Title: Special Topics in Astrophysics - Numerical Hydrodynamics - Lecture 14

Speakers: Daniel Siegel

Collection: Special Topics in Astrophysics - Numerical Hydrodynamics

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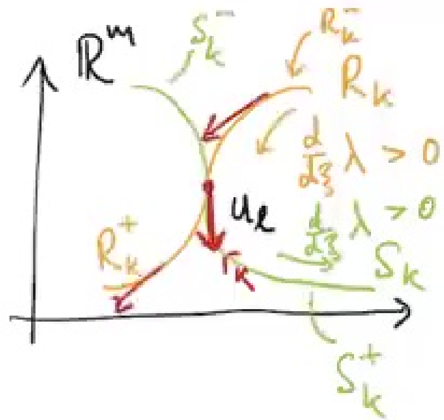
Recap:



RH jump conditions

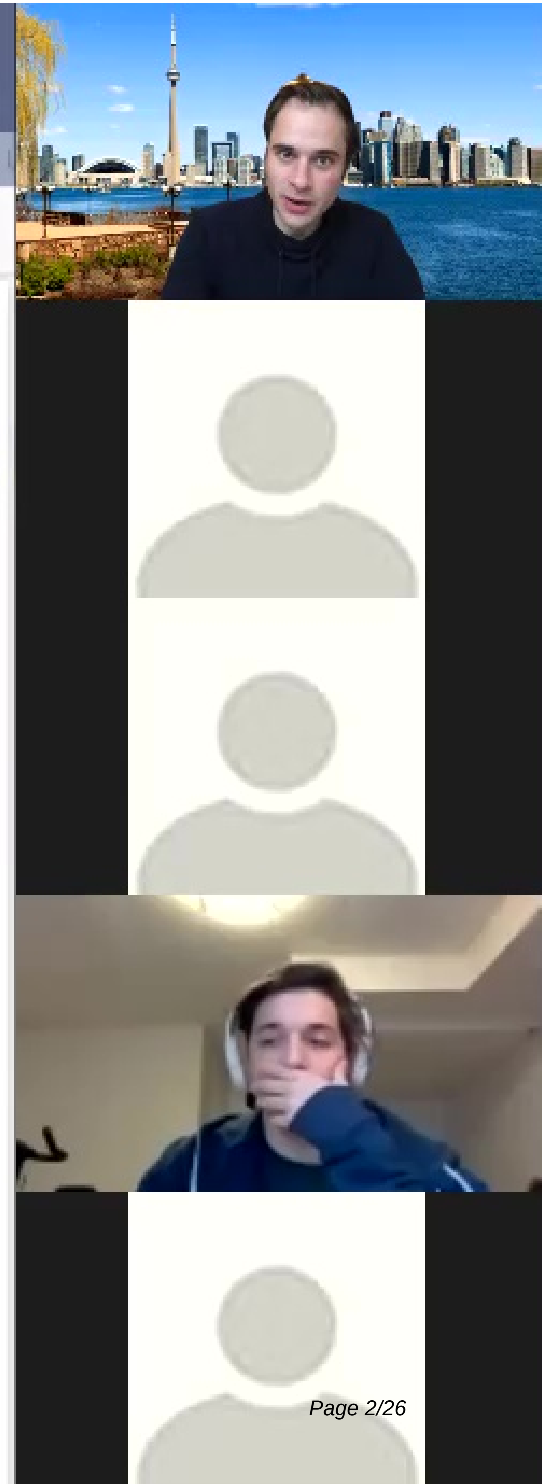
$$s(u_\ell - u_r) = f(u_\ell) - f(u_r)$$

m conditions

2m+1 unknowns

$$\rightarrow S_k^+(u_\ell) \equiv \{ z \in S_k(u_\ell) \mid \lambda_k(u_\ell) < s < \lambda_k(z) \}$$

$$S_k^-(u_r) \equiv \{ z \in S_k(u_\ell) \mid \lambda_k(z) < s < \lambda_k(u_\ell) \}$$

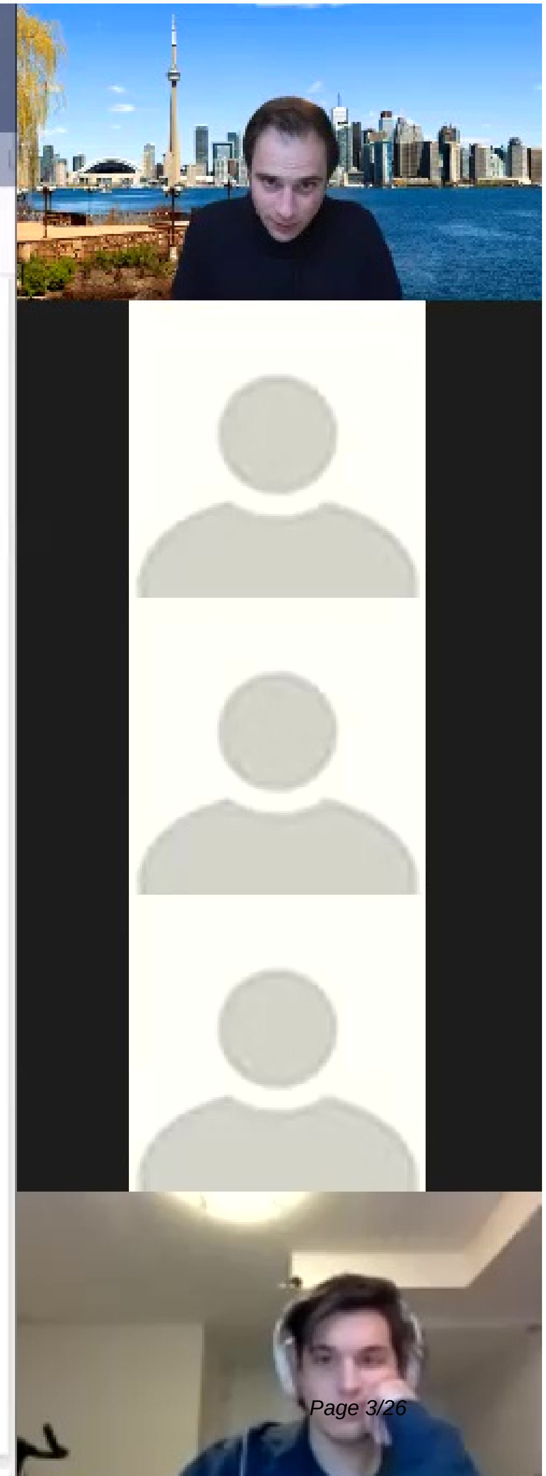


4.4.5 Contact Discontinuities

Theorem: Consider a strictly hyperbolic system of CLs $u_t + f(u)_x = 0$ and suppose the (λ_k, r_k) is linearly degenerate on $D \subseteq \mathbb{R}^m$ for $k \in \{1, \dots, m\}$. Then for all $u_\ell \in D$:

(i) $R_k(u_\ell) = S_k(u_\ell)$

(ii) $s(z, u_\ell) = \lambda_k(z) = \lambda_k(u_\ell)$



$\gamma' = \lambda_k = s$)

<u>Proof</u>: Consider the unique solution to

$$\begin{cases} v'(\xi) = r_k(v(\xi)) & \xi \in (-a, a) \\ v(0) = u_\ell \end{cases}$$

with $a > 0$ sufficiently small.

