Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 15

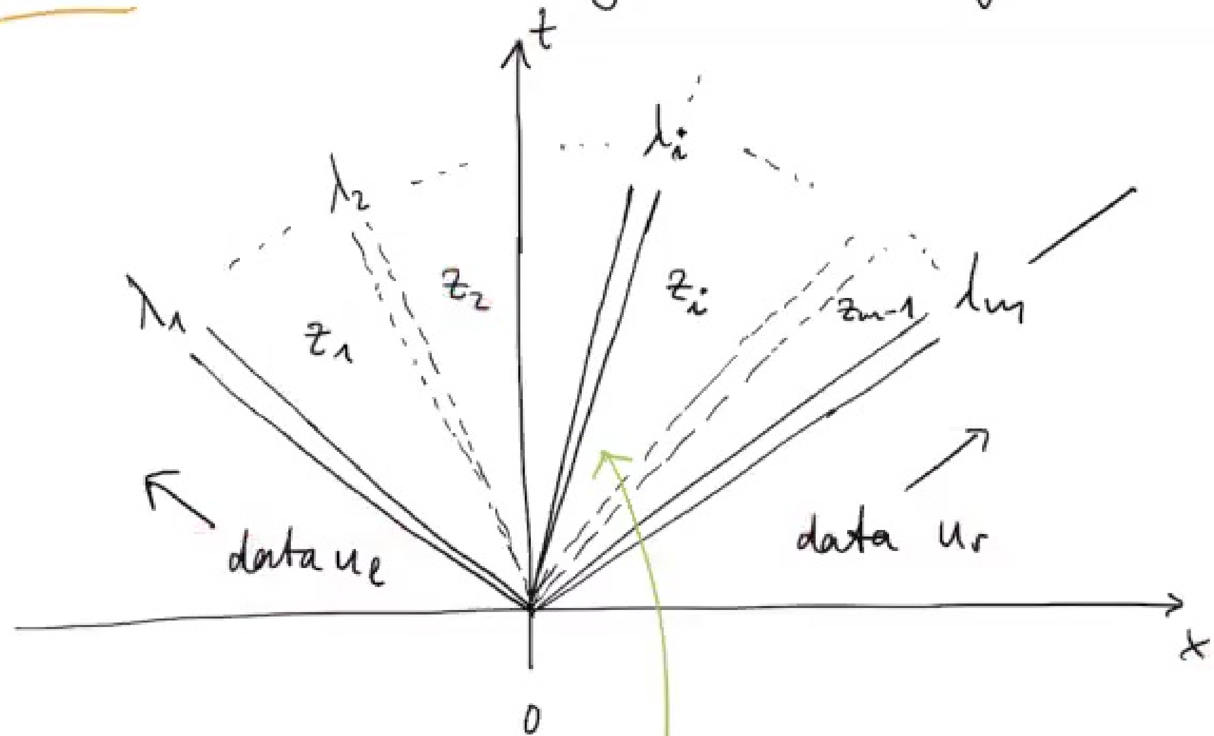
Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

Date: November 05, 2020 - 3:30 PM

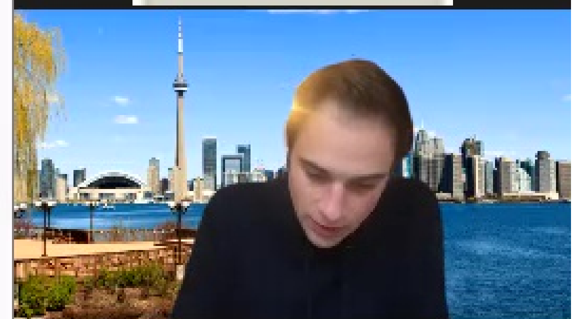
URL: http://pirsa.org/20110010

The global solution u is also unique as it satisfies the entropy condition by construction.



$\square$

solution constant on lines through the origin ($\frac{x}{t}$ = const.)



Consider Euler equs

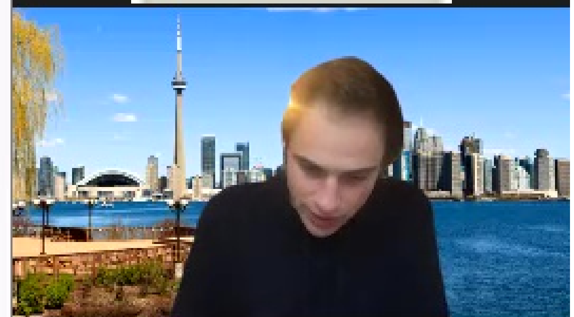
$$u_t + f(u)_x = 0$$

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}_x = 0$$

$$\lambda_1 = u - c \qquad \lambda_2 = u \qquad \lambda_3 = u + c$$

$$r_1 = \begin{pmatrix} 1 \\ u-c \\ H - uc \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 \\ u+c \\ H + uc \end{pmatrix}$$

- (λ_1, r_1), (λ_3, r_3) : genuinely non-linear
- (λ_2, r_2) : linearly degenerate



Consider Euler equs

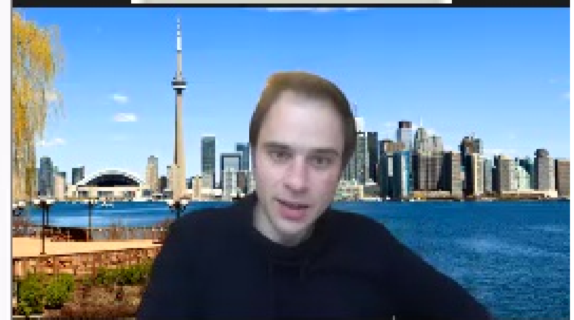
$$u_t + f(u)_x = 0$$

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}_x = 0$$

$$\lambda_1 = u - c \qquad \lambda_2 = u \qquad \lambda_3 = u + c$$

$$r_1 = \begin{pmatrix} 1 \\ u-c \\ H-uc \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 \\ u+c \\ H+uc \end{pmatrix}$$

- $(\lambda_1, r_1), (\lambda_3, r_3)$: genuinely non-linear
- (λ_2, r_2): linearly degenerate

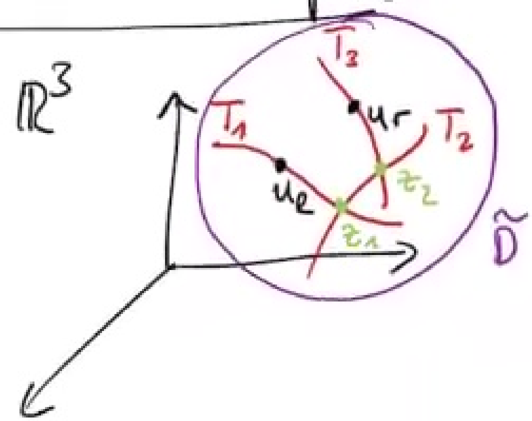


⇝ General structure of the RP for the Euler equations

Situation in state space:

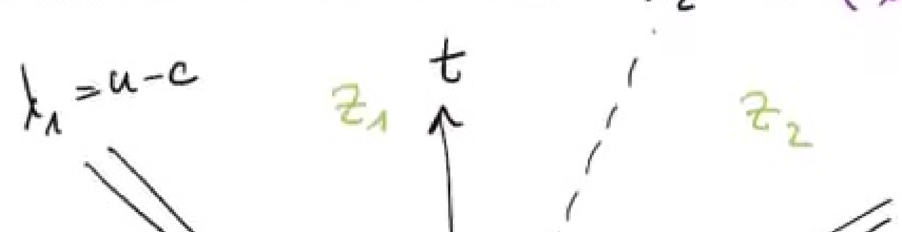
$m = 3$ equations

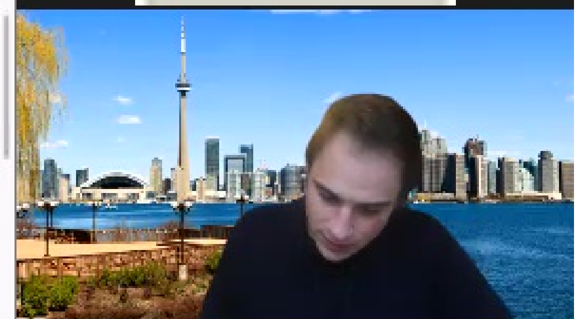
$$u = (u_1, u_2, u_3) = (\rho, \rho u, E)$$



Situation in real space:

contact discontinuity
$\lambda_2 = u$ ("entropy wave")





