Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 15

Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

Date: November 05, 2020 - 3:30 PM

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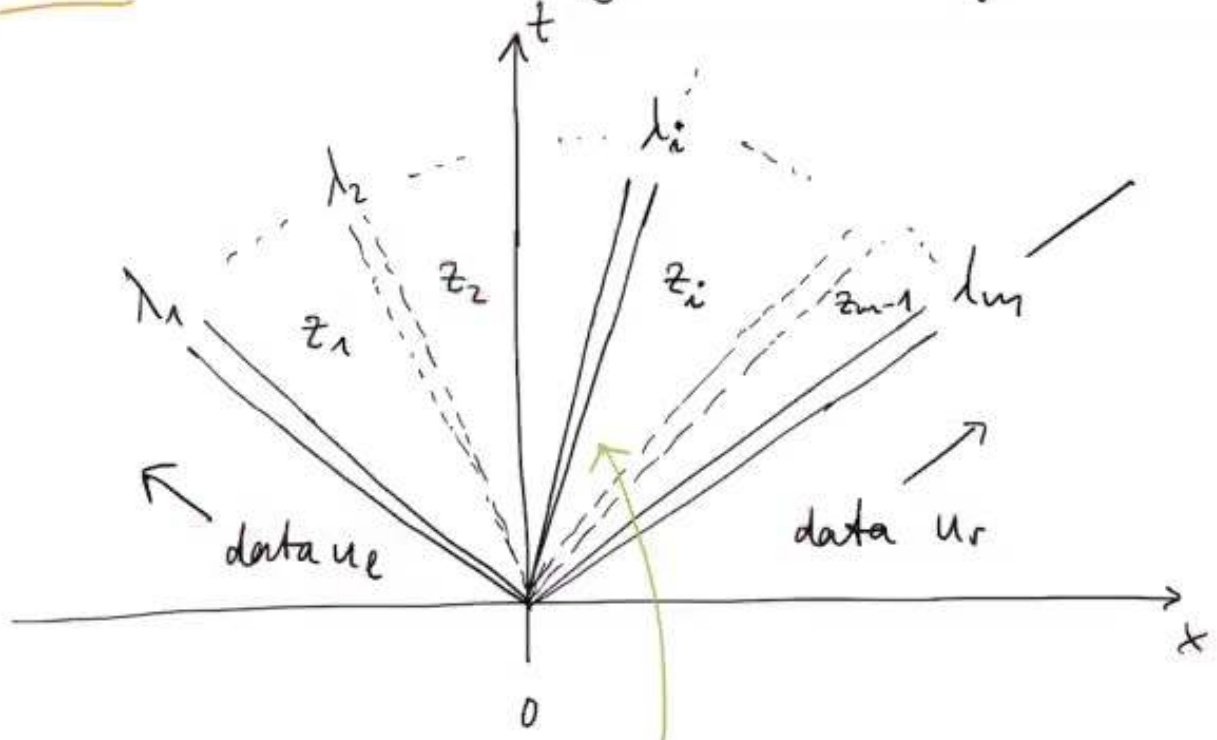
The global solution u is also unique as it satisfies the entropy condition by construction. □



solution constant on lines through the origin ($\frac{x}{t}$ = const.)

Consider Euler eqns

$$u_t + f(u)_x = 0$$

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}_x = 0$$

$$\lambda_1 = u - c \qquad \lambda_2 = u \qquad \lambda_3 = u + c$$

$$r_1 = \begin{pmatrix} 1 \\ u-c \\ H - uc \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 \\ u+c \\ H + uc \end{pmatrix}$$

- $(\lambda_1, r_1), (\lambda_3, r_3)$: genuinely non-linear
- (λ_2, r_2): linearly degenerate

Consider Euler eqns

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$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}_x = 0$$

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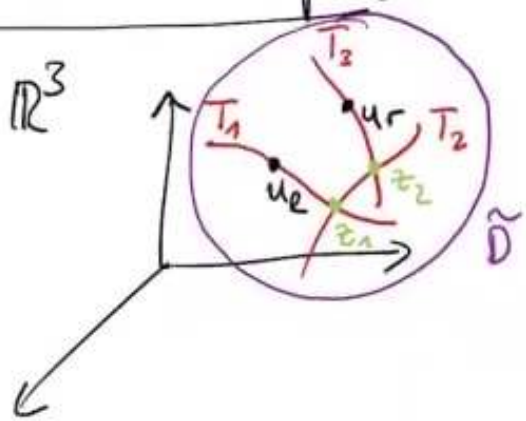
$$r_1 = \begin{pmatrix} 1 \\ u-c \\ H - uc \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 \\ u+c \\ H+uc \end{pmatrix}$$

- $(\lambda_1, r_1), (\lambda_3, r_3)$: generally non-linear
- (λ_2, r_2): linearly degenerate

General structure of the RP for the Euler equations

Situation in state space:

$\mathbb{R}^3$



$n = 3$ equations

$u = (u_1, u_2, u_3)$

$= (\rho, \rho u, E)$

Situation in real space:

contact discontinuity

$\lambda_2 = u$ ("entropy wave")

