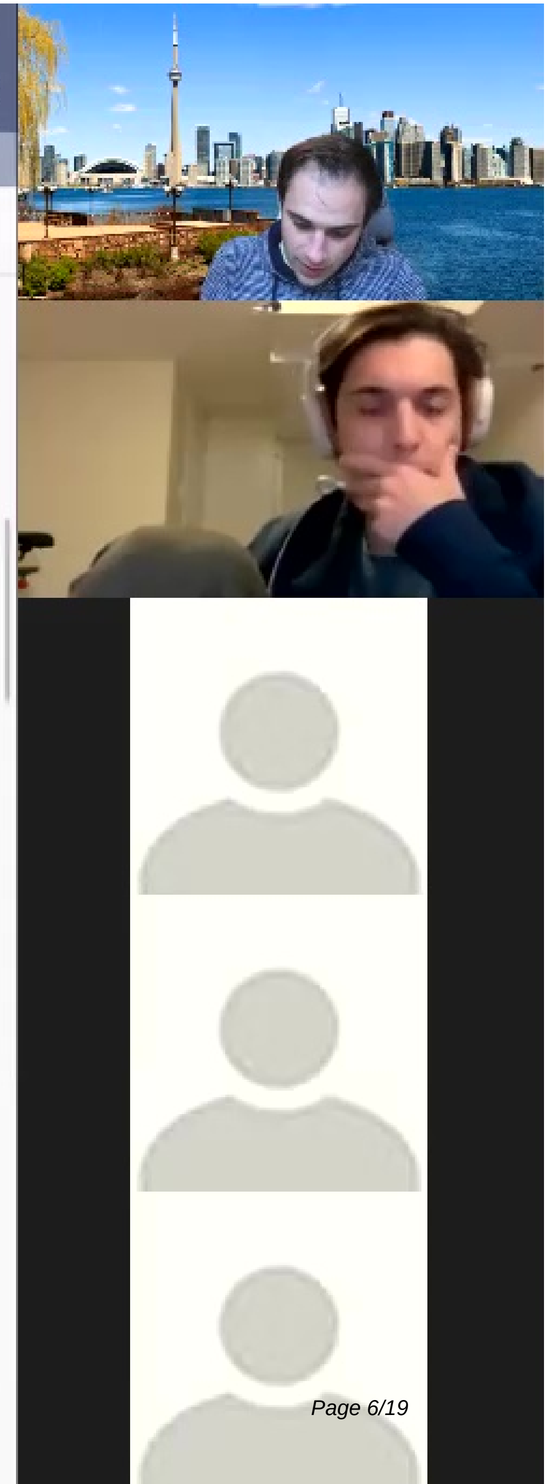
consider $u^0(x) = \text{const.}$

$\Rightarrow$ $u = u^0$ is a solution of CL

and $g^0_{i+\frac{1}{2}} = g(u^0_i, u^0_{i+1}) = g(u_0, u_0)$

$$g^0_{i+\frac{1}{2}} = \frac{1}{\Delta t}\int_0^{\Delta t} f(u(x_{i+\frac{1}{2}}, t))\, dt$$

$$= \frac{1}{\Delta t}\int_0^{\Delta t} f(u_0)\, dt = f(u_0)$$



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$$= \frac{1}{\Delta t}\int_0^{\Delta t} f(u_0)\,dt = f(u_0)$$

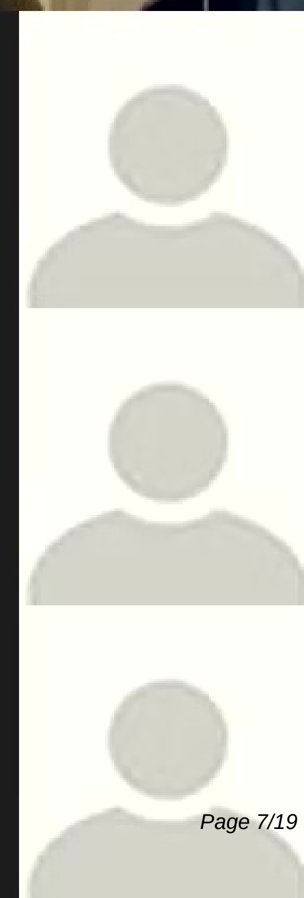
- Also need to expect continuity as u_{i+1}, u_i vary, i.e.