2-RI: u, p

3-RI: $s, \; u - \frac{2c}{\gamma - 1}$

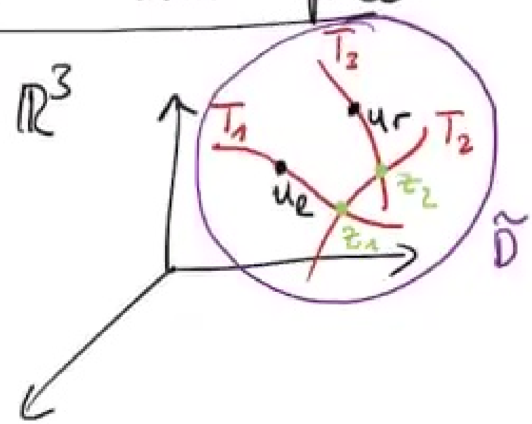
⇝ General structure of the RP for the Euler equations

Situation in state space: $m = 3$ equations

$u = (u_1, u_2, u_3)$

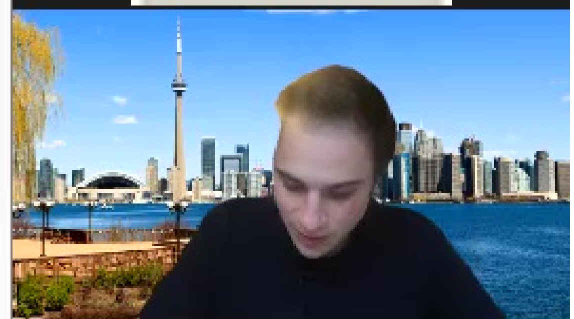
$= (\rho, \rho u, E)$

$w = (\rho, u, p)$

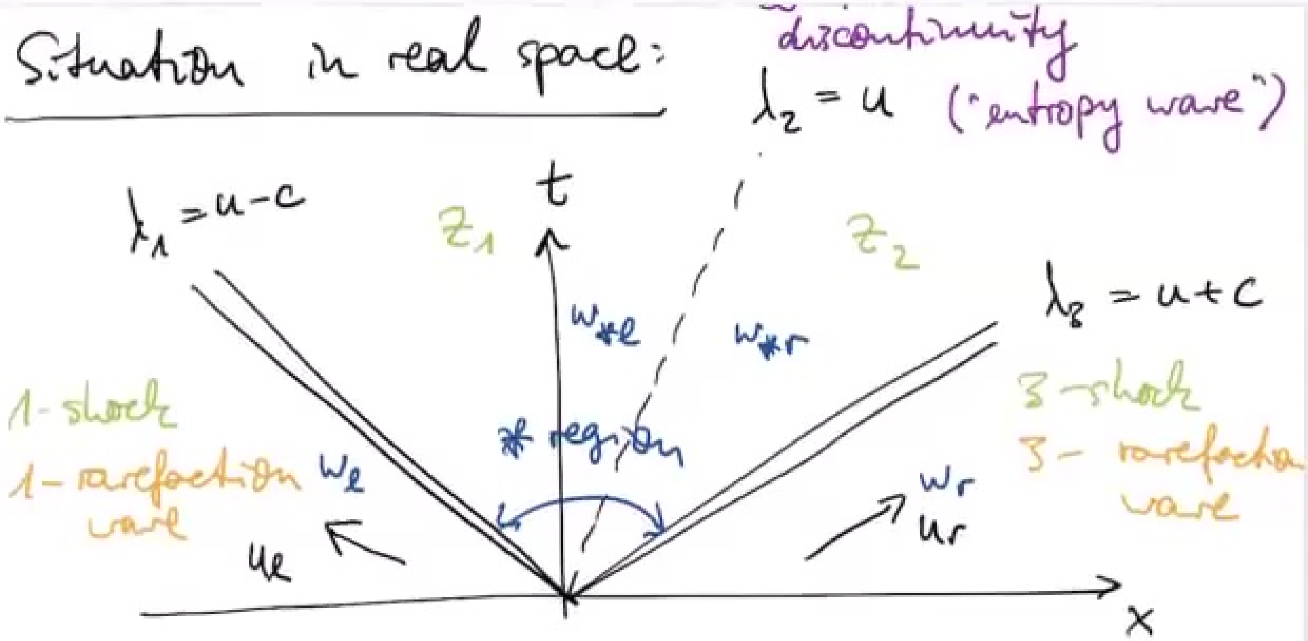


contact discontinuity

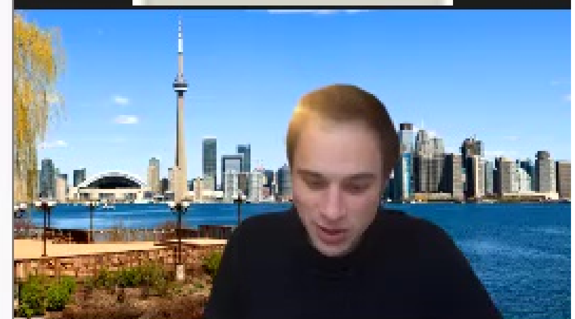
Situation in real space:



Situation in real space:

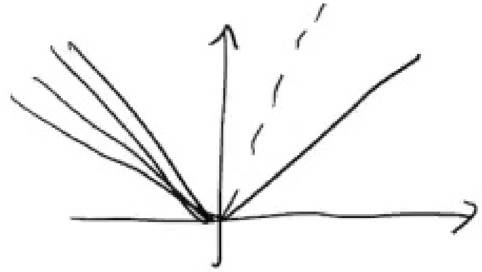


Remarks: 1) Since both u & p are constant across contact discontinuities it is easier to work with the primitive variables $w = (\rho, u, p)$

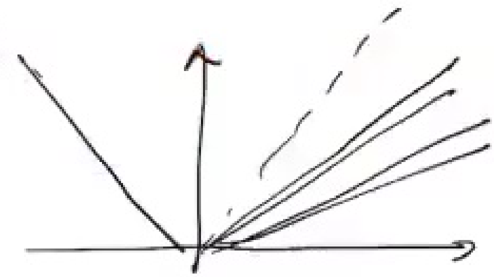


2) There are 4 possible patterns

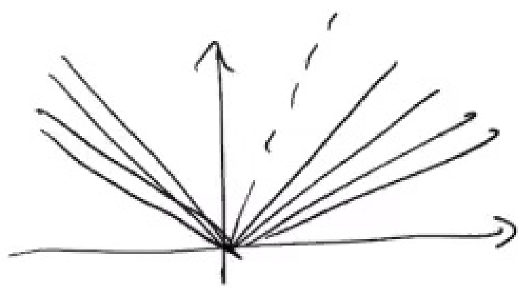
(i)



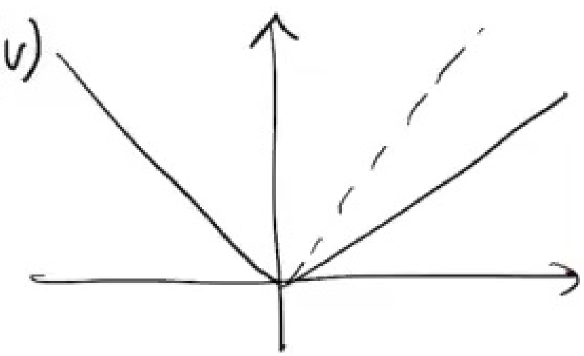
(ii)



(iii)