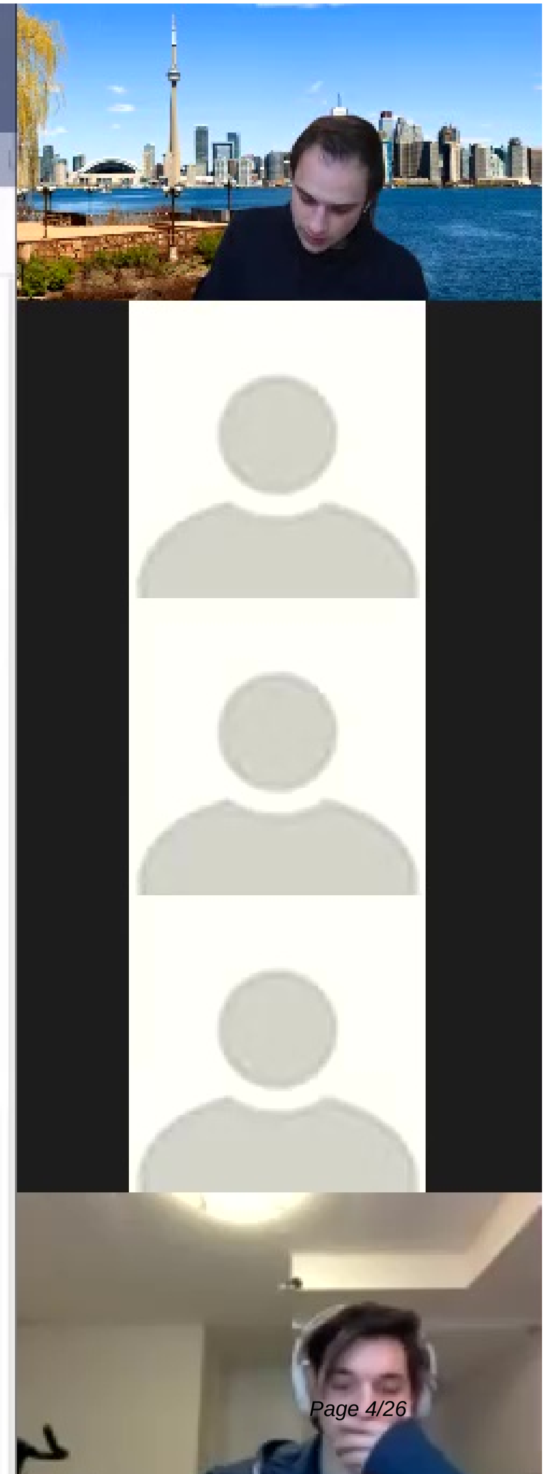
Then:

$$\frac{d}{d\xi} \lambda_k(v(\xi)) = \nabla \lambda_k(v(\xi)) \cdot v'(\xi)$$

$$= \nabla \lambda_k \cdot r_k = 0$$

(↑ assumption)

$\Rightarrow \lambda_k$ constant along



$$v(0) = u_l$$

with $a > 0$ sufficiently small.

Then:

$$\frac{d}{d\xi} \lambda_k(v(\xi)) = \nabla \lambda_k (v(\xi)) \cdot v'(\xi)$$

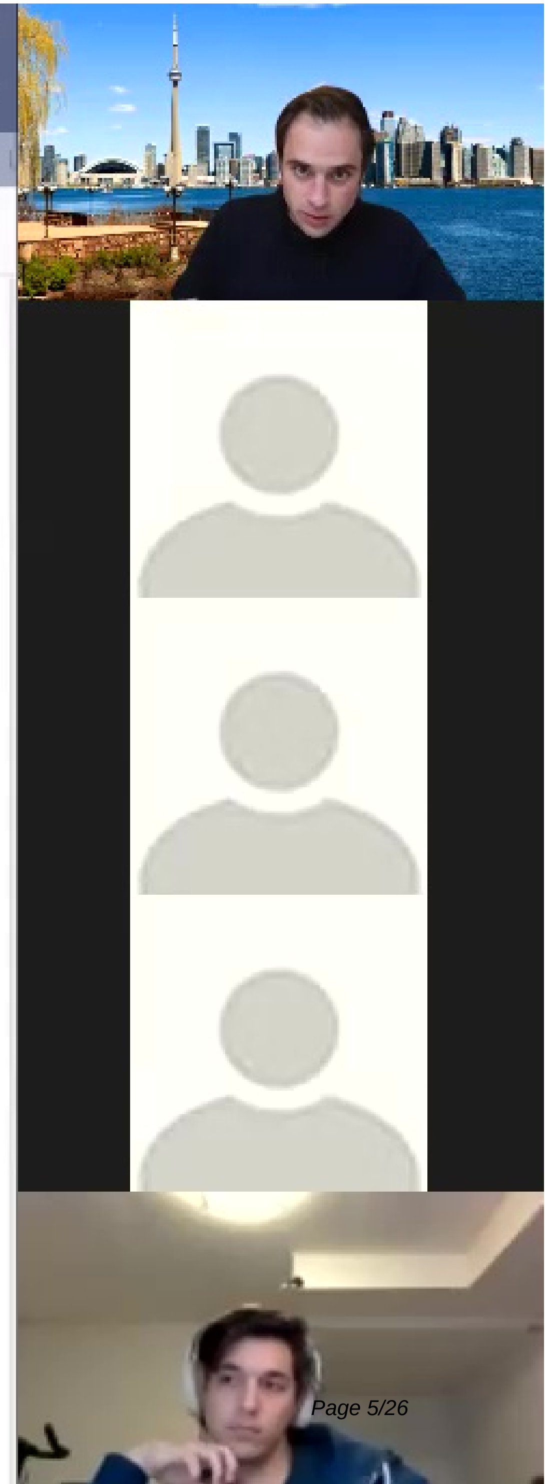
$$= \nabla \lambda_k \cdot r_k \underset{\substack{\uparrow \\ \text{assumption}}}{=} 0$$

$\Longrightarrow$ λ_k constant along $(*)$

R_k

$$\rightsquigarrow \quad f(\underbrace{v(\xi)}_{\text{"}u_r\text{"}}) - f(u_l) = \int_0^{\xi} Df(v(t))\, v'(t)\, dt$$

ξ



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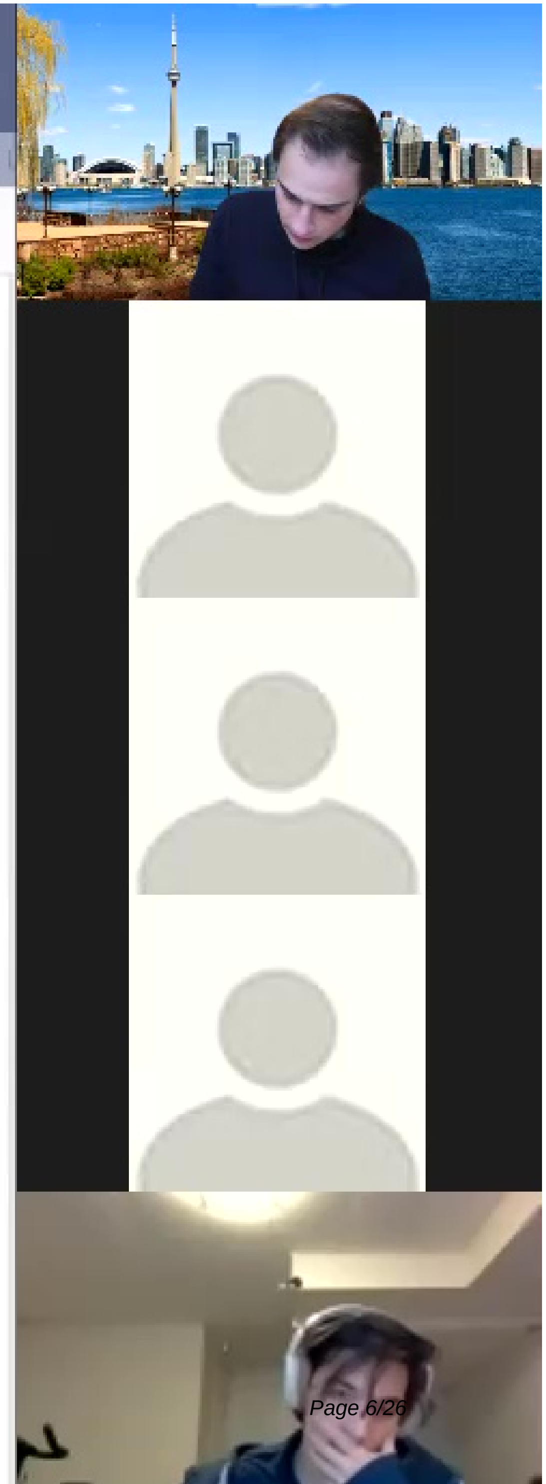