$$g(u_i, u_{i+1}) \longrightarrow f(v) \quad \text{as } u_{i+1}, u_i \to v$$

⇝ require Lipschitz continuity: there exist constants L_1, L_2 such that

$$\|g(u_{i+1}, u_i) - f(v)\| \le L_1 \|u_{i+1} - v\|^\alpha$$

$$+ L_2 \|u_i - v\|^\alpha$$



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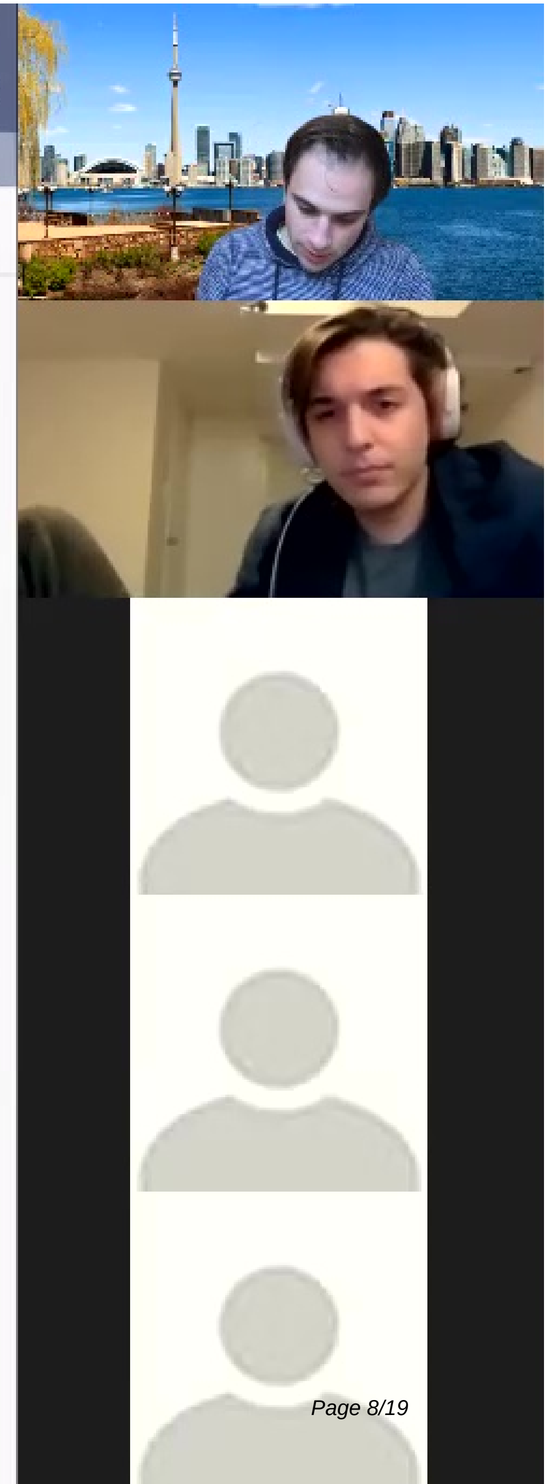

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there exist constants L_1, L_2 such that