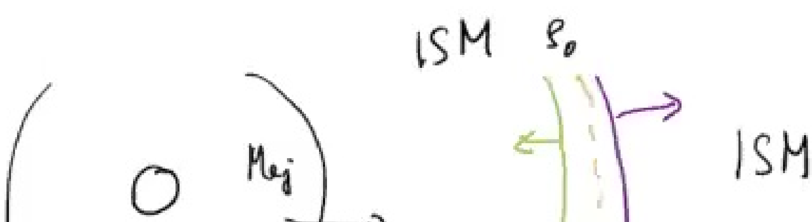


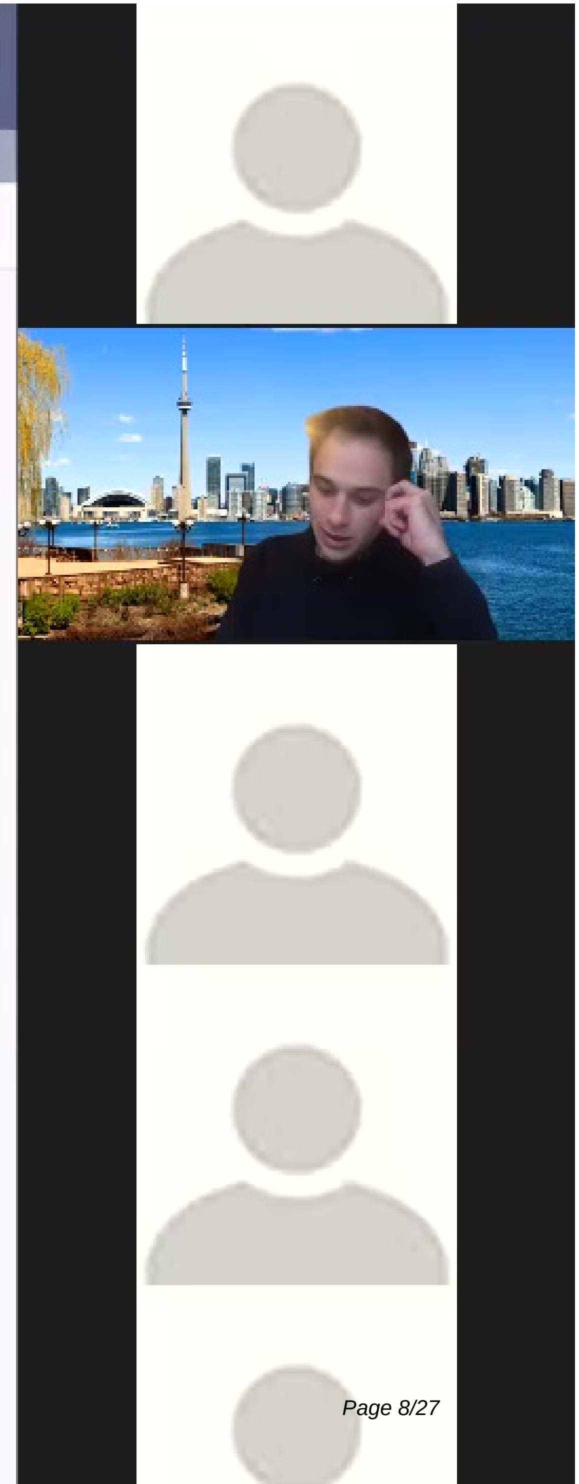
(iv)



Example:

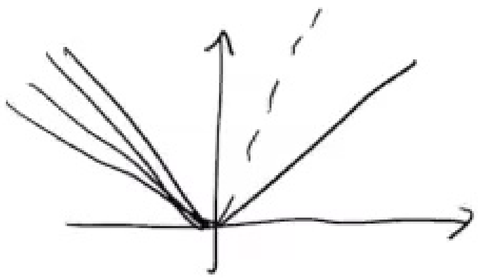
1)



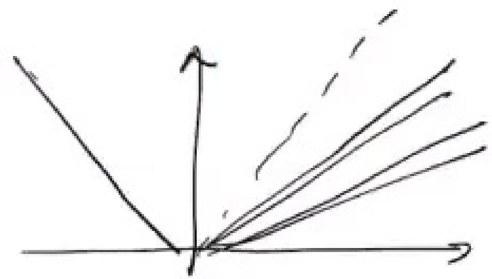


2) There are 4 possible patterns

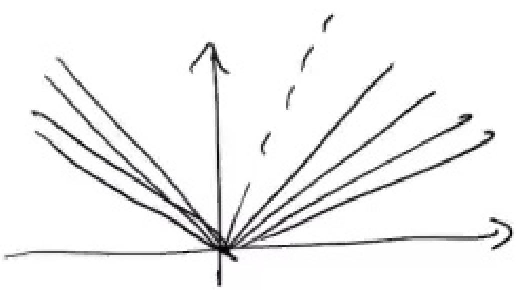
(i)



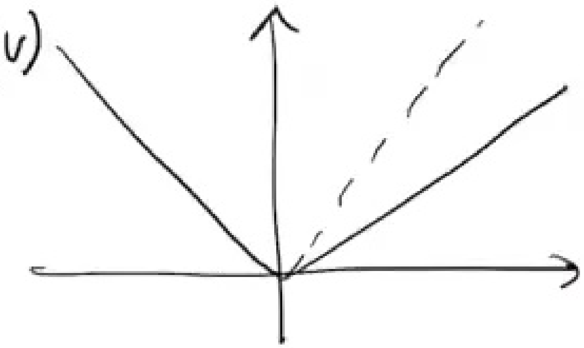
(ii)



(iii)

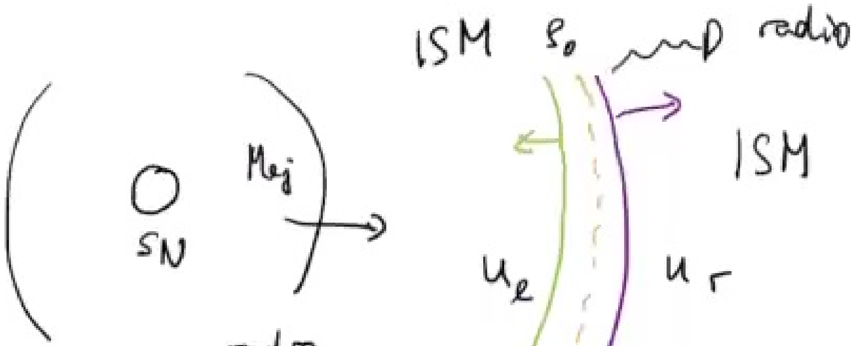


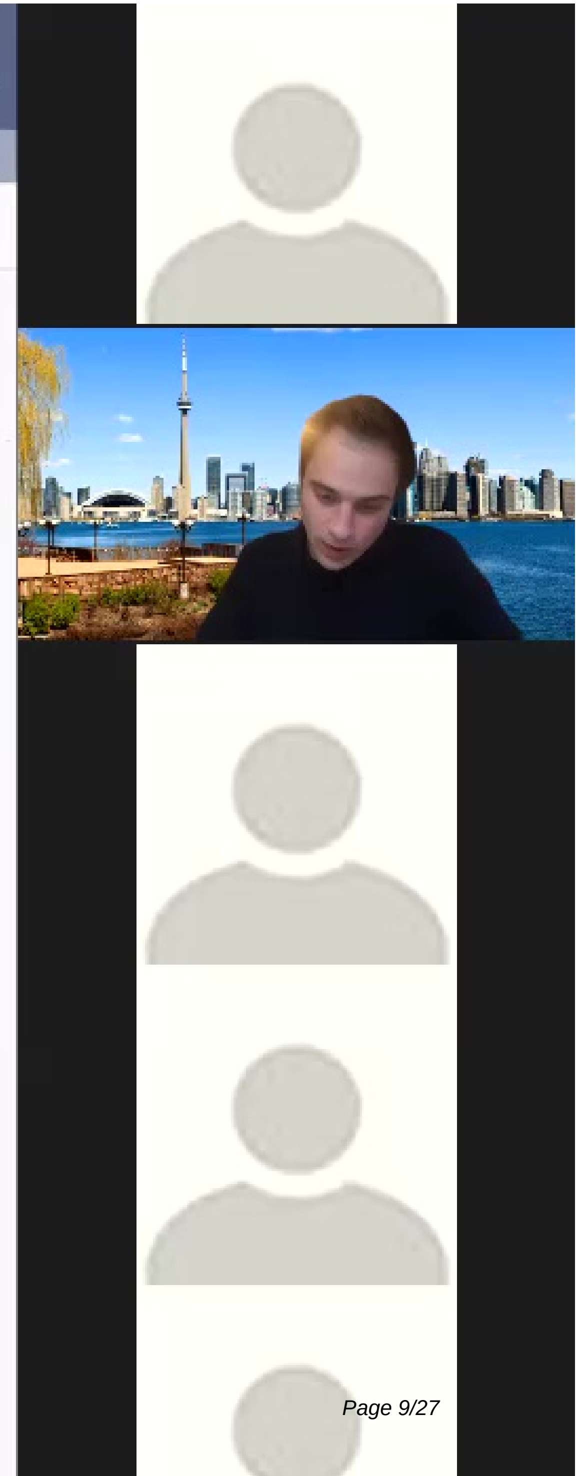
(iv)



Example:

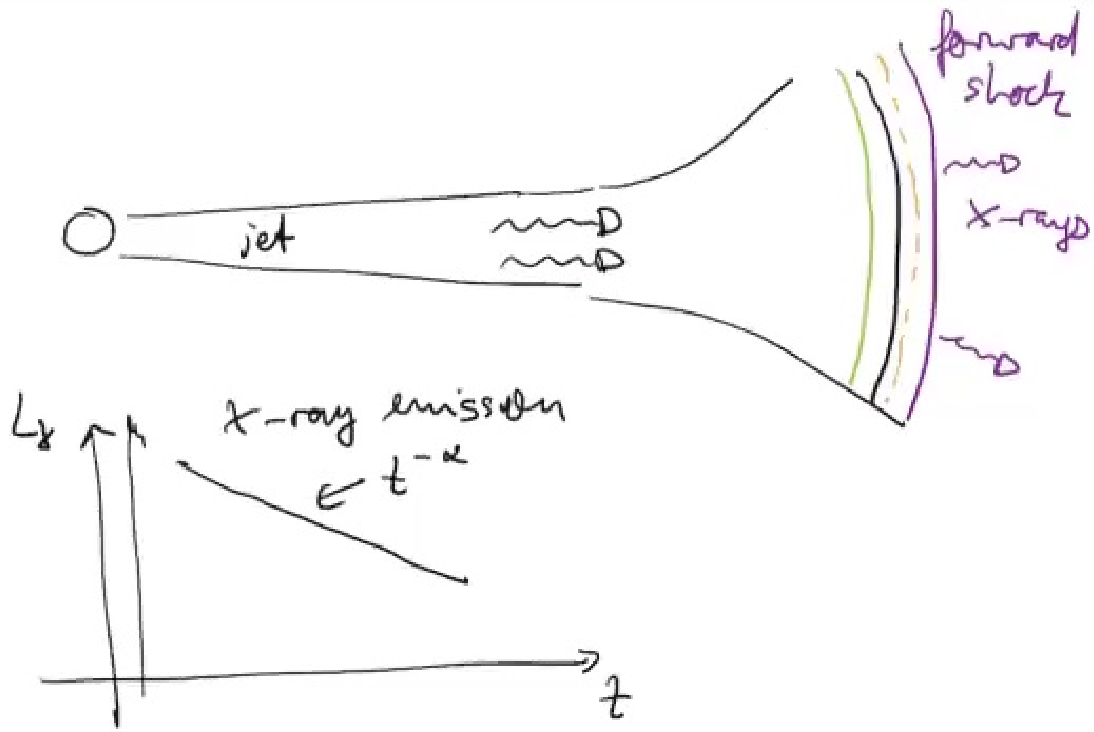
1)



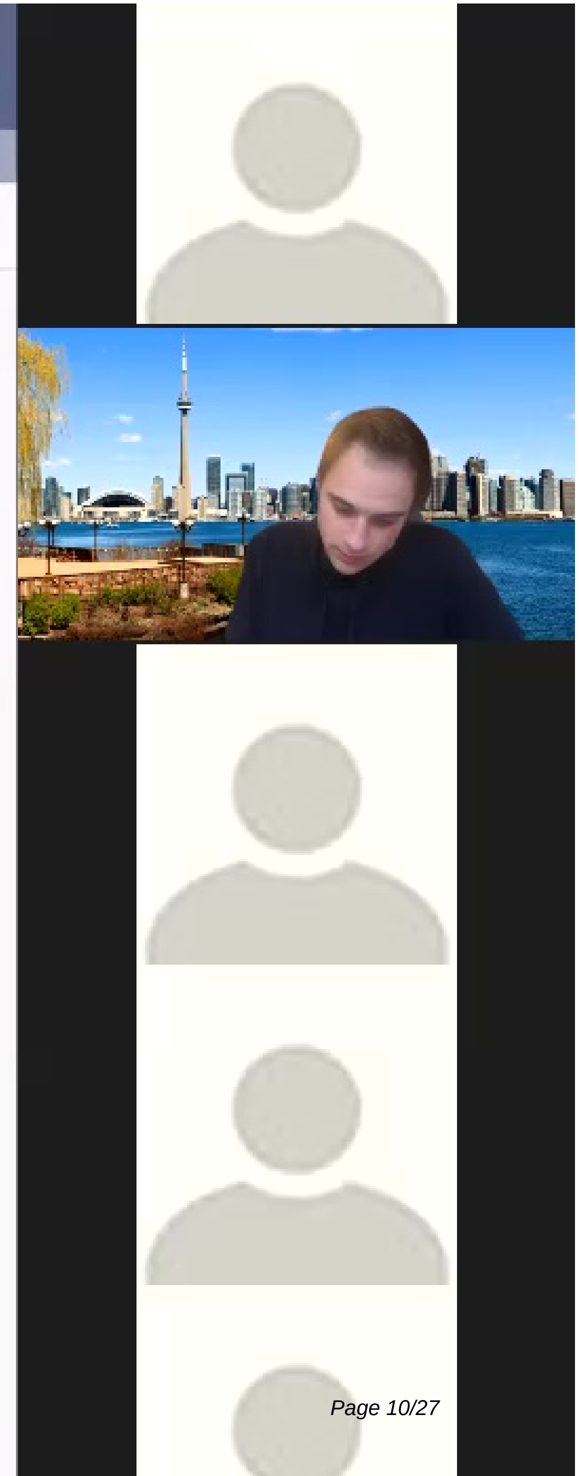


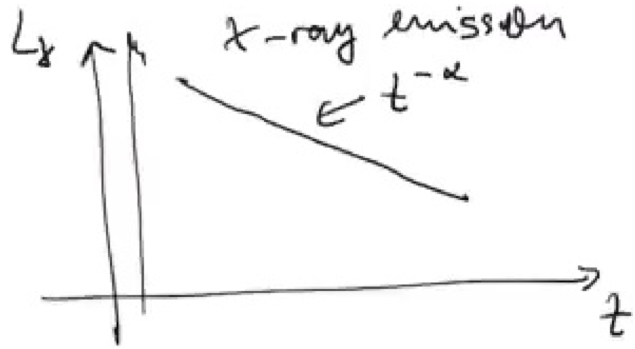
contact

2)



5.2 Solution strate



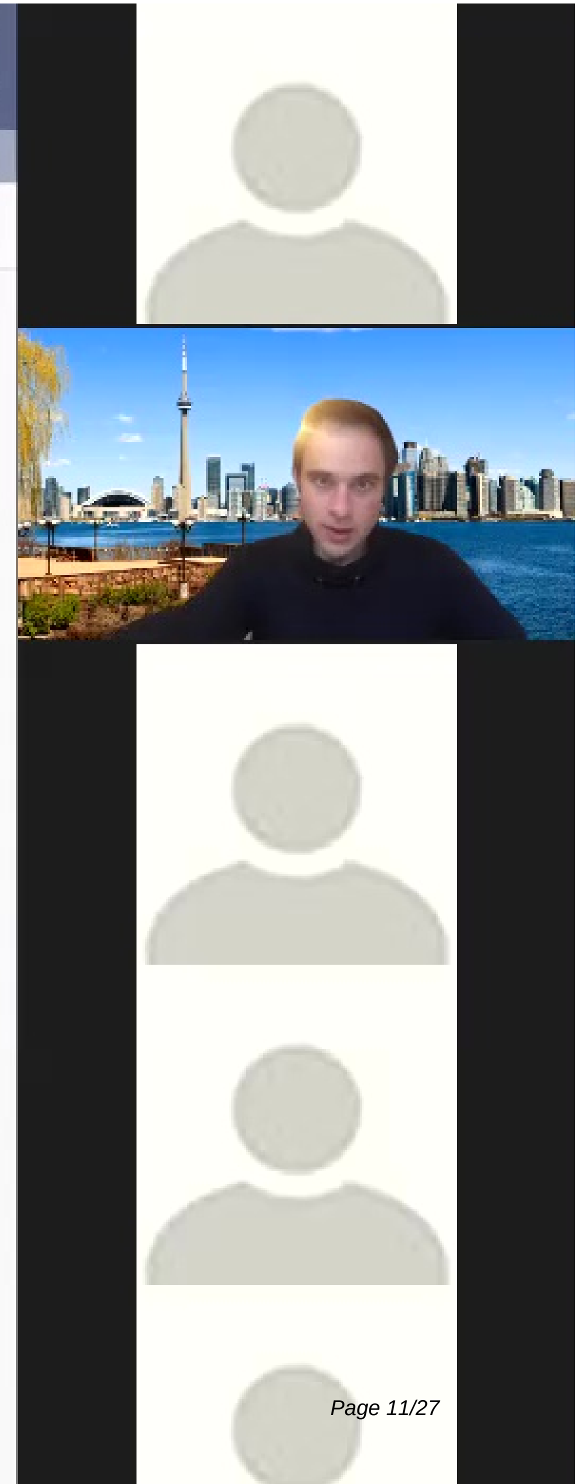


5.2 Solution strategy

① Assume we know u_*, p_*, evaluate the case (i)–(iv) that applies to this configuration of (u_*, p_*) and write down the combined solution

$$w = w(x,t).$$

② Given an initial guess for (u_*^{k}, p_*^{k})



9:41 AM Tue Jan 9

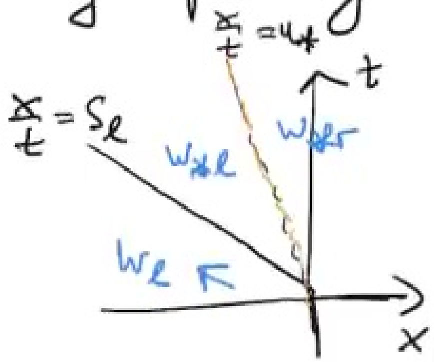
Lecture_15

Lecture_15 | Chap_4_Properties... | Lecture_14 | Chap_5_RiemannPr... | Chap_6_Numerical...