Def (Contact discontinuity)

Let u be a weak solution of $u_t + f(u)_x = 0$ with a discontinuity along $\Sigma = \{\sigma(t), t\} \subseteq \mathbb{R} \times (0, \infty)$. Let $\lambda_k(u^\pm(\sigma(t), t)) = s(t) = \sigma'(t)$ for some characteristic field λ_k of $Df(u)$,

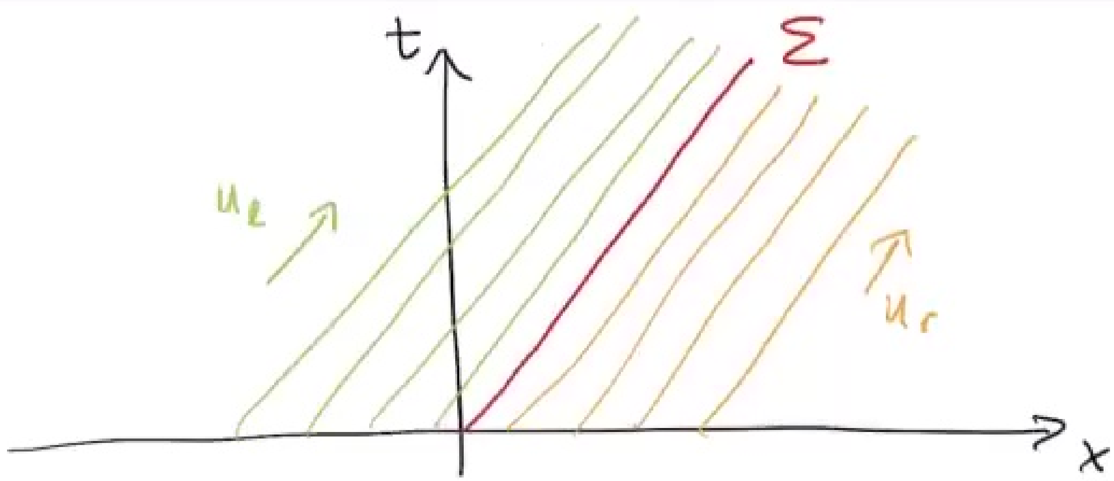
and $$u^\pm(\sigma(t), t) = \lim_{\varepsilon \to 0} u(s(t) \pm \varepsilon, t)$$

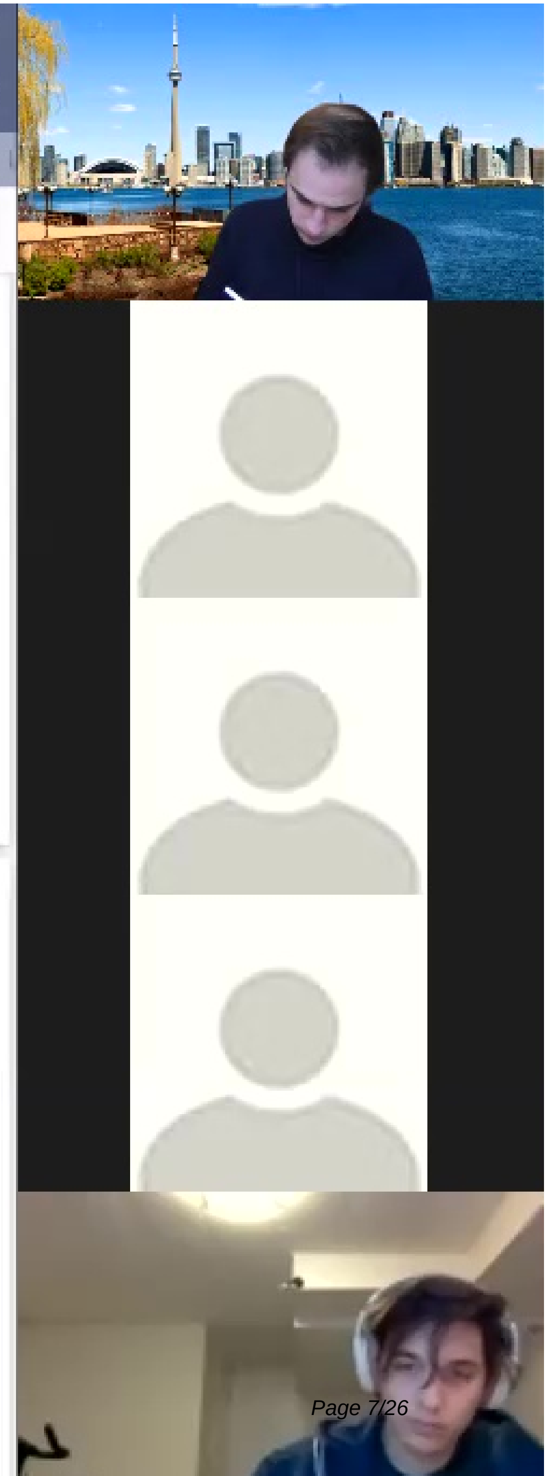
Then Σ is called a contact discontinuity w.r.t. λ_k.

Remark: 1) Note characteristic speed λ_k

Then Σ is called a contact discontinuity w.r.t. λ_k.

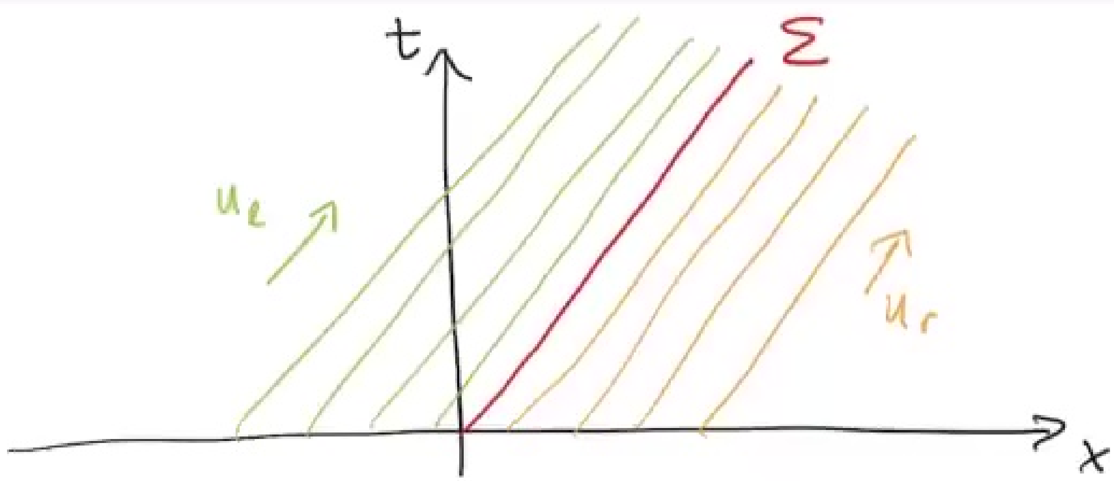
Remark: 1) Note characteristic speed λ_k on both sides of Σ are the same





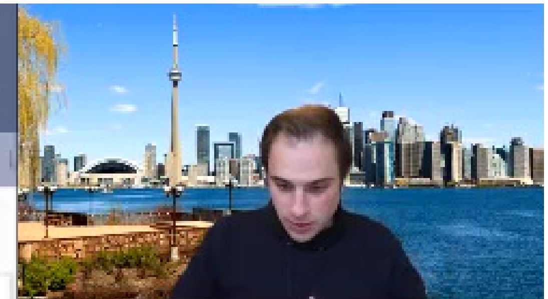
Structure: suppose (λ_k, r_k) are linearly