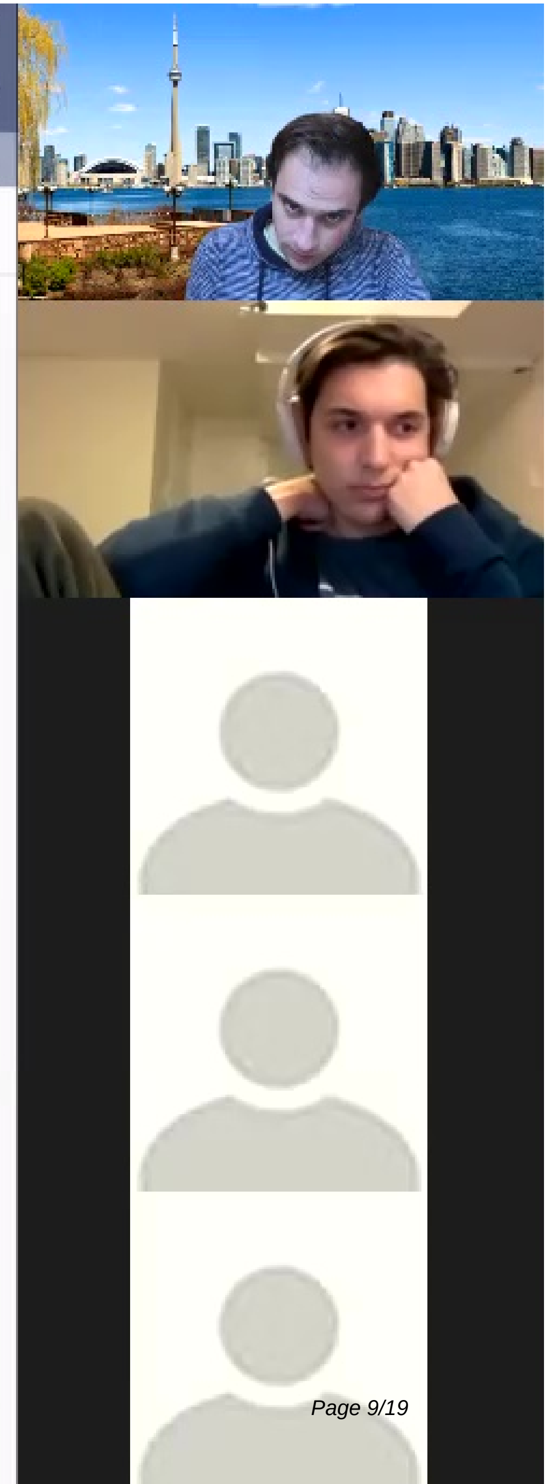
$$\|g(u_{i+1}, u_i) - f(v)\| \le L_1 \|u_{i+1} - v\|^\alpha + L_2 \|u_i - v\|^\alpha$$

typically: $\alpha = 1$

Def (Conservative scheme):

Let $f \in C^1(\mathbb{R}^m; \mathbb{R}^m)$ and $g \in C^{0,1}(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$ be consistent with the system of CLs (I), i.e. $g(v,v) = f(v)$ $\forall v \in \mathbb{R}^m$.

Assume we have a numerical grid



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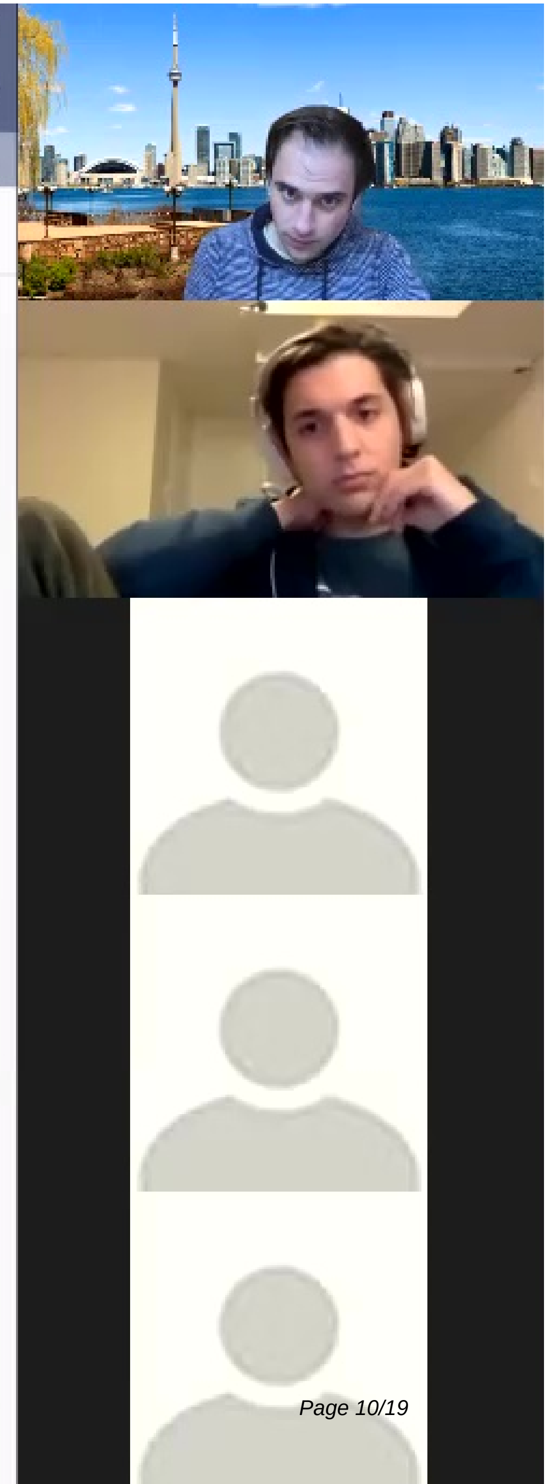