will follow with (1)

step (1): Treat regions left and right of the contact discontinuity separately

(1) $S \equiv \frac{x}{t} \le u_*$: left side

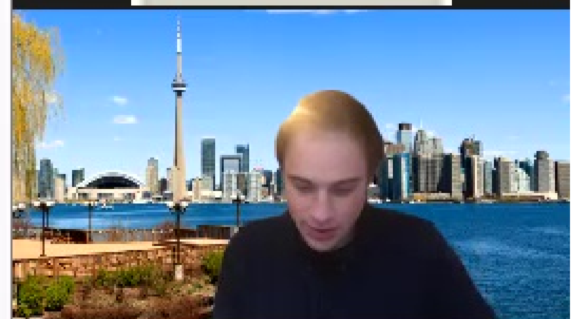


$p_* > p_\ell$: left shock (ii) or (iv)

$\left(\lambda(u_\ell) > \lambda(u_*)\right)$

$$w(x,t)_{left} = \begin{cases} W^{sh}_{*\ell} & \text{if } S_\ell \le \frac{x}{t} \le u_* \\ W_\ell & \text{if } \frac{x}{t} \le S_\ell \end{cases}$$

where $W^{sh}_{*\ell} = (\rho^{sh} \ldots$



shock

$p_* > p_\ell$: left shock

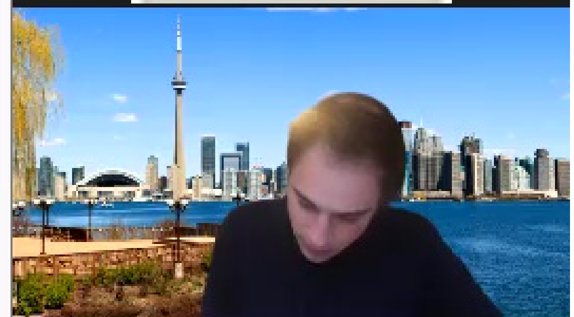
$\left(\lambda(u_\ell) > \lambda(u_*)\right)$ (ii) or (iv)

$$\boxed{w(x,t)_{left} = \begin{cases} w_{*\ell}^{sh} & \text{if } \ S_\ell \le \frac{x}{t} \le u_* \\ w_\ell & \text{if } \ \frac{x}{t} \le S_\ell \end{cases}}$$

where $w_{*\ell}^{sh} = (\rho_{*\ell}^{sh}, u_*, p_*)$

RH jump conditions

$$\rho_{*\ell} = \rho_\ell \frac{\frac{p_*}{p_\ell} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_\ell} + 1}$$



9:41 AM Tue Jan 9 79%

Lecture_15

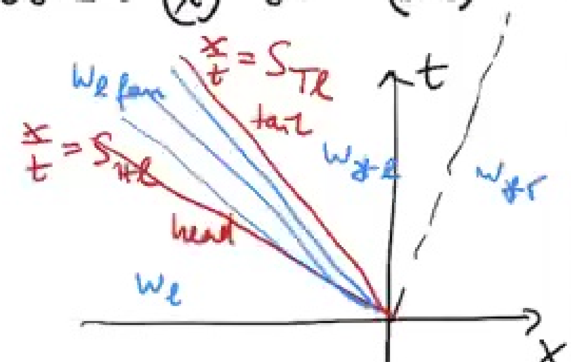
Lecture_15 | Chap_4_Properties... | Lecture_14 | Chap_5_RiemannPr... | Chap_6_Numerical...

$$S_{*l} = S_l \, \frac{\frac{p_*}{p_l} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1}\frac{p_*}{p_l} + 1}$$

$$S_l = u_l - c_l \sqrt{\frac{\gamma+1}{2\gamma}\frac{p_*}{p_l} + \frac{\gamma-1}{2\gamma}}$$

$\underline{p_* \le p_l :}$ $\Rightarrow$ left rarefaction wave: (i) or (iii)

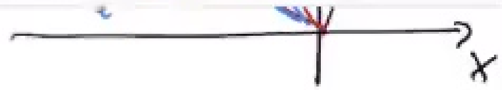
$(\lambda(u_l) < \lambda(u_*))$



9:41 AM Tue Jan 9 80%

Lecture_15

Lecture_15 | Chap_4_Properties... | Lecture_14 | Chap_5_RiemannPr... | Chap_6_Numerical...



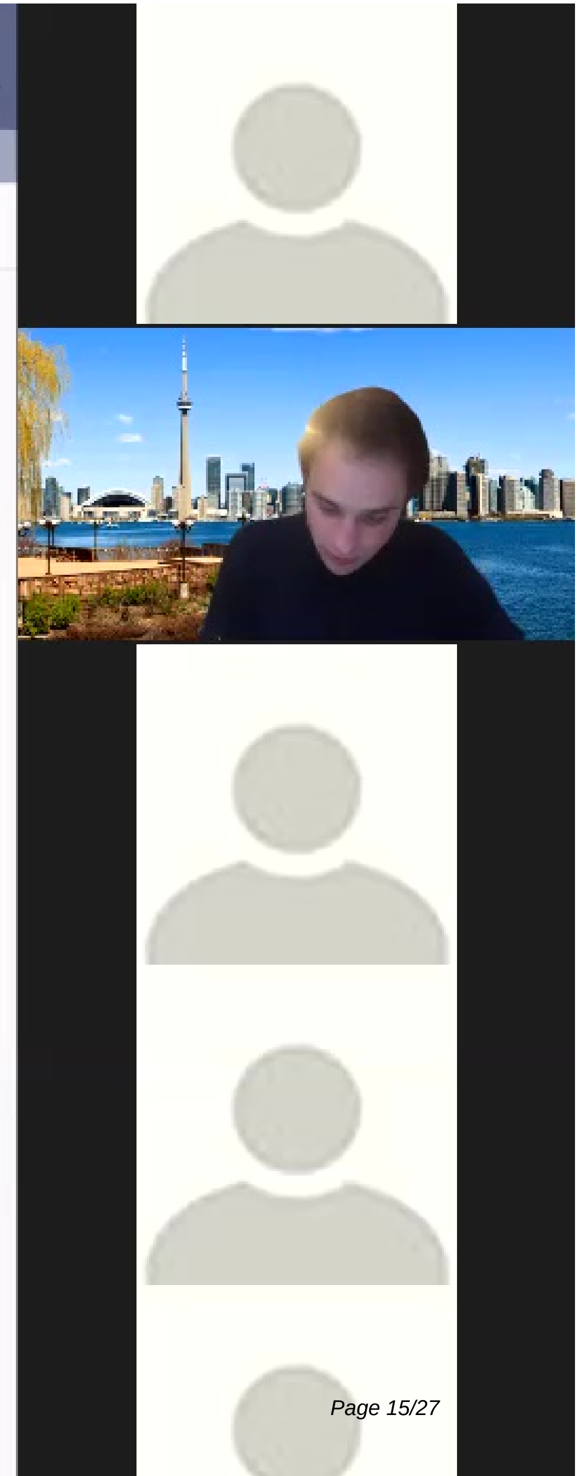
$$w(x,t)_{left} = \begin{cases} W_l \,, & \frac{x}{t} \le S_{Hl} \\ W_{l\,fan} \,, & S_{Hl} \le \frac{x}{t} \le S_{Tl} \\ W_{*l} \,, & S_{Tl} \le \frac{x}{t} \le u_* \end{cases}$$

where

$$W_{*l} = (\rho_{*l}, \underbrace{u_*, p_*}_{\text{known}})$$

$W_{l\,fan}$: state "inside" the rarefaction wave

Find: ρ_{*l}, S_{Hl}, S_{Tl},



where

$$W_{*l} = (\rho_{*l}, \underbrace{u_*, p_*}_{\text{known}})$$

$W_{l\,fan}$: state "inside" the rarefaction wave

Find: $\rho_{*l}, S_{Hl}, S_{Tl}, W_{l\,fan}$

$$\left.\begin{array}{ll} \mathrm{I} & S \\ \mathrm{II} & u + \dfrac{2c}{\gamma-1} \end{array}\right\} \text{1-Riemann invariants}$$

I
$\Rightarrow$ can use isentropic law $p = K\rho^{\gamma}$

adiabatic index of EOS

across wave

$$p_l = K\rho_l^{\gamma} \Rightarrow K = \frac{p_l}{\ldots}$$

$$\left.\begin{array}{ll} \mathrm{I} & s \\ \mathrm{II} & u + \dfrac{2c}{\gamma-1} \end{array}\right\}$$ 1-Riemann invariants