2-RI: u, p

3-RI: $s, \quad u - \frac{2c}{\gamma - 1}$

General structure of the RP for the Euler equations

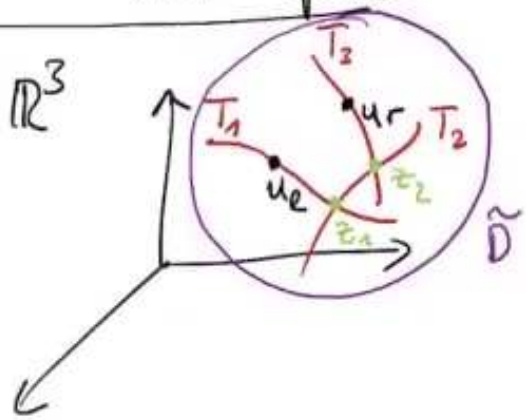
Situation in state space:

$n = 3$ equations

$u = (u_1, u_2, u_3)$

$= (\rho, \rho u, E)$

$w = (\rho, u, p)$



Situation in real space:

contact discontinuity

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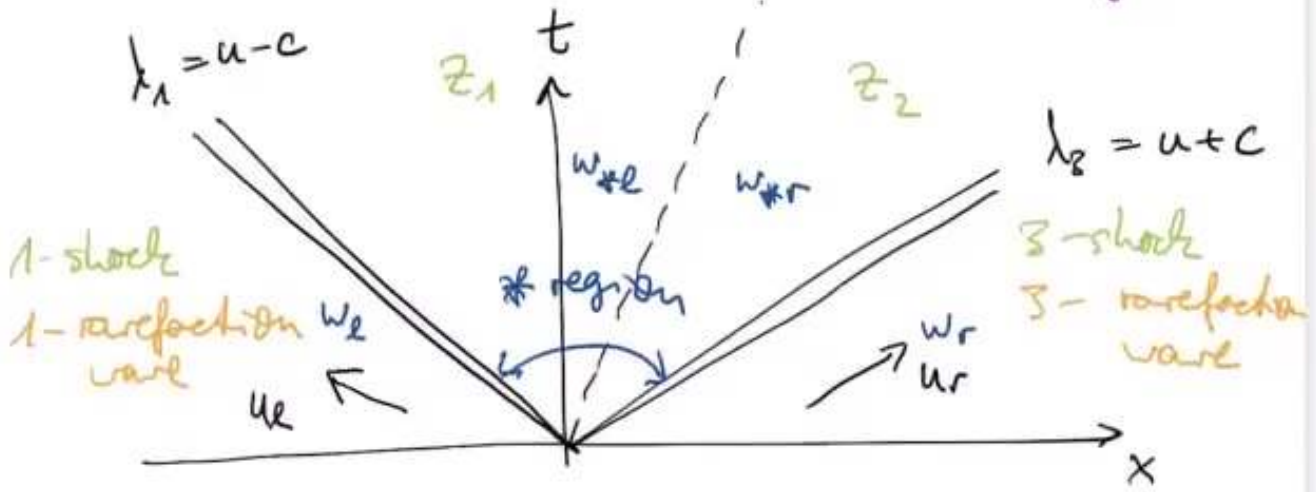
Situation in real space:

discontinuity

$\lambda_2 = u$ ("entropy wave")

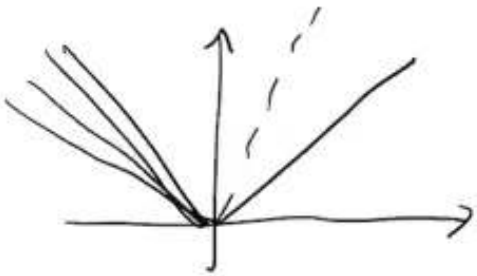


Remarks: 1) Since both u & p are constant across contact discontinuities

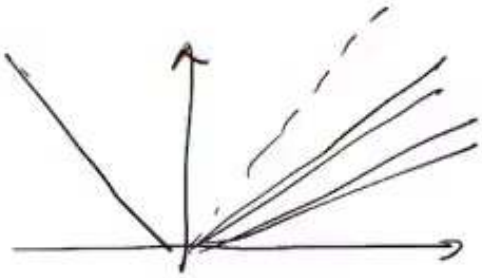
it is easier to work with the primitive variables $w = (\rho, u, p)$

2) There are 4 possible patterns

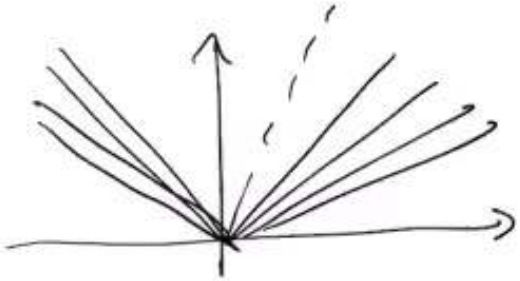
(i)



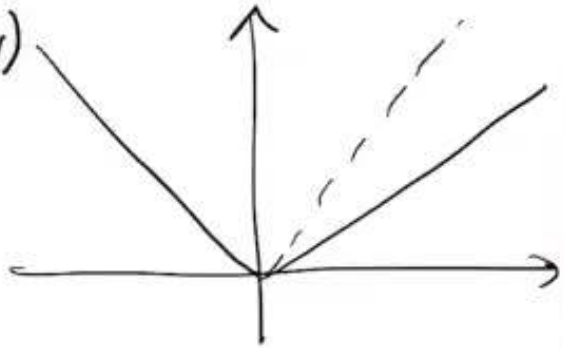
(ii)



(iii)