Assume we have a numerical grid

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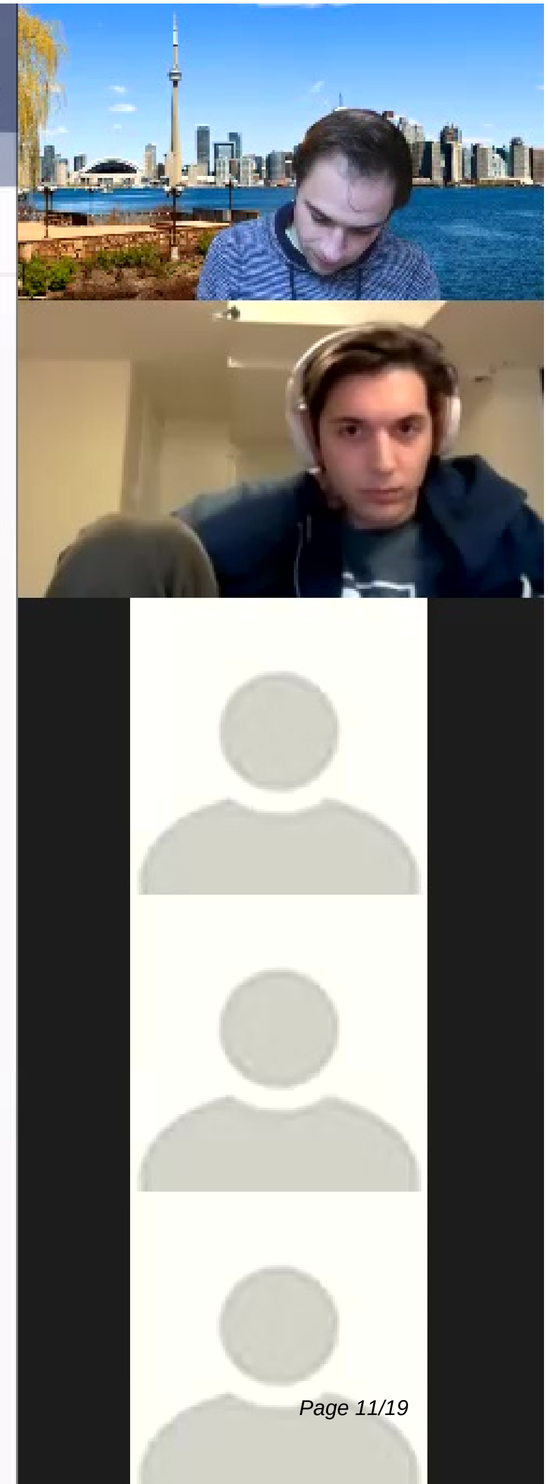
is said to be in <u>conservation form</u> with <u>numerical flux g</u>.

<u>Remarks:</u> 1) The definition can be generalized to

$$u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x}\Big\{\theta\left[g(u_i^{n+1}, u_{i+1}^{n+1}) - g(u_{i-1}^{n+1}, u_i^{n+1})\right] - (1+\theta)\left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\right]\Big\}$$

for $0 \le \theta \le 1$. For $\theta = 0$: explicit scheme

$\theta = 1$: implicit scheme



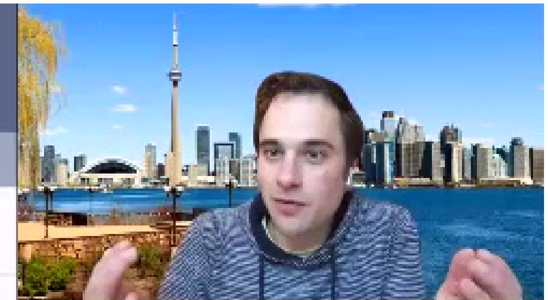
$\theta = 1$: implicit scheme

2) The important property of a scheme in conservative form is that it guarantees the conservation property of the solution on the discretized level:

$$\sum_i \left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}} \right) = \sum_i \left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n) \right]$$

counterterms cancel

$$= 0$$



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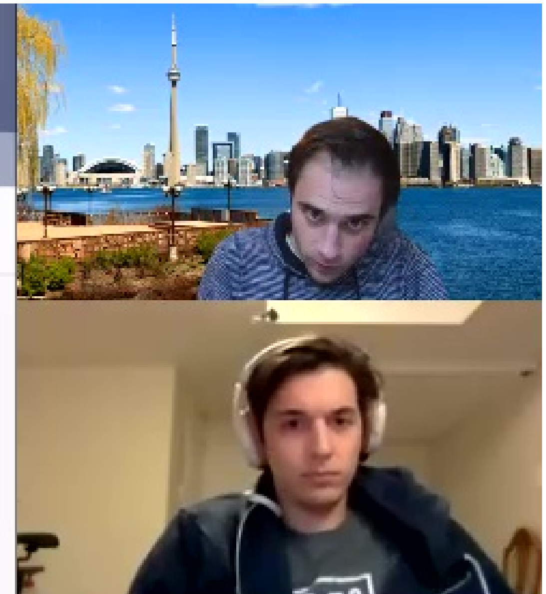
level

6.2 Convergence: Lax-Wendroff theorem & entropy condition

<u>Theorem</u>: (Lax-Wendroff 1960): Comm. Pure Appl. Math 13 217-237

Let $(u_\ell)_{\ell \in \mathbb{N}}$ be a sequence of discrete solutions of a scheme in conservation form w.r.t. $(h_\ell = \Delta x)_\ell$ and $(k_\ell = \Delta t)_\ell$, where $h_\ell, k_\ell \xrightarrow{\ell \to \infty} 0$ with



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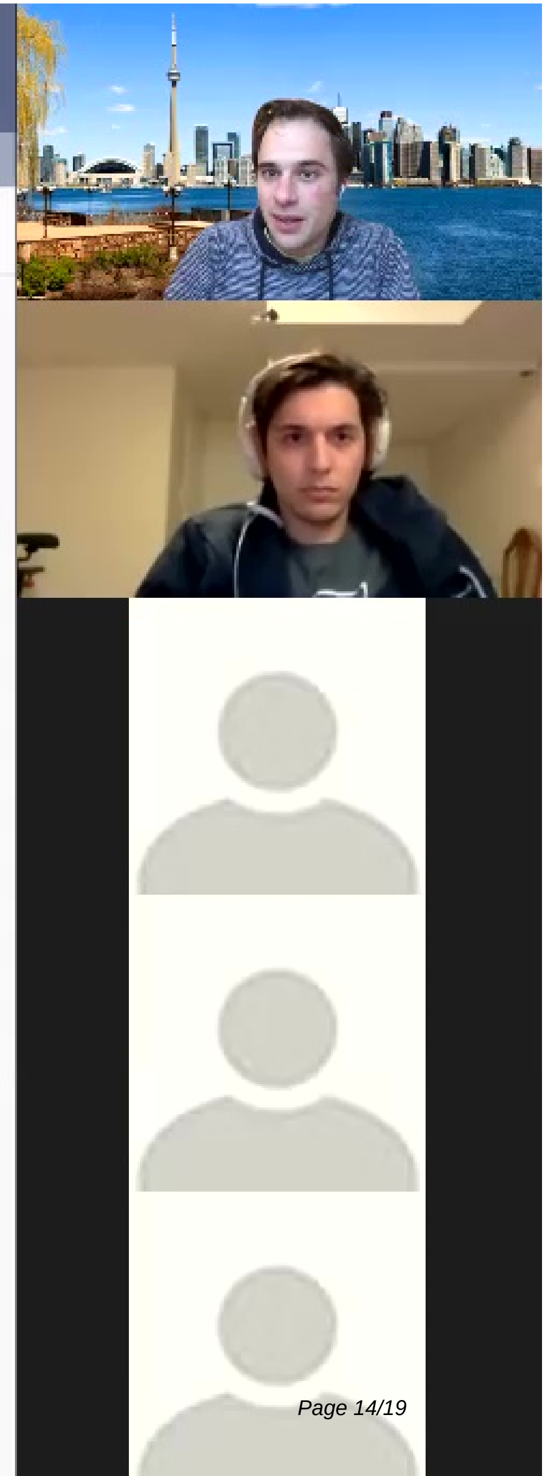
conservation form w.r.t. $(h_\ell = \Delta x)_\ell$ and $(k_\ell = \Delta t)_\ell$, where $h_\ell, k_\ell \xrightarrow{\ell \to \infty} 0$ with $\frac{k_\ell}{h_\ell} = \text{const.}$ Assume that there exists $C \in \mathbb{R}$ such that