I

→ can use isentropic law $p = K\rho^{\gamma}$

adiabatic index of EOS

across wave

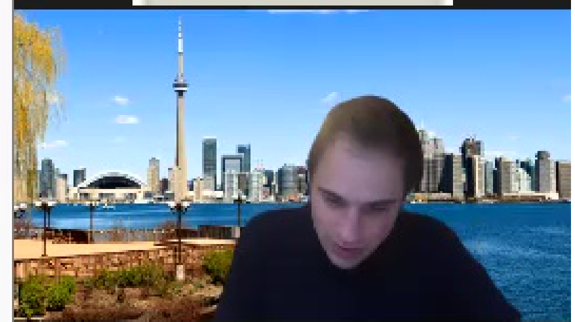
$$p_\ell = K\rho_\ell^{\gamma} \Rightarrow K = \frac{p_\ell}{\rho_\ell^{\gamma}}$$

$$\boxed{\rho_{*\ell} = \rho_\ell \left(\frac{p_*}{p_\ell}\right)^{\frac{1}{\gamma}}}$$

$$c = \sqrt{\gamma \frac{p}{\rho}}$$

$$\downarrow$$

$$c_{*\ell} = c_\ell \left(\frac{p_{*\ell}}{\ldots}\right)^{\frac{\gamma-1}{2\gamma}}$$



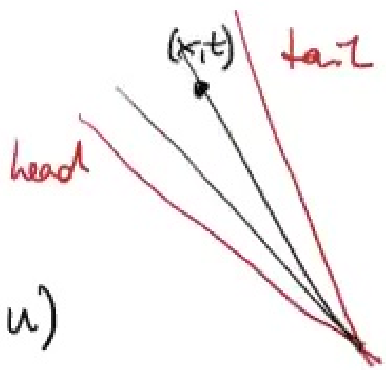
$$c_{*l} = c_l \left(\frac{p_{*l}}{p_l}\right)^{\frac{\gamma-1}{2\gamma}}$$

Head & tail speeds:

$$S_{Hl} = \lambda_1(u_l) = u_l - c_l$$

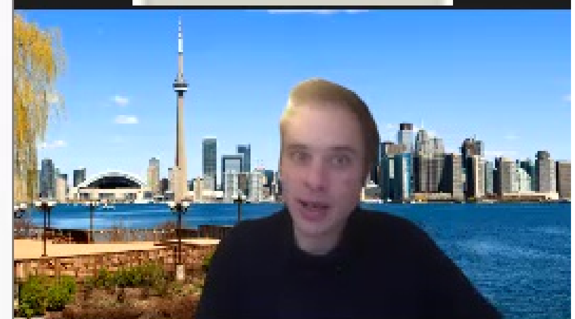
$$S_{Tl} = \lambda_1(u_*) = u_* - c_{*l}$$

W_l fan:



$$\frac{dx}{dt} = \frac{x}{t} = \lambda_1(u)$$

$$= u - c \qquad (1)$$



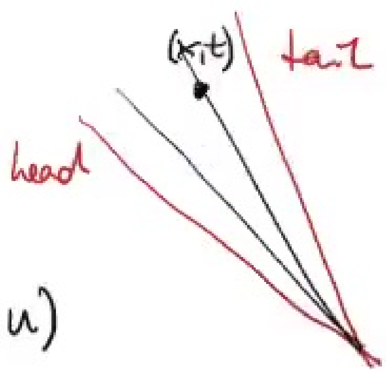
9:41 AM Tue Jan 9

Lecture_15

Lecture_15 | Chap_4_Properties... | Lecture_14 | Chap_5_RiemannPr... | Chap_6_Numerical...

$$S_{Tl} = \lambda_1(u_*) = u_* - c_{*l}$$

Wl fan:



$$\frac{dx}{dt} = \frac{x}{t} = \lambda_1(u) = u - c \qquad (1)$$

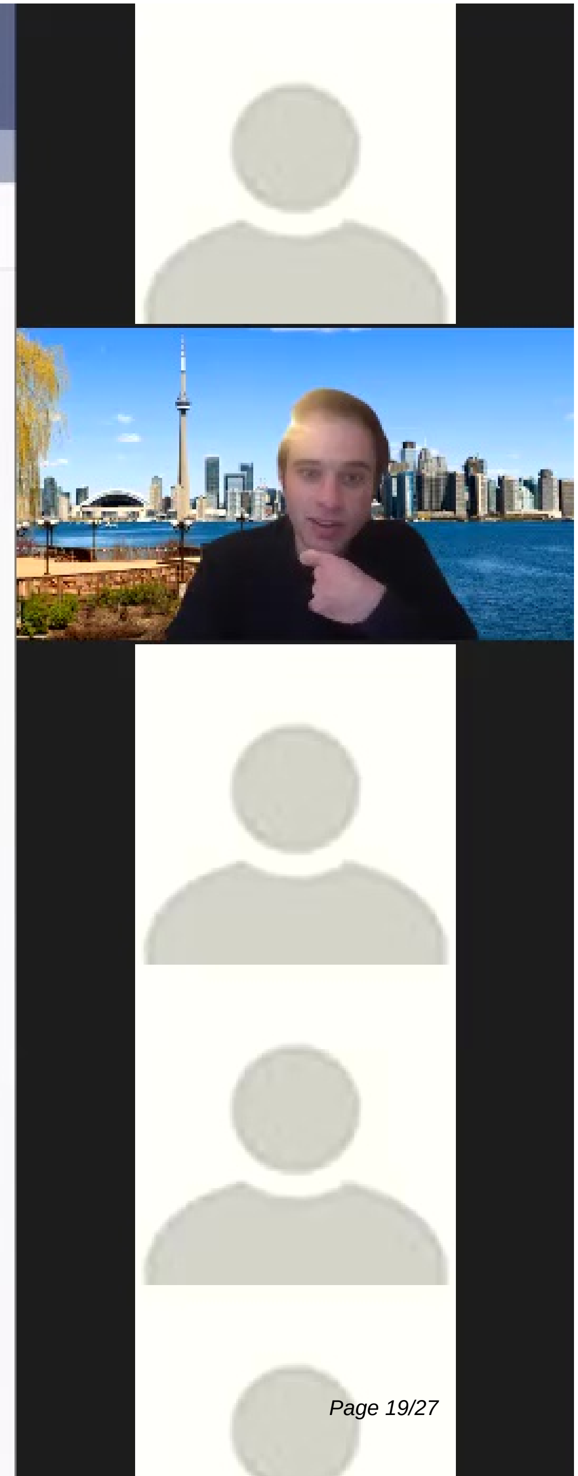
(u, c)

relate to u_l, c_l via II:

$$u_l + \frac{2c_l}{\gamma-1} = u + \frac{2c}{\gamma-1} \qquad (2)$$

combine to:

$$\rho = \rho_l \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_l}\left(u_l - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}}$$

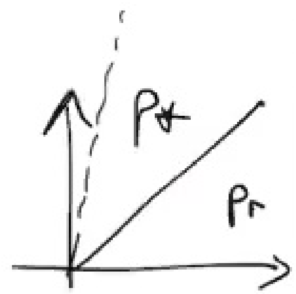


$$
W_{l\,\text{fan}} = \begin{cases} \rho = \rho_l \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_l}\left(u_l - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}} \\ u = \frac{2}{\gamma+1}\left[c_l + \frac{\gamma-1}{2}u_l + \frac{x}{t}\right] \\ p = p_l\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{(\gamma+1)c_l}\left(u_l - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}} \end{cases}
$$

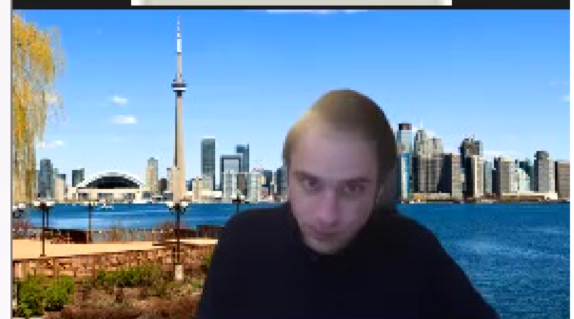
(2) $S \equiv \frac{x}{t} \geq u_*$: right side

analogous to above

$\underline{p_* > p_r}$: right shock (i) or (iv)



$$
w(x,t)_{\text{right}} = \begin{cases} w^{sh}_{*r} & u_* \leq \frac{x}{t} \leq S_r \\ w_r\,, & \frac{x}{t} \geq S_r \end{cases}
$$



$$w(x,t)_{\text{right}} = \begin{cases} w_{r\,\text{fan}}, & S_{Tr} \le \frac{x}{t} \le S_{Hr} \\ w_r, & \frac{x}{t} \ge S_{Hr} \end{cases}$$

② Find (u_*, p_*) given initial guess:

Proposition: The solution p_* is given by the root of the function (assuming an ideal gas EOS):

$$f(p, w_\ell, w_r) \equiv f_\ell(p, w_\ell) + f_r(p, w_r) + u_r - u_\ell$$

9:41 AM Tue Jan 9 89%

Lecture_15

Lecture_15 | Chap_4_Properties... | Lecture_14 | Chap_5_RiemannPr... | Chap_6_Numerical...