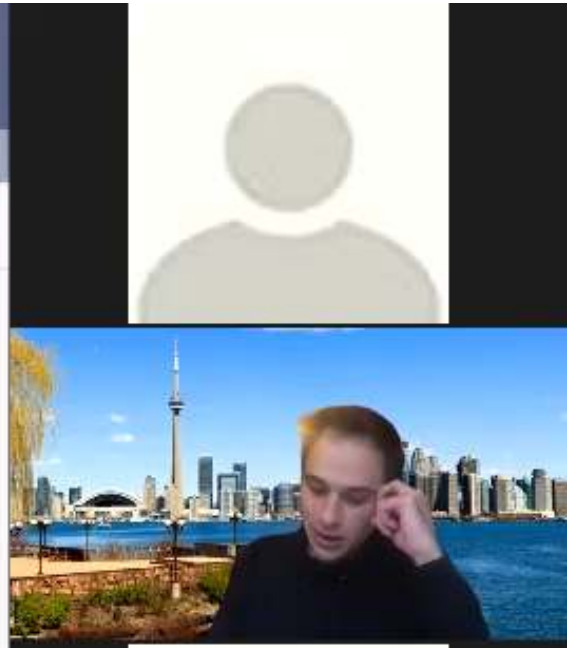


(iv)



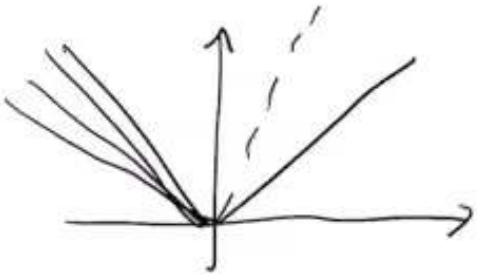
Example:

1)

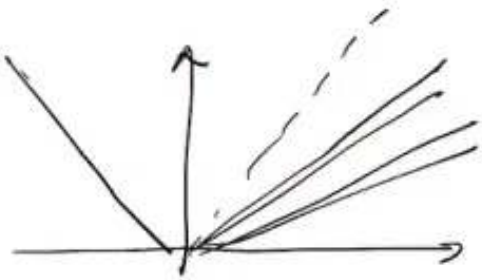


2) There are 4 possible patterns

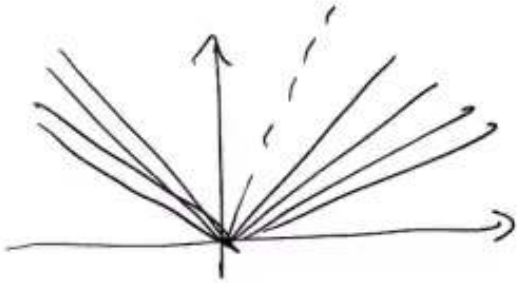
(i)



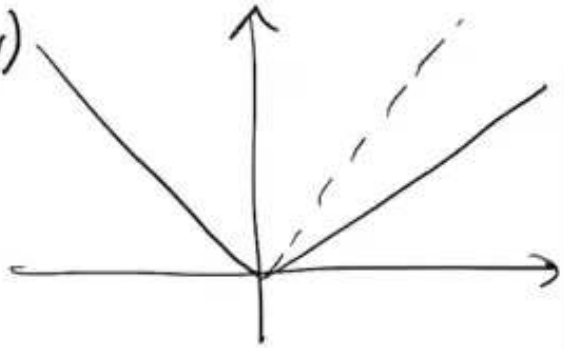
(ii)



(iii)

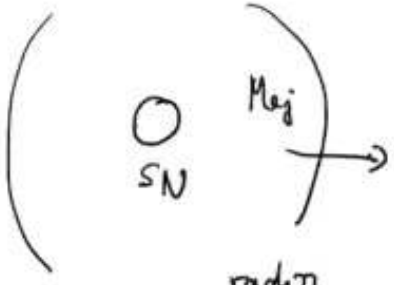


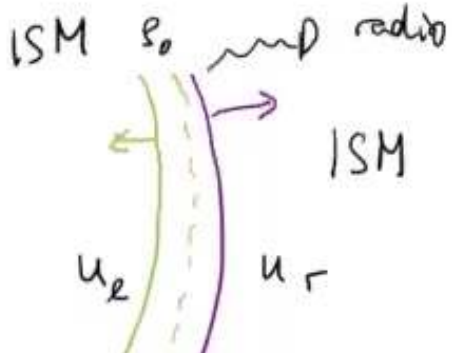
(iv)