$$\sup_\ell \sup_{\mathbb{R}\times(0,\infty)} |u_\ell(x,t)| \leq C$$

and $u_\ell \xrightarrow{\ell \to \infty} u$ almost everywhere in $\mathbb{R}\times(0,\infty)$. Then u is a weak solution of the system of CLS.



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<u>Proof</u>: Consider explicit case $(\theta = 0)$.

Choose $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ and multiply

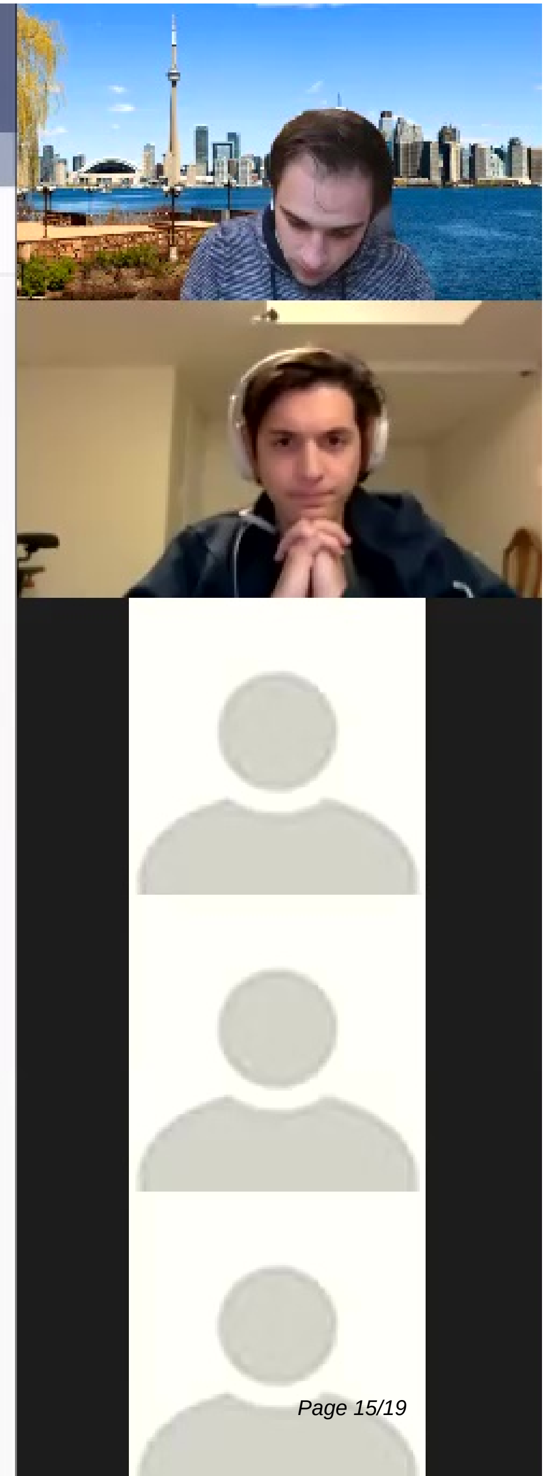
$\uparrow$ compact support

scheme by φ:

$$(*) \quad \Delta x\,(u_i^{n+1} - u_i^n)\,\varphi(x_i, t^n) = -\Delta t\,(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}})\,\varphi(x_i, t^n)$$

(1) Analyze convergence of LHS:

$$\sum_{n=0}^{\infty} (u_i^{n+1} - u_i^n)\,\varphi(x_i, t)$$



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scheme by φ :

$$(\circledast) \quad \Delta x \left(u_i^{n+1} - u_i^n\right) \varphi(x_i, t^n) = -\Delta t \left(g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n\right) \varphi(x_i, t^n)$$