$$\ldots \frac{\ldots}{\gamma-1}\left[\left(\frac{p}{p_\ell}\right)^{\ldots} - 1\right], \quad p \le p_\ell \ \text{(rarefaction wave)}$$

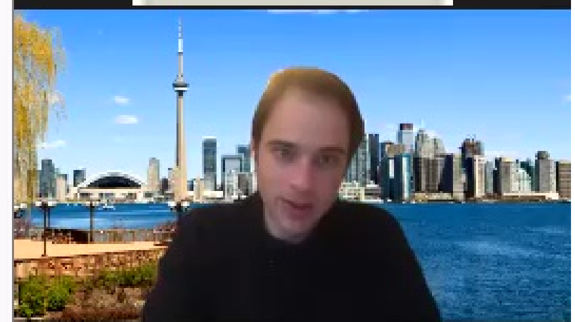
$$f_r(p, W_r) = \begin{cases} (p - p_r)\sqrt{\dfrac{A_r}{p + B_r}}, & p > p_r \ \text{(shock)} \\ \dfrac{2c_r}{\gamma-1}\left[\left(\dfrac{p}{p_r}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right], & p \le p_r \ \text{(rarefaction wave)} \end{cases}$$

with given constants

$$A_\ell \equiv \frac{2}{(\gamma+1)\rho_\ell}, \quad B_\ell \equiv \frac{\gamma-1}{\gamma+1} p_\ell$$

$$A_r \equiv \frac{2}{(\gamma+1)\rho_r}, \quad B_r \equiv \frac{\gamma-1}{\gamma+1} p_r$$

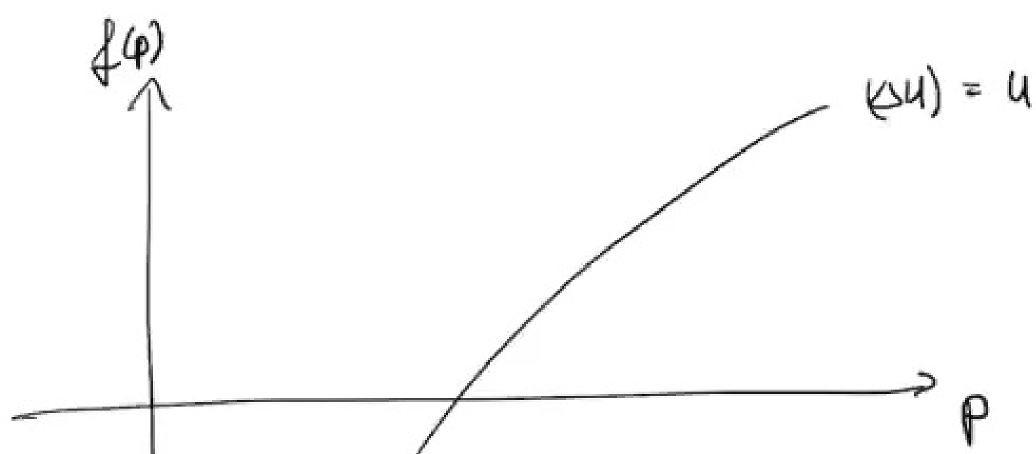
The solution for the velocity ...

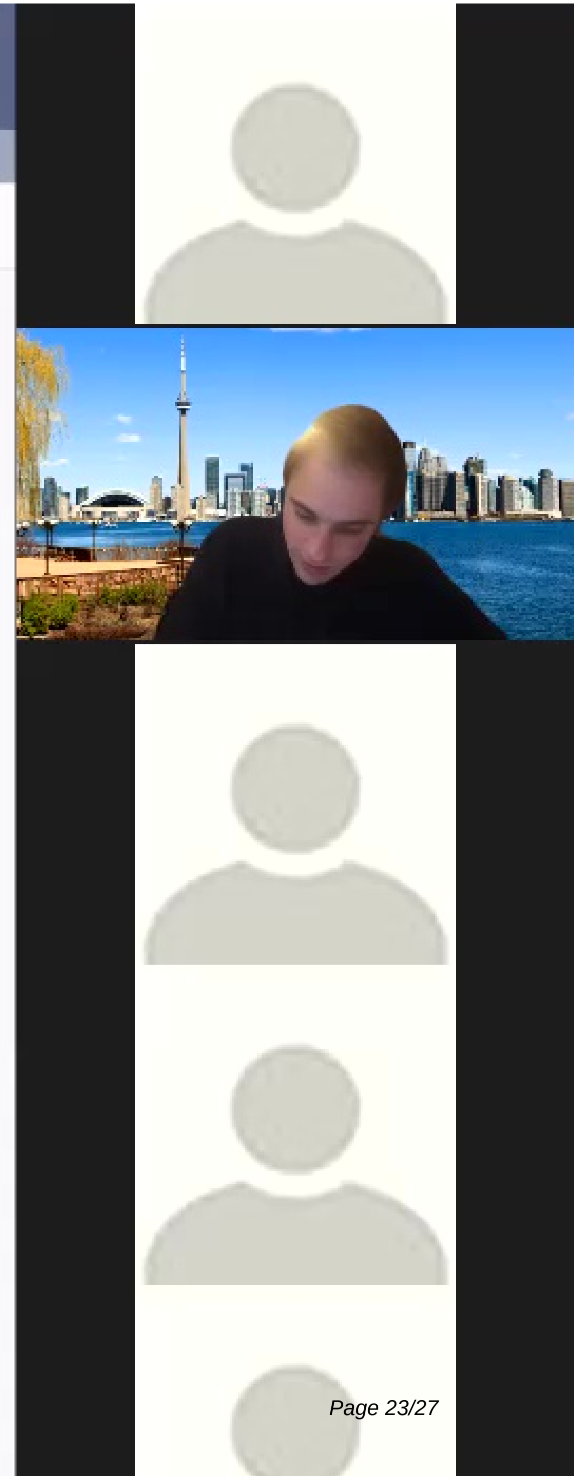


Remarks: 1) Behavior of $f(p)$

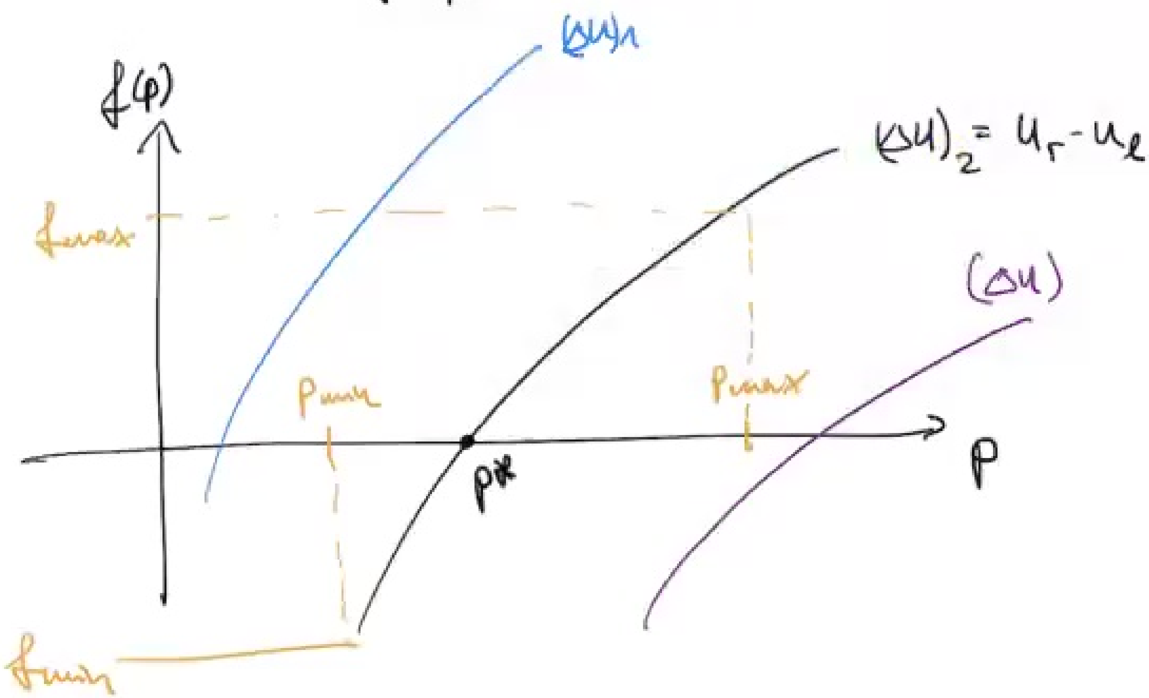
Can show:

- $f'(p) > 0 \rightarrow$ monotonically increasing
- $f''(p) < 0 \rightarrow$ concave
- $\left\{ \begin{matrix} f'(p) \\ f''(p) \end{matrix} \right\} \xrightarrow{p \to \infty} 0$