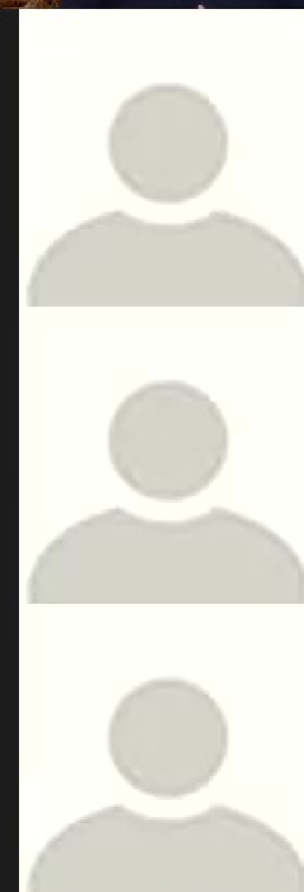
Remark: 1) Note characteristic speed λ_k on both sides of Σ are the same

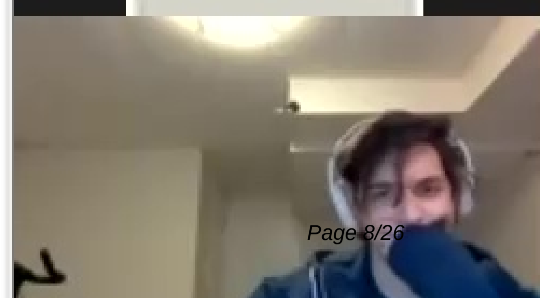


Structure: suppose (λ_k, r_k) are linearly degenerate and $u_r \in S_k(u_\ell)$, then

$$u(x,t) = \begin{cases} u_\ell, & x < st \\ \ldots \end{cases}$$







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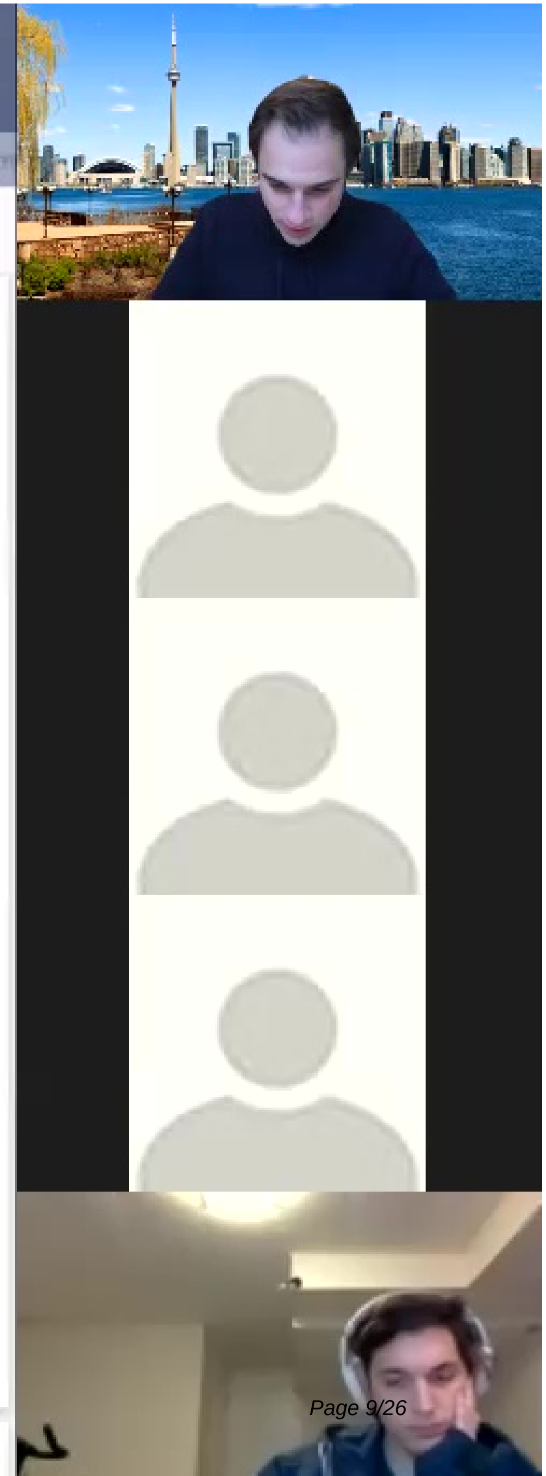
$$u(x,t) = \begin{cases} u_l, & x < st \\ u_r, & x > st \end{cases}$$

with $s = s(u_r, u_l) = \lambda_k(u_r) = \lambda_k(u_l)$ is a weak solution of the Riemann problem.

4.4.6 Local solution of Riemann's problem



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Riemann1860

Material zu beherrschen, oder wenigstens um so weniger wesentlich pflegt für das Kunstwerk dasselbe zu seyn. Bei Gefässen, welche zur Aufbewahrung von Flüssigkeiten, oder zur Bereitung von Speisen dienen sollen, ist es nicht gleichgültig, ob sie aus Thon, Stein oder Metall bestehen; sobald es aber nur darauf ankommt, schön geformte Gefässe die zur Zierde dienen sollen zu verfertigen, ist es gleichgültiger, ob man Porphyr oder Alabaster, Thon oder Bronze dazu nimmt. Indessen kann auch die schöne Kunst sich nie ganz von dem Einflusse des Materials frei machen. Das Material schreibt der zurichtenden Kraft bald mehr bald weniger den Weg vor, ist nicht selten eine Hemmung für das freie Walten der Kunstidee; und hat oft auf den Eindruck den ein Kunstwerk macht, einen nicht unbedeutenden Einfluss. Thon muss anders behandelt werden als Stein; und ein grosser Unterschied ist es, ob ein harter Porphyr, oder ein weicher Alabaster zu bearbeiten ist. Von der dünnen zarten Ausbildung Griechischer Thongefässe hielt sich im Alterthum die Darstellung von Gefässen aus hartem Stein sehr fern; und nicht einmal ist es durch die in neueren Zeiten so sehr vervollkommneten mechanischen Hülfsmittel, wie sie z. B. in der Schleiferei zu Elfdalen in Schweden angewandt werden, gelungen, aus hartem Porphyr Gefässe zu bilden, welche in jener Eigenschaft den Griechischen Thongefässen gleich kommen, so vollkommen auch übrigens die Formen derselben nachgeahmt werden. Der weiche Thon gehorcht unter der Hand des bildenden Künstlers willig den Eingebungen der Phantasie; der starre Marmor, der nur dem Meissel und der Feile nachgiebt, hemmt dagegen ihren Flug. Der Eindruck den eine bronzene Statue macht, ist sehr abweichend von dem eines Bildwerks aus Marmor.

Wenn man nun gleich der Natur einen bedeutenden Einfluss auf die Kunst einräumen darf, so ist doch grosse Vorsicht nöthig, damit man jener nicht zu viel zutraue. Hin und wieder ist man in dieser Hinsicht offenbar zu weit gegangen, indem man z. B. bald in einem altdeutschen Götterhaine, oder einem Palmenwalde, bald in den Säulen des Basaltes den Prototyp der sogenannten

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Lax - 1957 - Hyperbolic systems of conservation l...