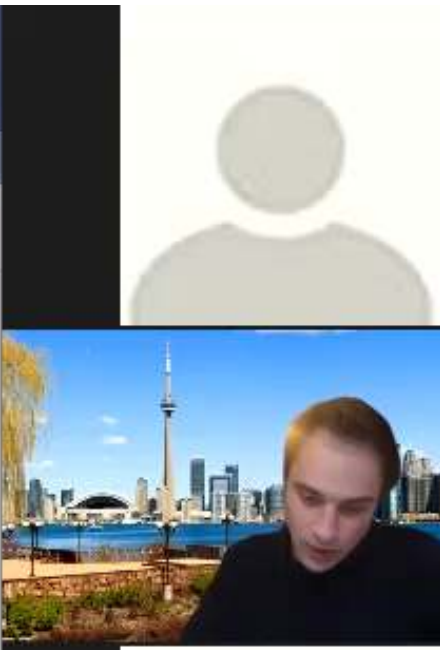


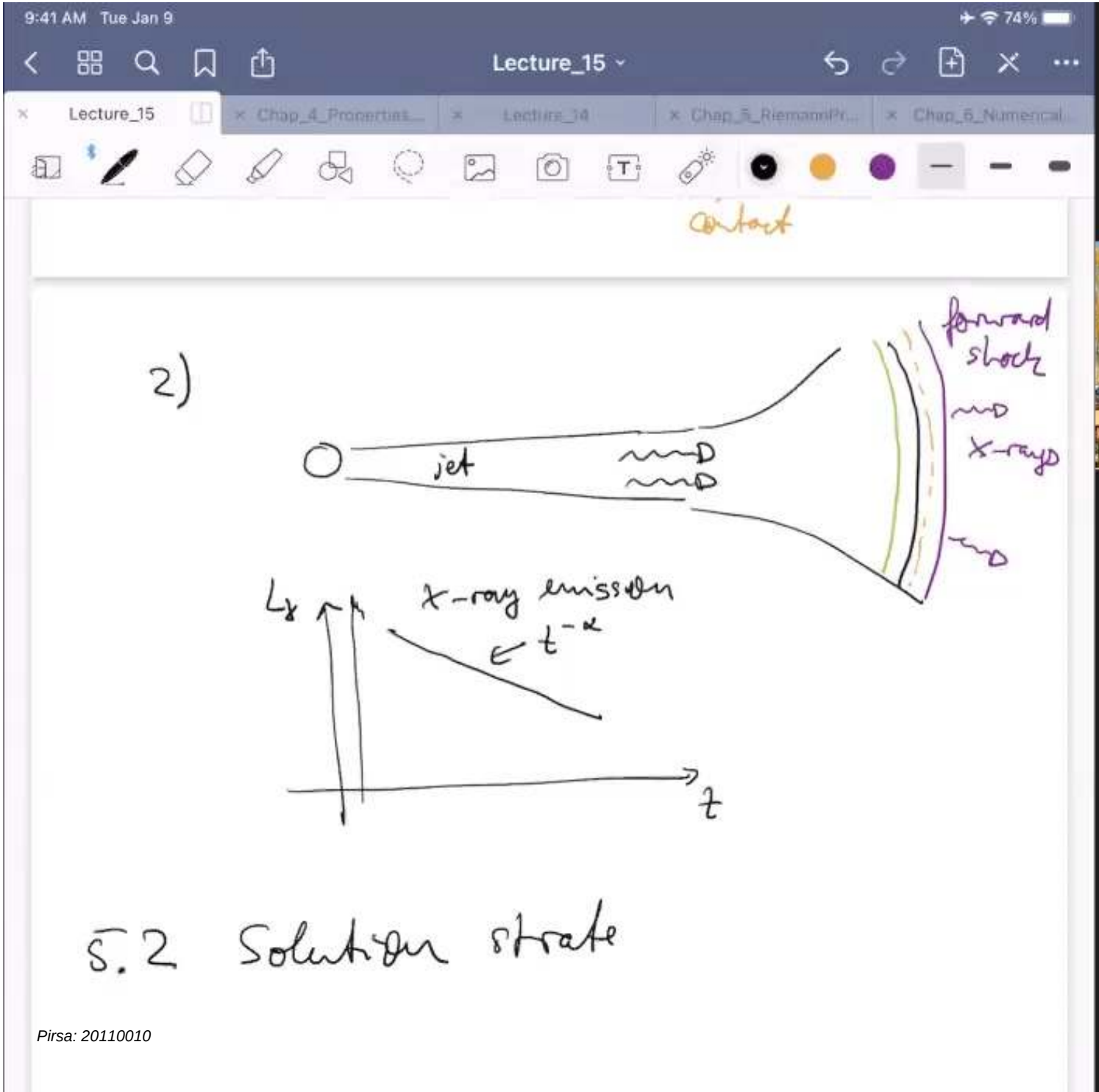
Example:

1)