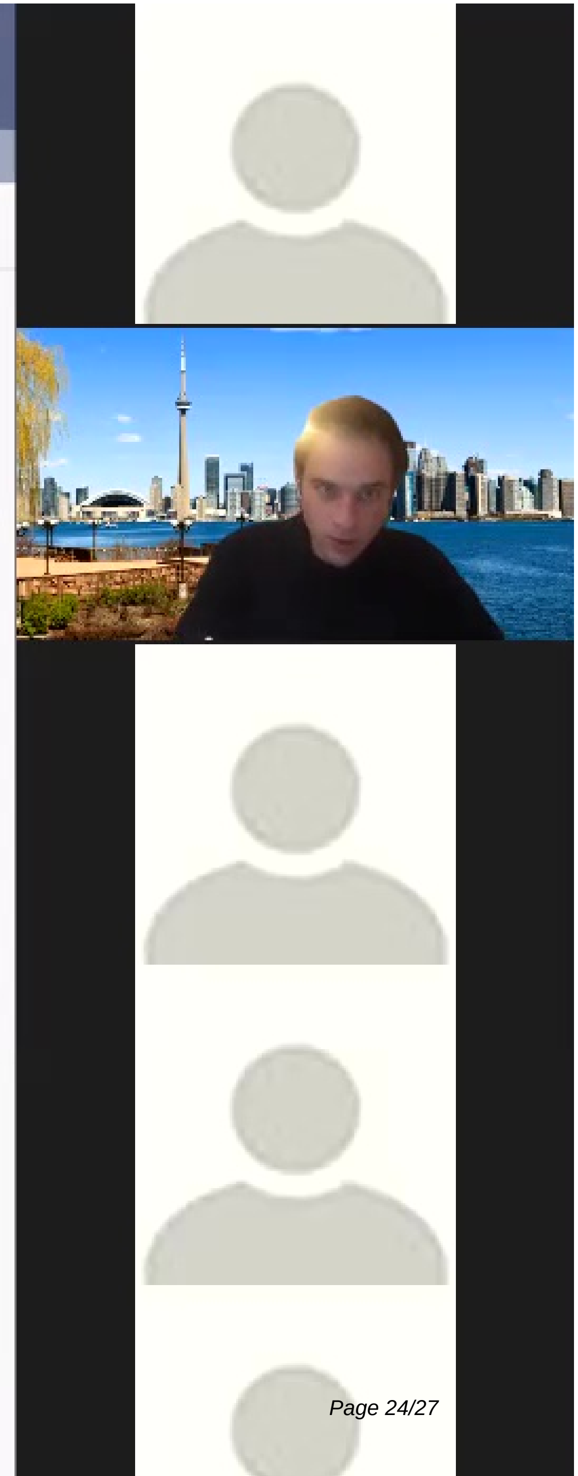


- $\left\{ \begin{array}{l} f'(p) \\ f''(p) \end{array} \right\} \xrightarrow{p \to \infty} 0$



$$p_{min} = \min(p_l, p_r), \quad f_{min} = f(p_{min})$$

$$p_{max} = \max(p_l, p_r), \quad f_{max} = f(p_{max})$$



3 regimes: $f_{min}, f_{max} > 0 \Rightarrow p_* \in I_1$

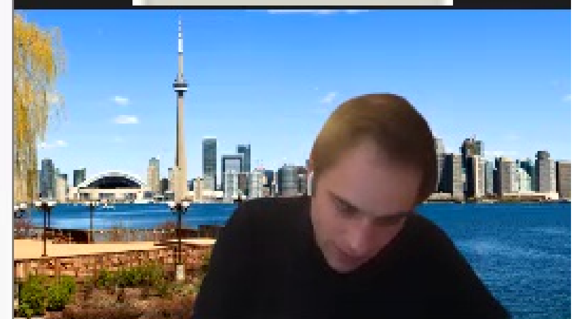
$f_{min} \leq 0,\ f_{max} \geq 0 \Rightarrow p_* \in I_2$

$f_{min}, f_{max} < 0 \Rightarrow p_* \in I_3$

I_1: $p_* < p_l, p_r \Rightarrow$ 2 rarefaction waves

I_2: $\left.\begin{array}{l} p_l < p_* < p_r \\ p_r < p_* < p_l \end{array}\right\}$ one rarefaction wave & one shock

I_3: $p_* > p_l, p_r \Rightarrow$ 2-shock solution



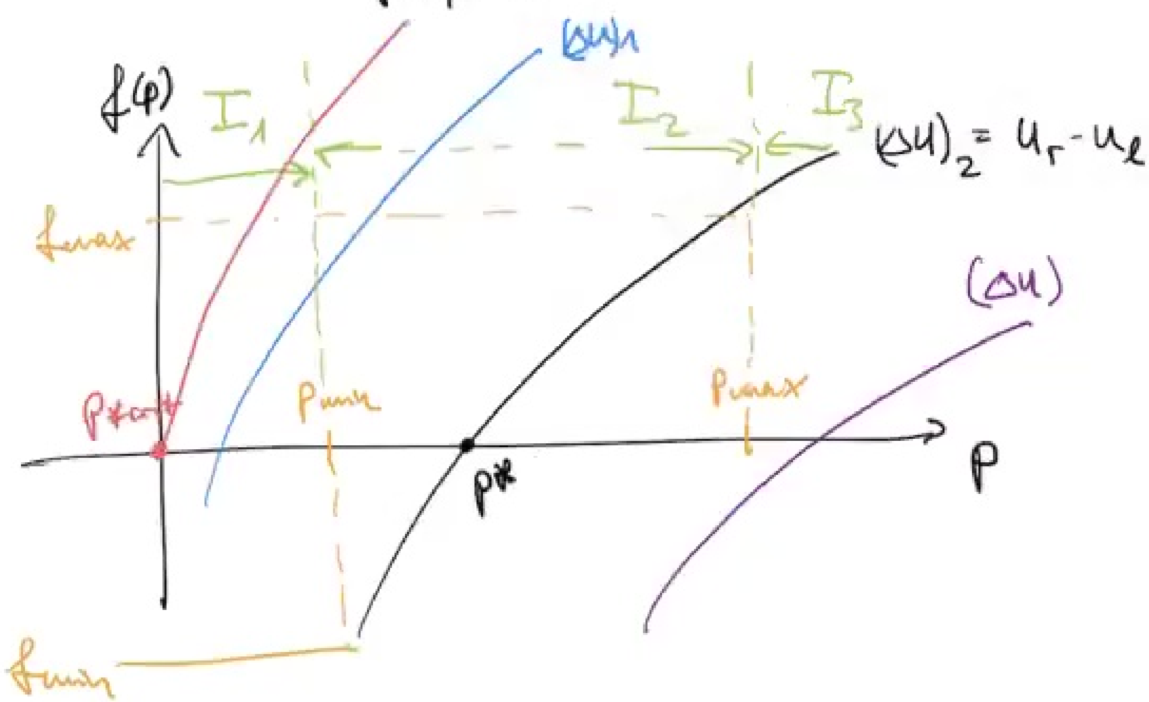
9:41 AM Tue Jan 9 93%

Lecture_15

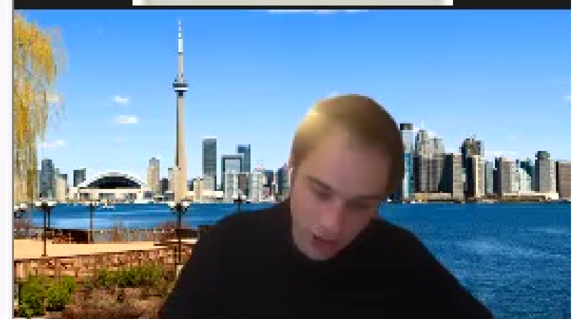
Lecture_15 | Chap_4_Properties... | Lecture_14 | Chap_5_RiemannPr... | Chap_6_Numerical...

- $f''(p) < 0 \to$ concave

- $\begin{Bmatrix} f'(p) \\ f''(p) \end{Bmatrix} \xrightarrow{p\to\infty} 0$



$p_{min} = \min(p_l, p_r)$, $f_{min} = f(p_{min})$



9:41 AM Tue Jan 9

Lecture_15

Lecture_15 | Chap_4_Properties... | Lecture_14 | Chap_5_RiemannPr... | Chap_6_Numerical...

2) Vacuum:

A physical solution to $f(p)$ requires

$f(0) < 0$ (monotonicity property)

For $p_* = p_{crit} = 0$ we have $p_* \le p_l, p_r$

$\Rightarrow$ left & right rarefaction

$$\rightsquigarrow \underline{(u_r - u_l)_{crit}} \overset{f(p_{*crit})=0}{=} -f_l(0) - f_r(0)$$

$$= \frac{2}{\gamma - 1}(c_l + c_r) \quad (> u_r - u_l)$$