Chap_5_Riemann... | Riemann - 1860 - ... | Lax - 1957 - H... | Riemann1860 | CFD_problem

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. X, 537-566 (1957)

Hyperbolic Systems of Conservation Laws II*

P. D. LAX

1. Introduction

A conservation law is an equation in divergence form, i.e.

$$u_t + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} f_j = 0.$$

It expresses the fact that the quantity of u contained in any domain G of x-space changes at a rate equal to the flux of the vector-field (f_1, f_2, f_3) into G:

$$\frac{d}{dt}\iiint_G u\, dx = \iint_{BG} f \cdot n\, dS.$$

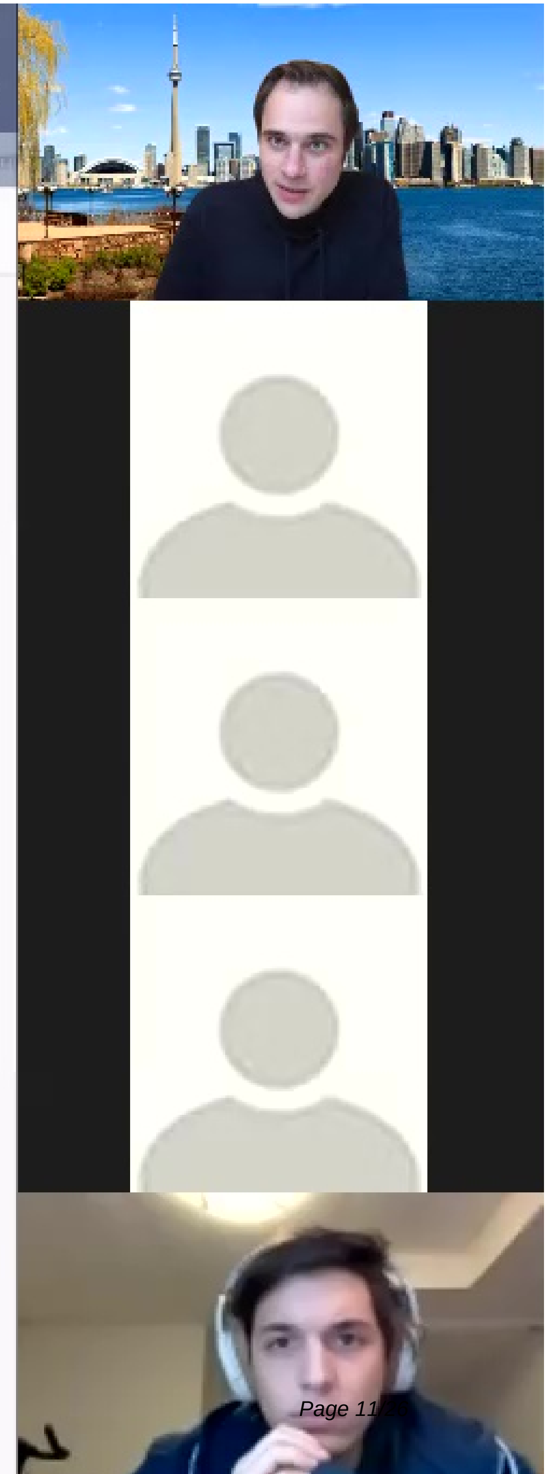
Many physical laws are conservation laws; the quantities u and f depend on the variables describing the state of a physical system, and on their derivatives. In theories which ignore mechanisms of dissipation such as viscous stresses, heat conduction, ohmic loss, the conservation laws are of first order, i.e., the quantities u and f are functions of the state variables but not of their derivatives. In this paper we shall consider such systems of first order conservation laws in one space variable. The components of u shall be chosen as state variables so that the system is in the form

(1.1) $$u_t + f_x = 0,$$

where u is a vector of n components and $f = f(u)$ a vector valued function of u.

When the differentiation in (1.1) is carried out, a quasilinear system of first order results:

(1.1′) $$u_t + A(u)u_x = 0, \qquad A = \text{grad } f.$$



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This can be summarized in

THEOREM 9.1. *Every state u_0, has a neighborhood such that, if u_n belongs to this neighborhood, the Riemann initial value problem* (9.1) *has a solution. This solution consists of $n+1$ constant states connected by centered waves. There is exactly one solution of this kind, provided the intermediate states are restricted to lie in a neighborhood of u_0.*

In order to find such a solution when the state u_n is *not* near u_0, i.e. for large values of the parameters ε_i, one must study the n-parameter family (9.2) and verify that u_n is in its range.

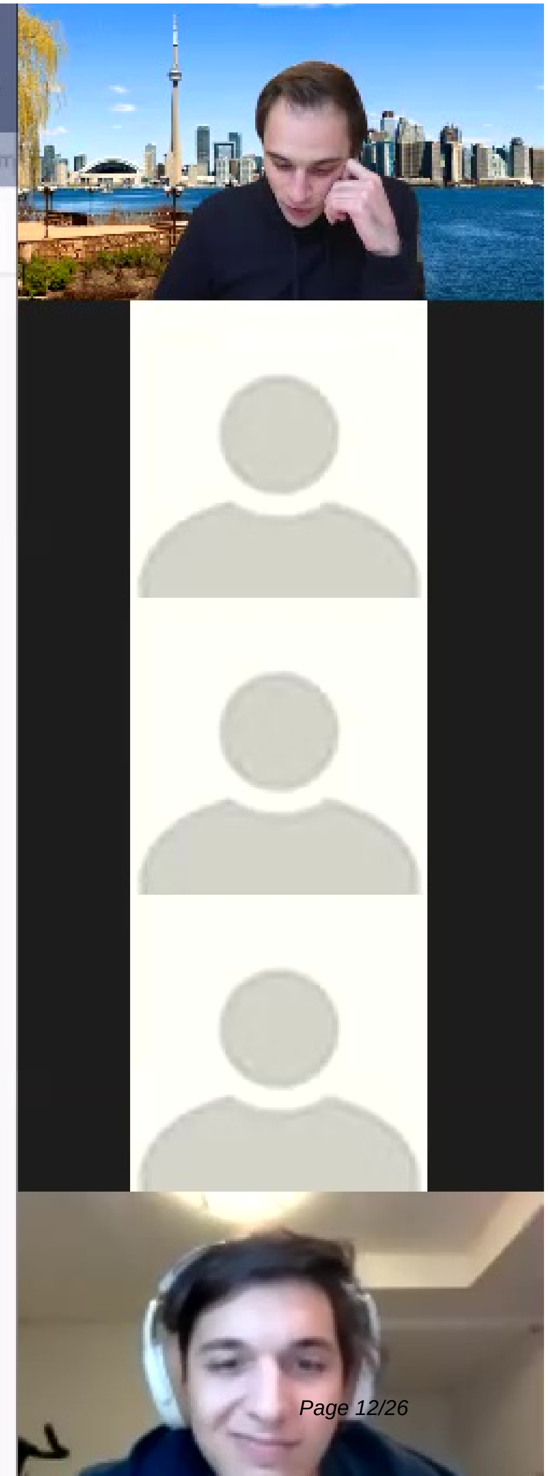
Within terms of order ε, the intermediate states $u_1, u_2, \cdots, u_{n-1}$ can be found by decomposing the initial discontinuity as follows:

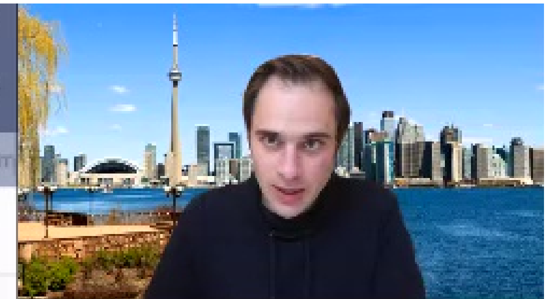
$$u_n - u_0 = \sum_{k=1}^{n} \varepsilon_k r_k. \tag{9.3}$$

Then

$$u_j = u_0 + \sum_{k=1}^{j} \varepsilon_k r_k + o(\varepsilon). \tag{9.4}$$

The expression (9.4) resembles the solution of Riemann's initial value problem for a linear equation with constant coefficients. In that case, the solution consists of $n+1$ constant states separated by characteristics, and the intermediate states are given by (9.3) and





is hyperbolic. This means that the matrix A has n real, distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ which are functions of u and which are arranged in increasing order. The corresponding right and left eigenvectors of A are denoted by $r_1, r_2, \cdots, r_n$ and $l_1, l_2, \cdots, l_n$, and are also functions of u. Later on, we shall give a convenient normalization for them.