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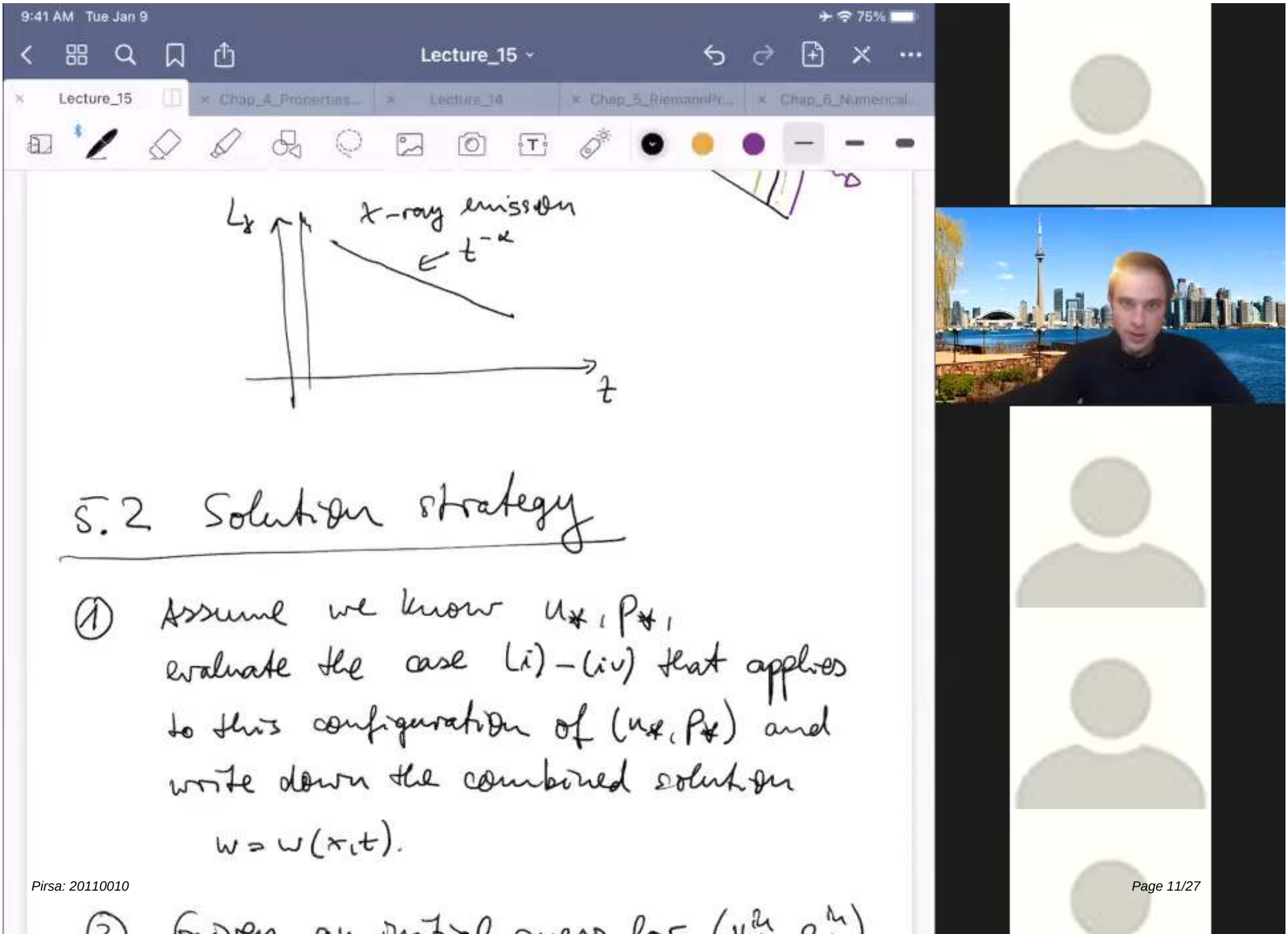
2)
jet
forward shock
X-rays
X-ray emission
$t^{-\alpha}$
L_γ
t
5.2 Solution strate



5.2 Solution strategy

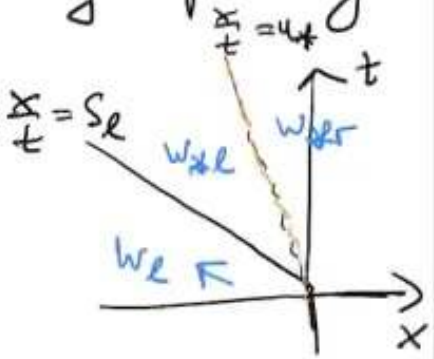
① Assume we know u_*, p_*, evaluate the case (i)–(iv) that applies to this configuration of (u_*, p_*) and write down the combined solution

$$w = w(x,t).$$

② Given an initial guess for (u^{in}, p^{in})

will follow with (1)

step (1): Treat regions left and right of the contact discontinuity seperately



(1) $S \equiv \frac{x}{t} \leq u_*$: left side

$p_* > p_L$: left shock

$(\lambda(u_L) > \lambda(u_*))$ (ii) or (iv)

$$w(x,t)_{left} = \begin{cases} w_{*L}^{sh} & \text{if } S_L \leq \frac{x}{t} \leq u_* \\ w_L & \text{if } \frac{x}{t} \leq S_L \end{cases}$$

where $w_{*L}^{sh} = (\rho_{*L}^{sh}$

$p_* > p_\ell$: left shock (ii) or (iv)

$(\lambda(u_\ell) > \lambda(u_*))$

$$w(x,t)_{left} = \begin{cases} w^{sh}_{*\ell} & \text{if } s_\ell \le \frac{x}{t} \le u_* \\ w_\ell & \text{if } \frac{x}{t} \le s_\ell \end{cases}$$

where $w^{sh}_{*\ell} = (s^{sh}_{*\ell}, u_*, p_*)$

RH jump conditions

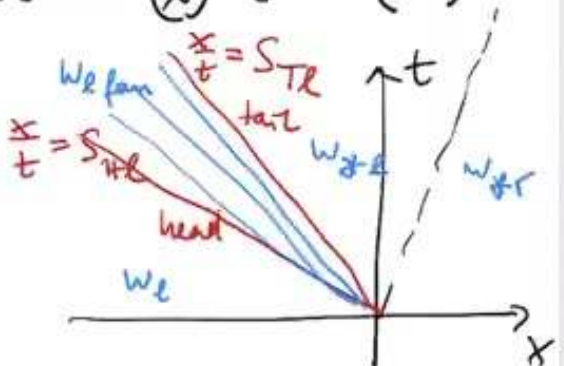
$$s_{*\ell} = s_\ell \frac{\frac{p_*}{p_\ell} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_\ell} + 1}$$

$$S_{*\ell} = S_\ell \frac{\frac{p_*}{p_\ell} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_\ell} + 1}$$

$$S_\ell = u_\ell - c_\ell \sqrt{\frac{\gamma+1}{2\gamma}\frac{p_*}{p_\ell} + \frac{\gamma-1}{2\gamma}}$$