① Analyze convergence of LHS:

$$\sum_{n=0}^{\infty} \left(u_i^{n+1} - u_i^n\right) \varphi(x_i, t) \underset{\substack{\uparrow \\ \text{partial summation}}}{=} \sum_{n=1}^{\infty} u_i^n \left[\varphi(x_i, t^{n-1}) - \varphi(x_i, t^n)\right] - u_i^0 \varphi(x_i, 0)$$

$$= -\int_0^{\infty} u_\ell(x_i, t)\, \partial_t \varphi(x_i, t)\, dt \; - u_i^0 \varphi(x_i, 0)$$

$u_\ell(x_i, t) = u_i^n$ at any

$u_\ell(x_i,t) = u_i^n$ at any given $\ell \in \mathbb{N}$

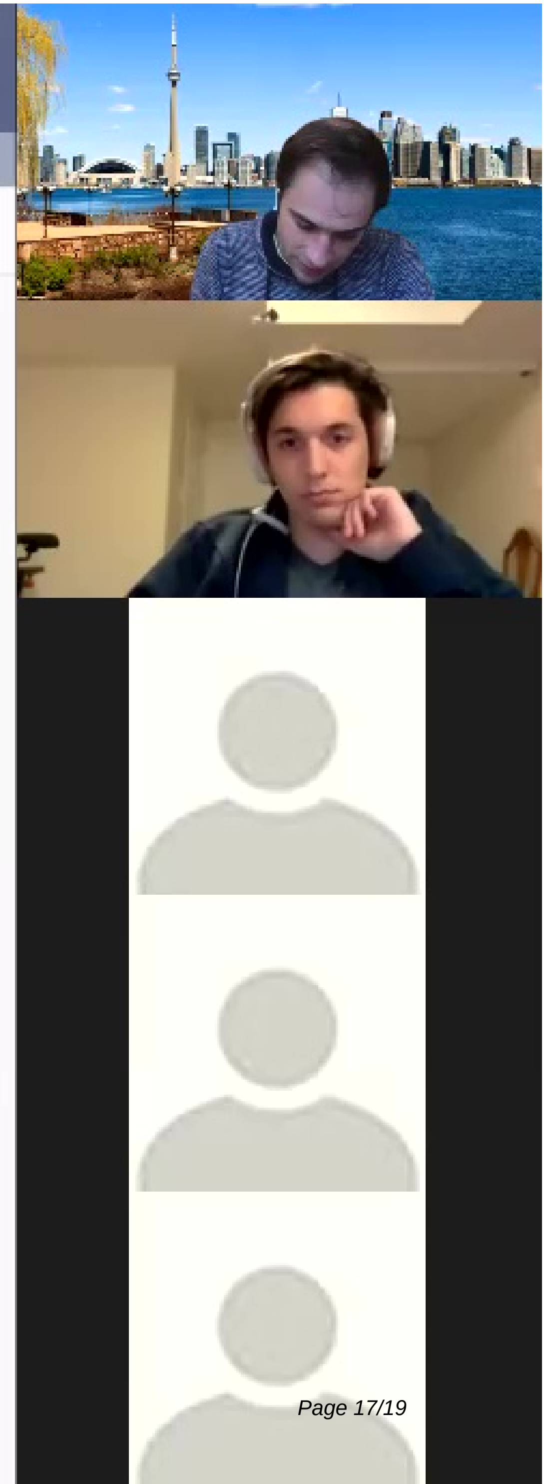
Add sum over x_i:

$$\Delta x \sum_i \sum_{n=0}^{\infty} \left(u_i^{n+1} - u_i^n\right) \varphi(x_i, t^n)$$

$$= -\sum_i \Delta x \left\{ \int_0^{\infty} u_\ell(x_i,t)\, \partial_t \varphi(x_i,t)\, dt - u_i^0 \varphi(x_i,0) \right\}$$

$$= -\int_{\mathbb{R}} \int_0^{\infty} u_\ell(x,t)\, \partial_t \varphi(x,t)\, dt\, dx - \int_{\mathbb{R}} u^0(x) \varphi(x,0)\, dx + \mathcal{O}(\Delta x)$$