Throughout this section we shall consider piecewise smooth weak solutions of the system of conservation laws (1.1). These are, we recall from Lem-

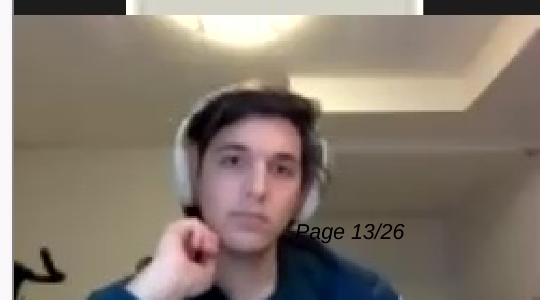
ma 2.1, completely characterized by requiring that the Rankine-Hugoniot condition

$$s[u] = [f]$$

hold across the lines of discontinuity. Let P be a point on a line of discontinuity C and let u_l and u_r be the values at P of the solution on the left and right side of the discontinuity C, respectively. Draw, issuing from P in the positive t-direction, those characteristics with respect[7] to the state u_l which stay to the left of C, and those with respect to the state u_r which stay to the right of C.

DEFINITION 7.1. A jump discontinuity in a weak solution is called a shock if the total number of characteristics drawn in this fashion is $n-1$.

The analytical expression for the shock requirement is: *for some index k, $1 \leq k \leq n$, the inequalities*



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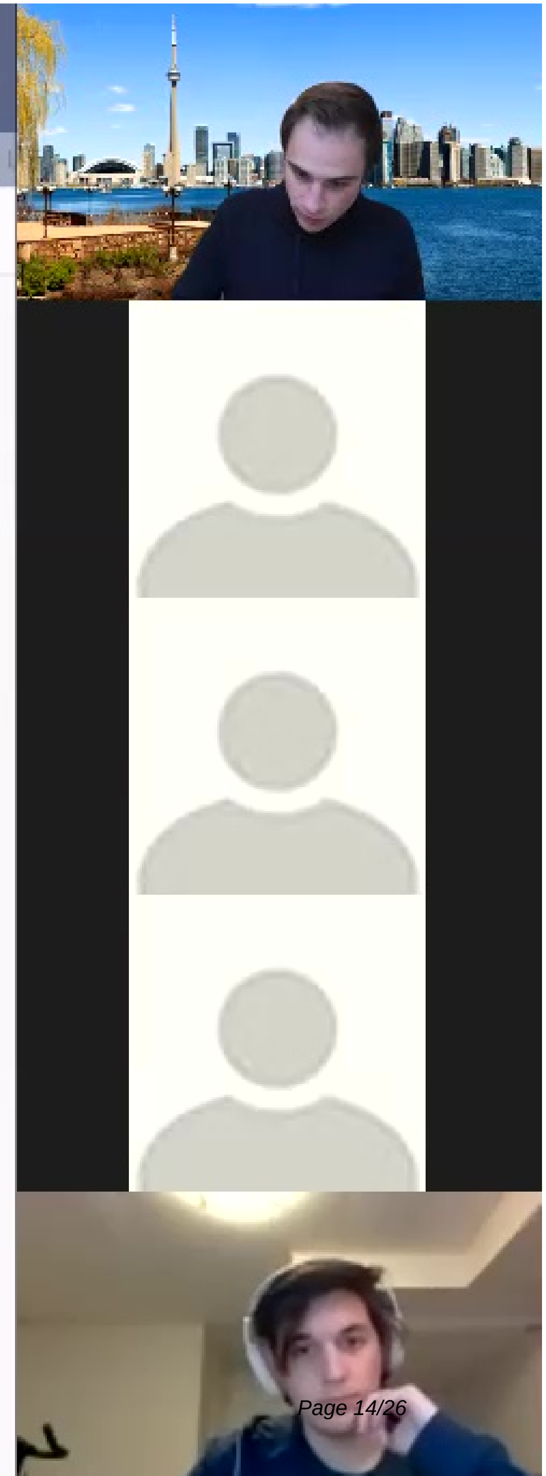
Def: Let $u_e \in D \subseteq \mathbb{R}^m$ and (λ_k, r_k) be an eigenvalue - eigenvector pair of a strictly hyperbolic system of CLs with $S(u_e)$ and integral curve $R_k(u_e)$. Then:

$$T_k(u_e) \equiv \begin{cases} R_k^{+}(u_e) \cup \{u_e\} \cup S_k^{-}(u_e) & \text{if } (\lambda_k, r_k) \text{ genuinely non linear} \\ R_k(u_e) = S_k(u_e) & \text{if } (\lambda_k, r_k) \text{ is linearly degenerate in } D \end{cases}$$

(R_k^{+}: existence of rarefaction waves; S_k^{-}: existence of k-shocks)



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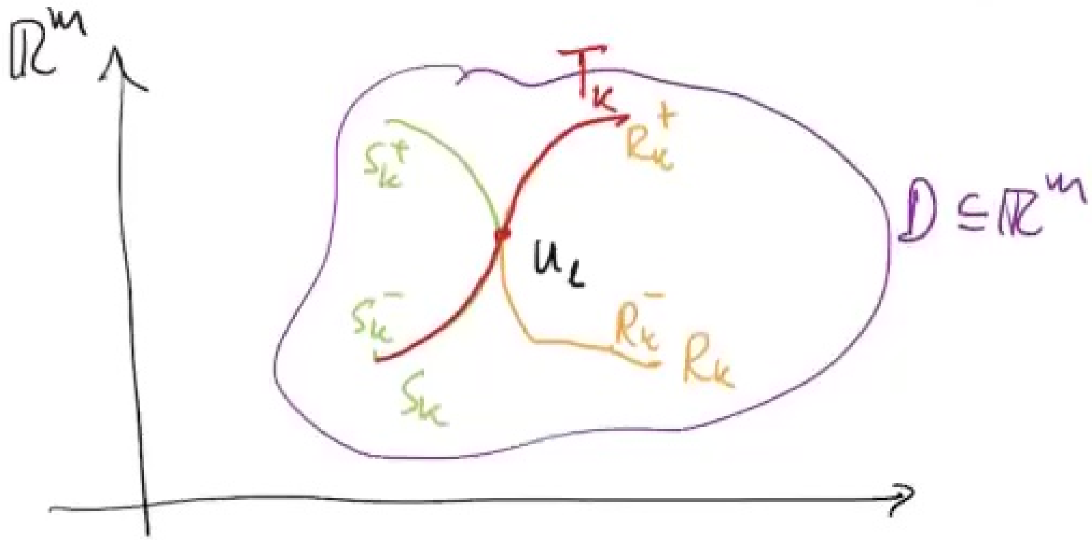
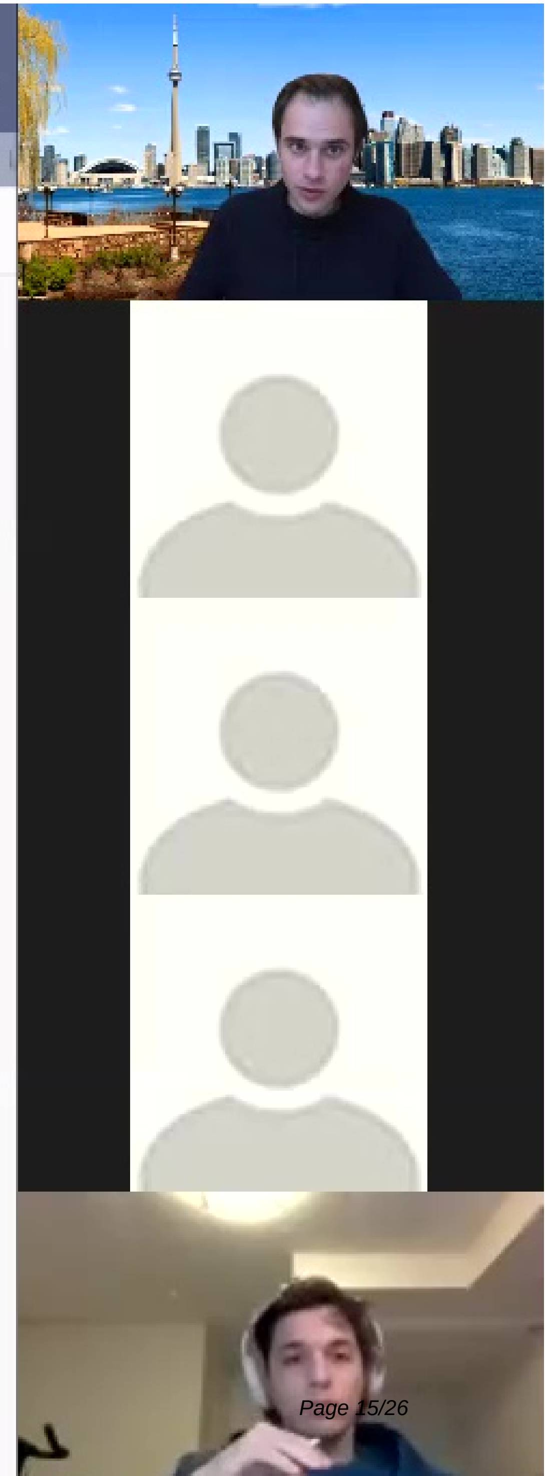
non linear