$p_* \le p_\ell$: $\Rightarrow$ left rarefaction wave: (i) or (iii)

$(\lambda(u_\ell) < \lambda(u_*))$



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$$w(x,t)_{left} = \begin{cases} w_l, & \frac{x}{t} \le s_{Hl} \\ w_{l\,fan}, & s_{Hl} \le \frac{x}{t} \le s_{Tl} \\ w_{*l}, & s_{Tl} \le \frac{x}{t} \le u_* \end{cases}$$

where $w_{*l} = (\rho_{*l}, u_*, p_*)$

known

$w_{l\,fan}$: state "inside" the rarefaction wave

Find: $\rho_{*l}, s_{Hl}, s_{Tl}$

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where $W_{*L} = (\rho_{*L}, u_*, p_*)$

known

We can : state "inside" the rarefaction wave

Find: $\rho_{*L}, s_{HL}, s_{TL}, W_{L\,fan}$

I $\quad s$

II $\quad u + \frac{2c}{\gamma - 1}$

} 1-Riemann invariants

I

we can use isentropic law $p = k\rho^{\gamma}$ ← adiabatic index of EOS

across wave

$p_L = k\rho_L^{\gamma} \Rightarrow k = \frac{p_L}{\rho_L^{\gamma}}$

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$$\left.\begin{array}{ll} \mathrm{I} & s \\ \mathrm{II} & u + \dfrac{2c}{\gamma - 1} \end{array}\right\} \text{ 1-Riemann invariants}$$

I

$\Rightarrow$ can use isentropic law $p = K\rho^{\gamma}$

adiabatic index of EOS

across wave

$$p_\ell = K\rho_\ell^{\gamma} \Rightarrow K = \frac{p_\ell}{\rho_\ell^{\gamma}}$$

$$\boxed{\rho_{*\ell} = \rho_\ell \left(\frac{p_*}{p_\ell}\right)^{\frac{1}{\gamma}}}$$

$$c = \sqrt{\gamma \frac{p}{\rho}}$$

$$\downarrow$$

$$c_{*\ell} = c_\ell \left(\frac{p_{*\ell}}{\ldots}\right)^{\frac{\gamma-1}{2\gamma}}$$

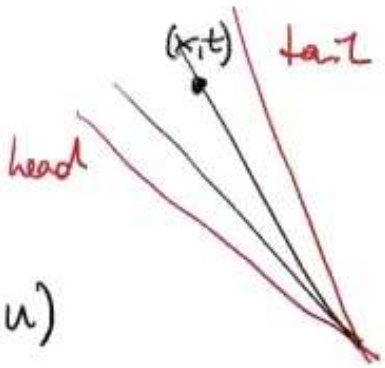
$$c_{*\ell} = c_\ell \left(\frac{p_{*\ell}}{p_\ell}\right)^{\frac{\gamma-1}{2\gamma}}$$

Head & tail speeds:

$$S_{H\ell} = \lambda_1(u_\ell) = u_\ell - c_\ell$$

$$S_{T\ell} = \lambda_1(u_*) = u_* - c_{*\ell}$$

Wave:

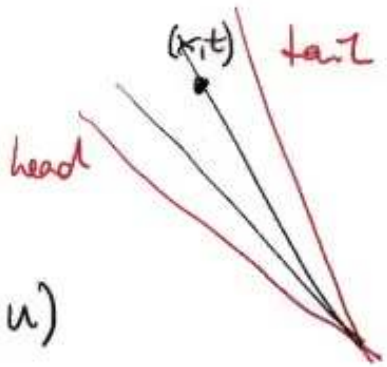


$$\frac{dx}{dt} = \frac{x}{t} = \lambda_1(u)$$

$$= u - c \qquad (1)$$

$$S_{TL} = \lambda_1(u_*) = u_* - c_{*L}$$

We have:



$$\frac{dx}{dt} = \frac{x}{t} = \lambda_1(u)$$

$$= u - c \qquad (1)$$

relate to u_L, c_L via II: (u, c)

$$u_L + \frac{2c_L}{\gamma-1} = u + \frac{2c}{\gamma-1} \qquad (2)$$

combine to:

$$\begin{cases} S = S_L \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_L}(u_L - \frac{x}{t})\right]^{\frac{2}{\gamma-1}} \\ u = \end{cases}$$

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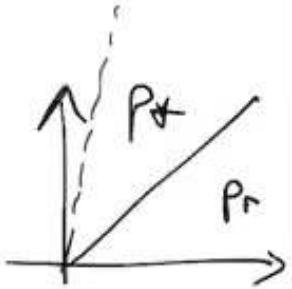
$$W_{left} = \begin{cases} S = u_l - c_l\sqrt{\frac{\gamma+1}{2\gamma}\left(\frac{p}{p_l}\right) + \frac{\gamma-1}{2\gamma}} \\ u = \frac{2}{\gamma+1}\left[c_l + \frac{\gamma-1}{2}u_l + \frac{x}{t}\right] \\ p = p_l\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_l}\left(u_l - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}} \end{cases}$$