$\Big\downarrow \; \ell \to \infty$



② Analyze convergence of RHS:

$$-\Delta t \sum_i \sum_{n=0}^{\infty} \left(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}\right) \varphi(x_i, t^n)$$

$$= -\Delta t \sum_i \sum_{n=0}^{\infty} g^n_{i+\frac{1}{2}} \left[\varphi(x_i, t^n) - \varphi(x_{i+1}, t^n)\right]$$

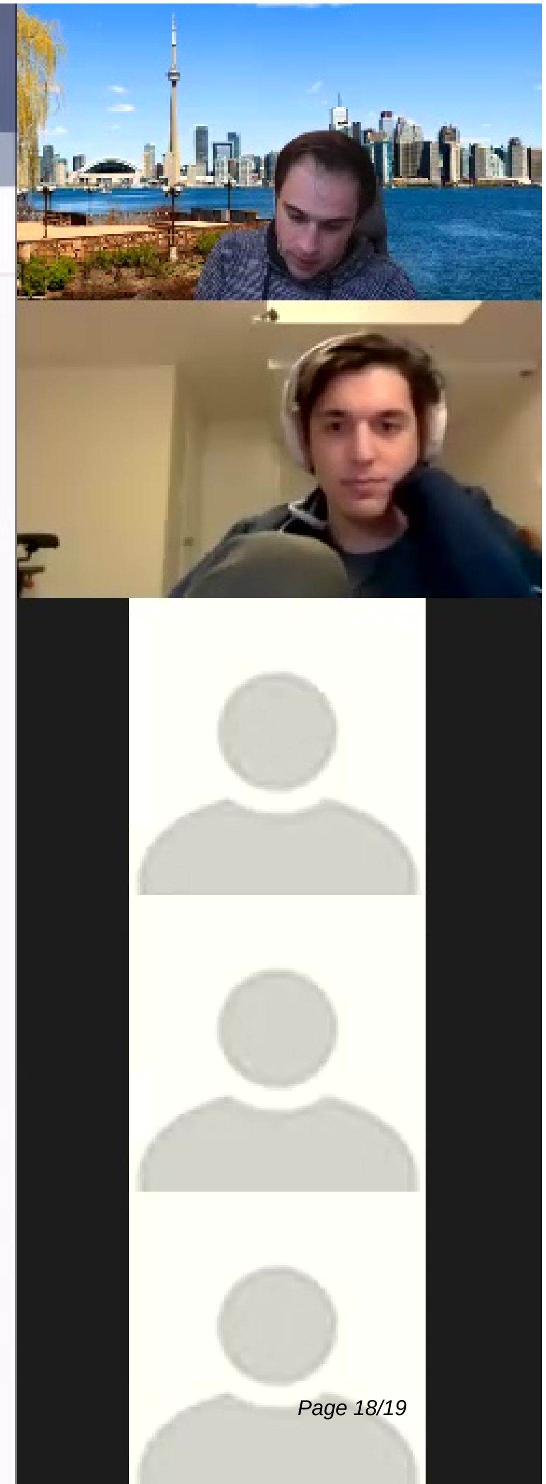
partial summation

$$= \Delta x \, \Delta t \sum_i \sum_{n=0}^{\infty} g^n_{i+\frac{1}{2}} \, \partial_x \varphi(x_i, t^n) + O(\Delta x)$$

$\downarrow$

Set: $g_\ell(x,t) \equiv g^n_{i+\frac{1}{2}}$ $\quad x_i \le x < x_{i+1}$, $\quad t^n < t \le t^{n+1}$

using $u = u_\ell$



$\downarrow$ using $u = u_\ell$ $\quad t^n < t \le t^{n+1}$

$$= \int_{\mathbb{R}} \int_0^\infty g_\ell(x,t)\, \partial_x \varphi(x,t)\, dt\, dx + \mathcal{O}(\Delta x)$$

$\Big\downarrow \ell \to \infty$ $\quad$ g consistent & Lipschitz continuous

$$\int_{\mathbb{R}} \int_0^\infty \underbrace{g(u(x,t), u(x,t))}_{= f(u)}\, \partial_x \varphi(x,t)\, dt\, dx$$

$\underline{①+②} \implies (*)$

$\Big\downarrow \ell \to \infty$