(2) $S \equiv \frac{x}{t} \geq u_*$: right side

analogous to above

$p_* > p_r$: right shock (i) or (iv)



$$w(x,t)_{right} = \begin{cases} w_{*r}^{sh} & u_* \leq \frac{x}{t} \leq S_r \\ w_r & \frac{x}{t} \geq S_r \end{cases}$$

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$$w(x,t)_{right} = \begin{cases} w_{rfan}, & s_{Tr} \le \frac{x}{t} \le s_{Hr} \\ w_r, & \frac{x}{t} \ge s_{Hr} \end{cases}$$

(2) Find (u_*, p_*) given initial guess:

Proposition: The solution p_* is given by the root of the function (assuming an ideal gas EOS):

$$f(p, w_l, w_r) \equiv f_l(p, w_l) + f_r(p, w_r) + u_r - u_l$$

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$$\left[\ \gamma-1\left[(p_l)\ \ -1\right]\right],\ p \le p_l$$

(rarefaction wave)

$$f_r(p, w_r) = \begin{cases} p - p_r\sqrt{\dfrac{A_r}{p+B_r}}, & p > p_r \text{ (shock)} \\ \dfrac{2c_r}{\gamma-1}\left[\left(\dfrac{p}{p_r}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right], & p \le p_r \text{ (rarefaction wave)} \end{cases}$$

with given constants

$$A_l \equiv \frac{2}{(\gamma+1)\rho_l}, \quad B_l \equiv \frac{\gamma-1}{\gamma+1}\, p_l$$

$$A_r \equiv \frac{2}{(\gamma+1)\rho_r}, \quad B_r \equiv \frac{\gamma-1}{\gamma+1}\, p_r$$

The solution for the velocity u_*

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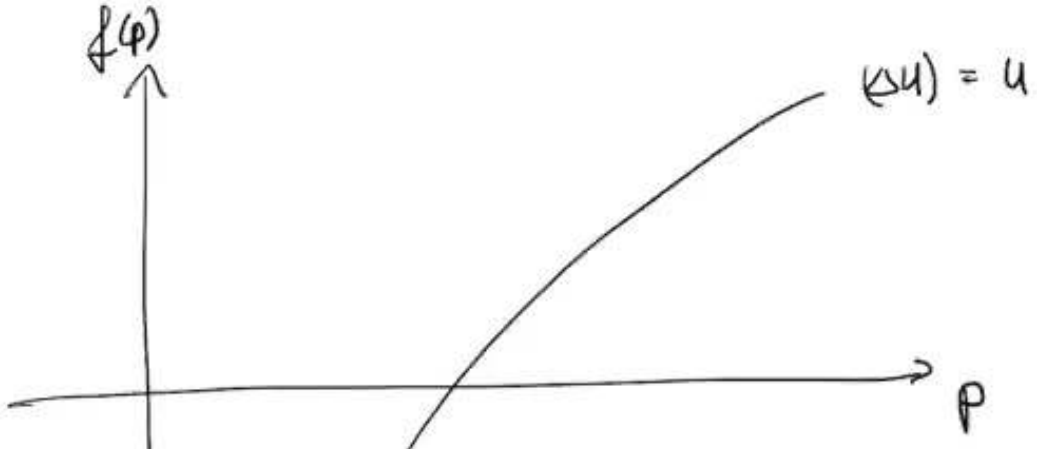
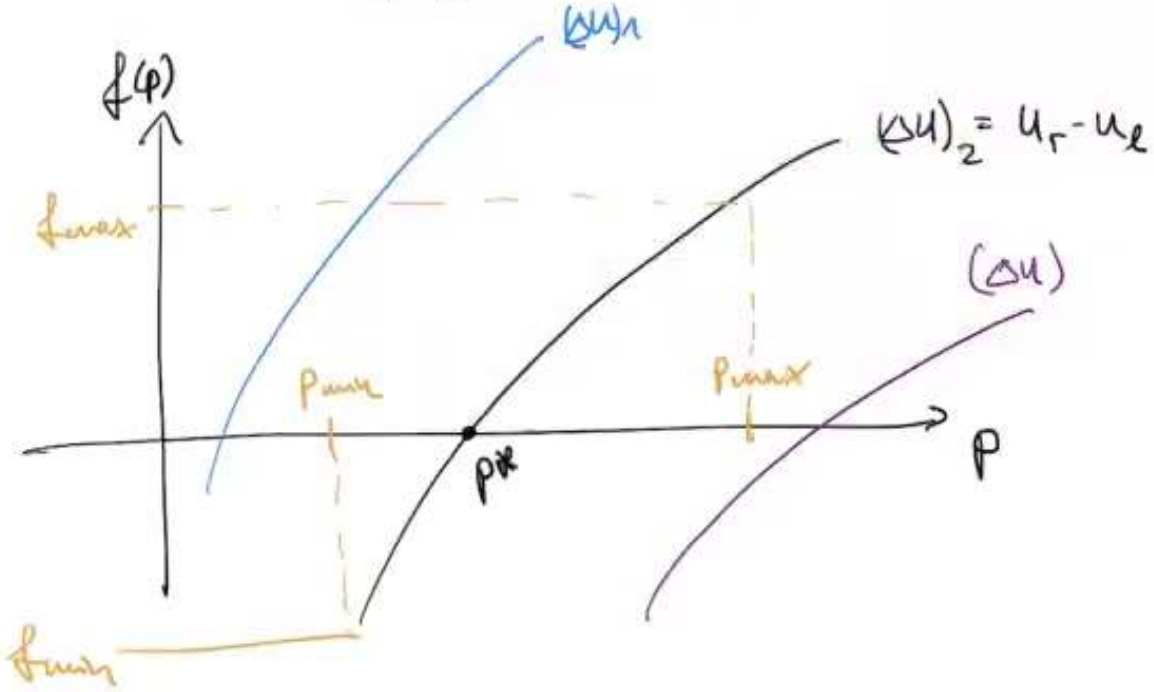
Remarks: 1) Behavior of $f(p)$

Can show:

- $f'(p) > 0 \rightarrow$ monotonically increasing
- $f''(p) < 0 \rightarrow$ concave
- $\left\{ \begin{matrix} f'(p) \\ f''(p) \end{matrix} \right\} \xrightarrow{p \to \infty} 0$



$$\cdot \begin{Bmatrix} f'(\rho) \\ f''(\rho) \end{Bmatrix} \quad \rho \to \infty \quad 0$$



$$\rho_{min} = \min(\rho_l, \rho_r), \quad f_{min} = f(\rho_{min})$$

$$\rho_{max} = \max(\rho_l, \rho_r), \quad f_{max} = f(\rho_{max})$$

3 regimes: $f_{min}, f_{max} > 0 \Rightarrow p_* \in I_1$

$f_{min} \leq 0, f_{max} \geq 0 \Rightarrow p_* \in I_2$

$f_{min}, f_{max} < 0 \Rightarrow p_* \in I_3$

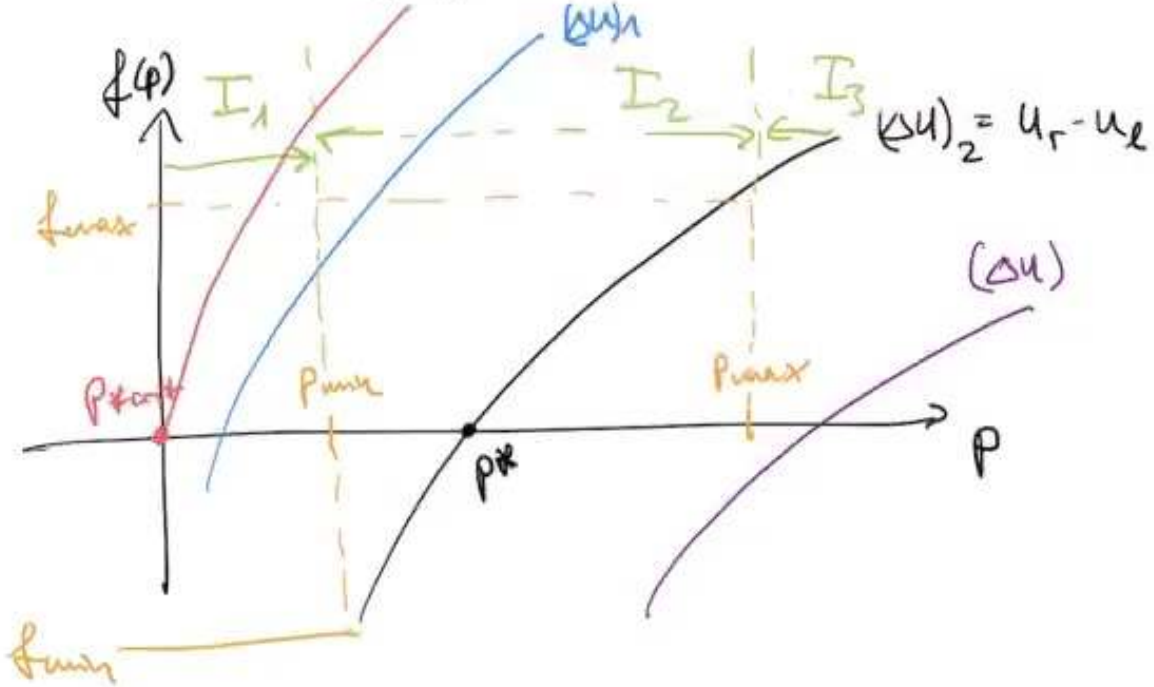
I_1: $p_* < p_l, p_r \Rightarrow$ 2 rarefaction waves

I_2: $p_l < p_* < p_r$ } one rarefaction wave & one shock

$p_r < p_* < p_l$ }

I_3: $p_* > p_l, p_r \Rightarrow$ 2-shock solution

- $f''(\rho) < 0 \rightarrow$ concave
- $\left\{ \begin{matrix} f'(\rho) \\ f''(\rho) \end{matrix} \right\} \rho \rightarrow \infty \rightarrow 0$



$\rho_{min} = \min(\rho_l, \rho_r)$, $f_{min} = f(\rho_{min})$

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2) Vacuum:

A physical solution to $f(p)$ requires

$f(0) < 0$ (monotonicity property)

For $p_* = p_{crit} = 0$ we have $p_* \leq p_l, p_r$

$\Rightarrow$ left & right rarefaction

$$\rightsquigarrow \underline{(u_r - u_l)_{crit}} \overset{f(p_{*crit}) = 0}{\equiv} -f_l(0) - f_r(0)$$

$$= \frac{2}{\gamma - 1}(c_l + c_r) \left(> u_r - u_l \right)$$