$R_k(u_l) = S_k(u_l)$ if (λ_k, r_k) is linearly degenerate in D



Note: u_l can be joined with u_r by a k-rarefaction wave, a k-shock, or a

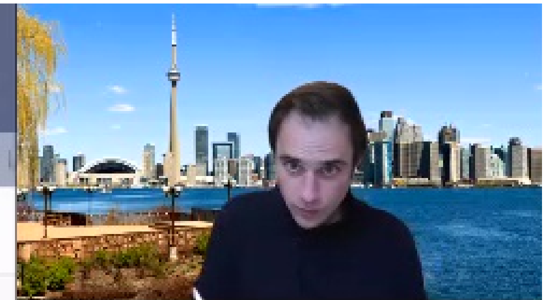
k contact discontinuity if $u_r \in T_k(u_l)$

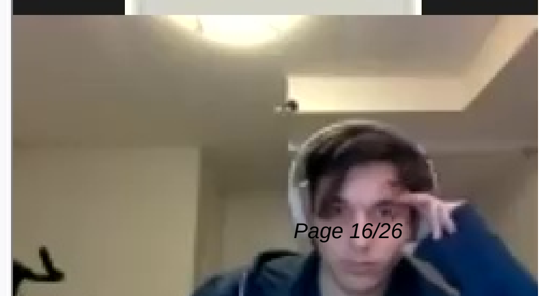


Now: consider general case u_r & $T_k(u_\ell)$
for some $k \in \{1, \dots, m\}$

Theorem (Lax 1957)

Let $u_\ell \in D \subseteq \mathbb{R}^m$. Consider the strongly hyperbolic system $u_t + f(u)_x = 0$ and assume that every pair (λ_k, r_k) is either genuinely non-linear or linearly degenerate on D. Then there exists a $\tilde{D} \subseteq D$ of u_ℓ such that for all $u_r \in \tilde{D}$ there exists a unique weak solution $u(x,t)$ to





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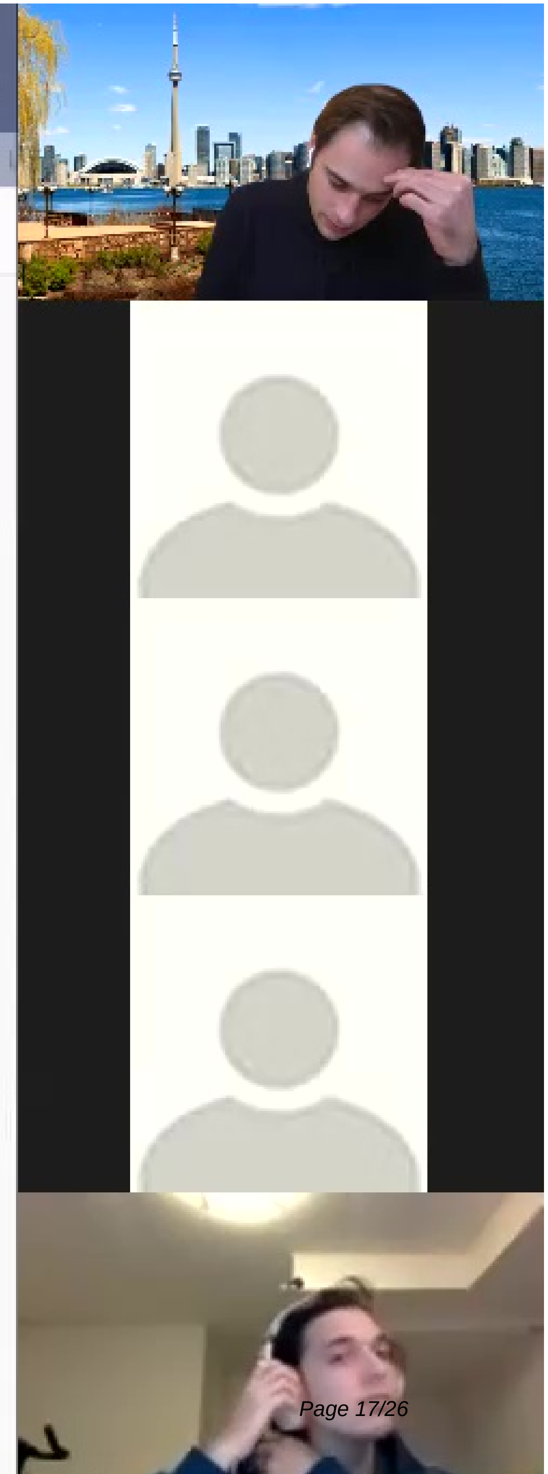
Lecture_14

the Riemann problem

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

which is constant along lines through the origin (at most $m+1$ constant states connected by shocks, rarefaction waves & contact discontinuities).

Idea:



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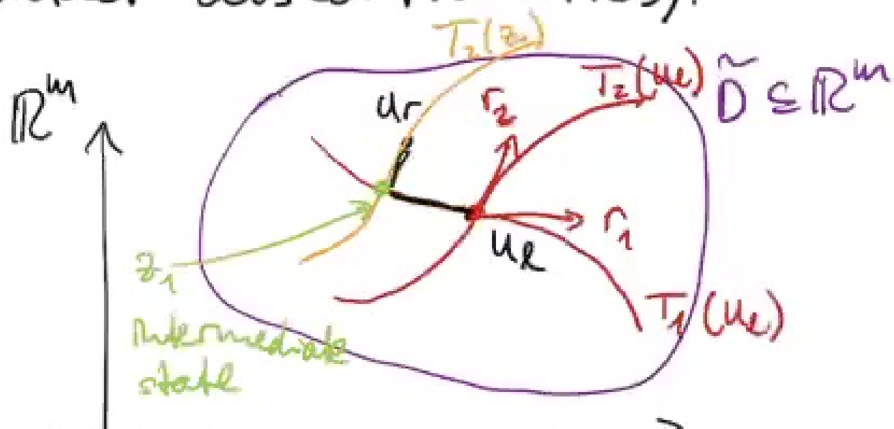
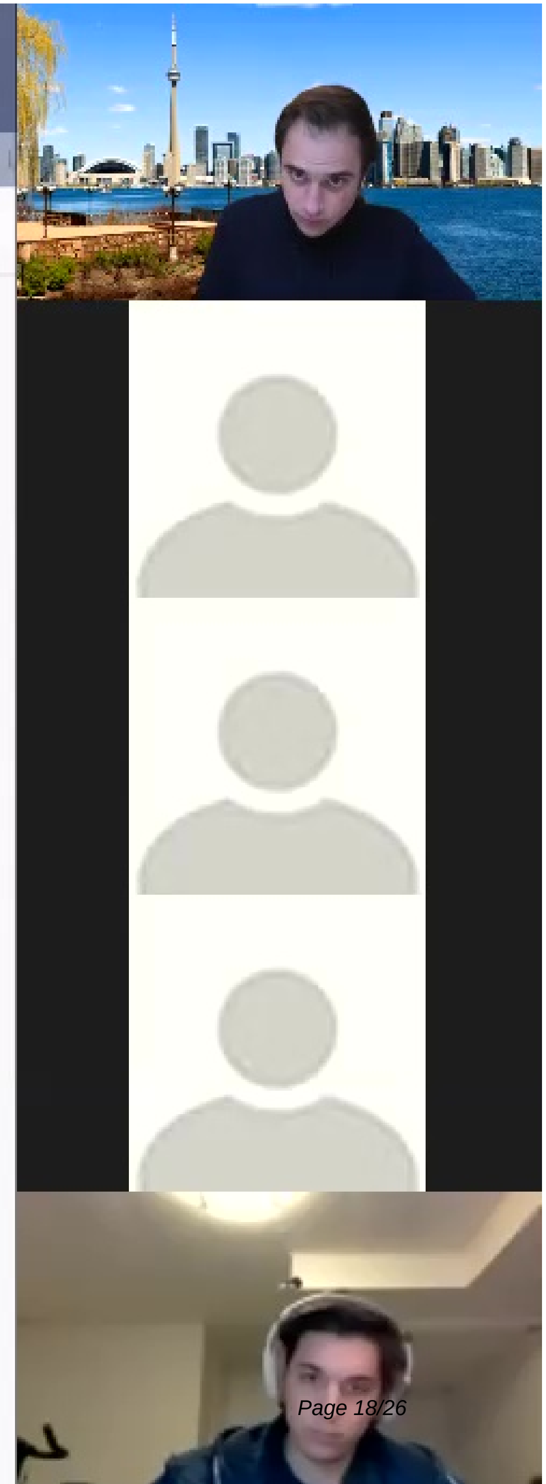
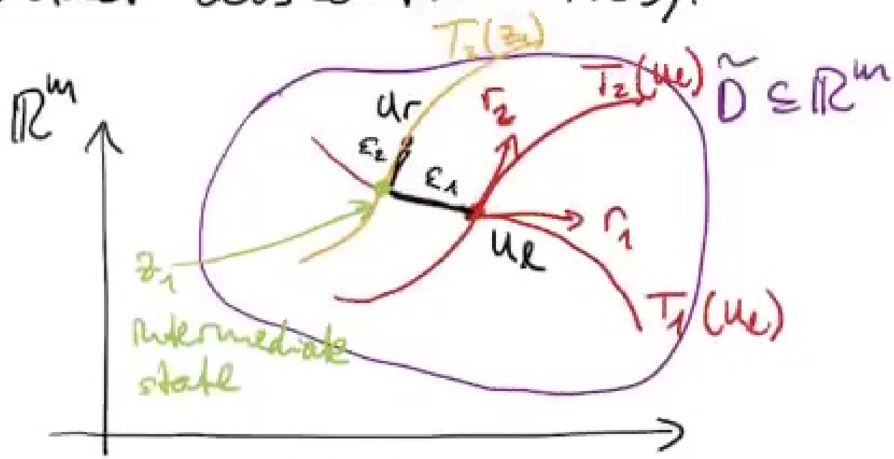




which is constant along lines through

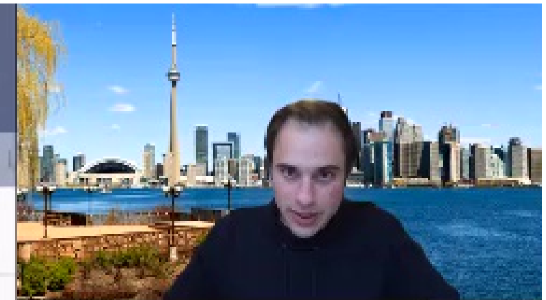
the origin (at most $m+1$ constant states connected by shocks, rarefaction waves & contact discontinuities).

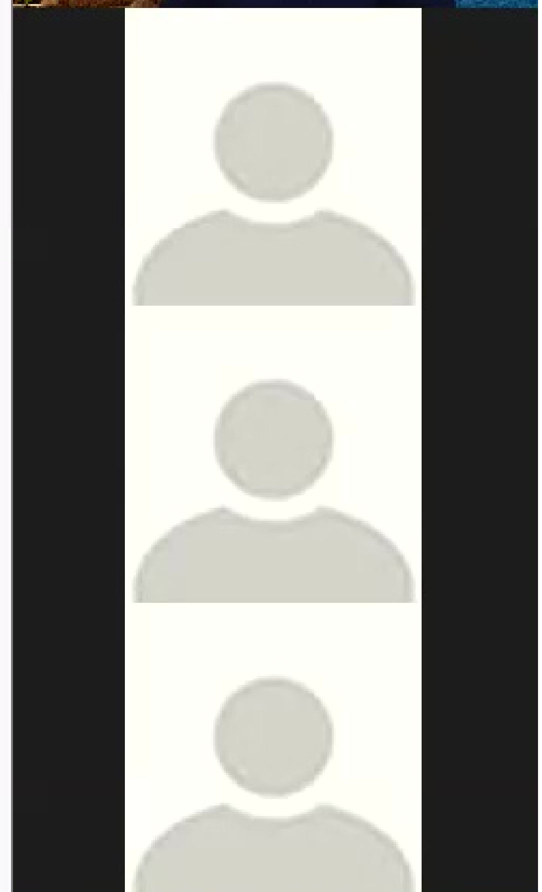
Idea:

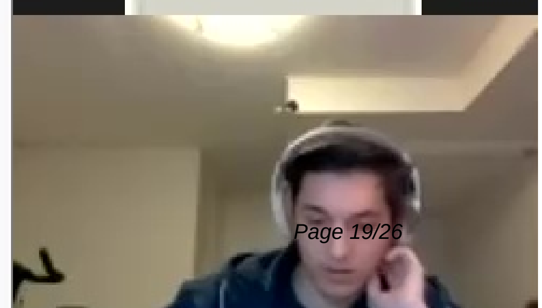


$\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$

connect (u_r, u_ℓ) by introducing intermediate states following



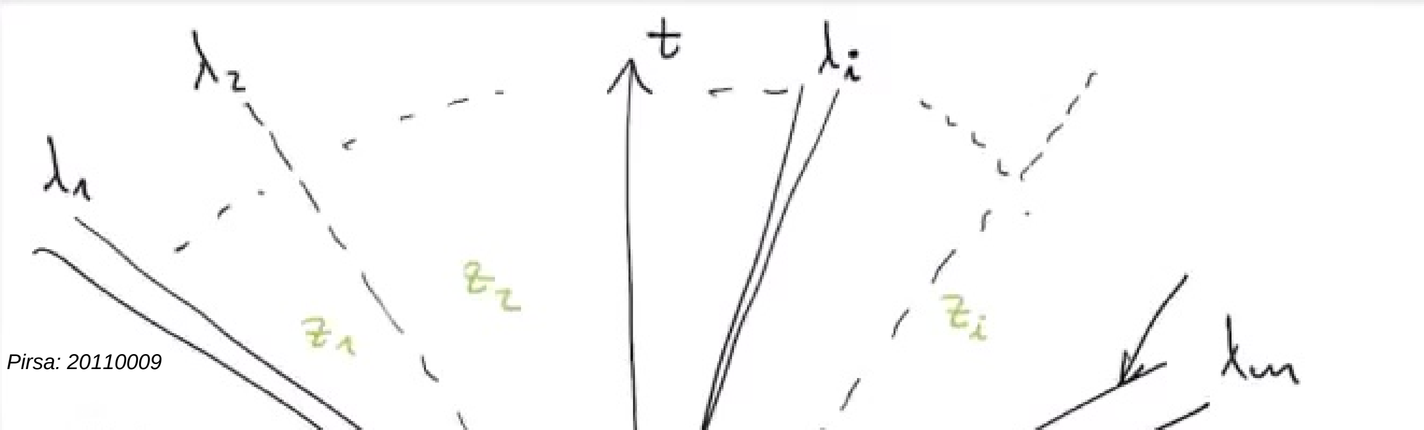


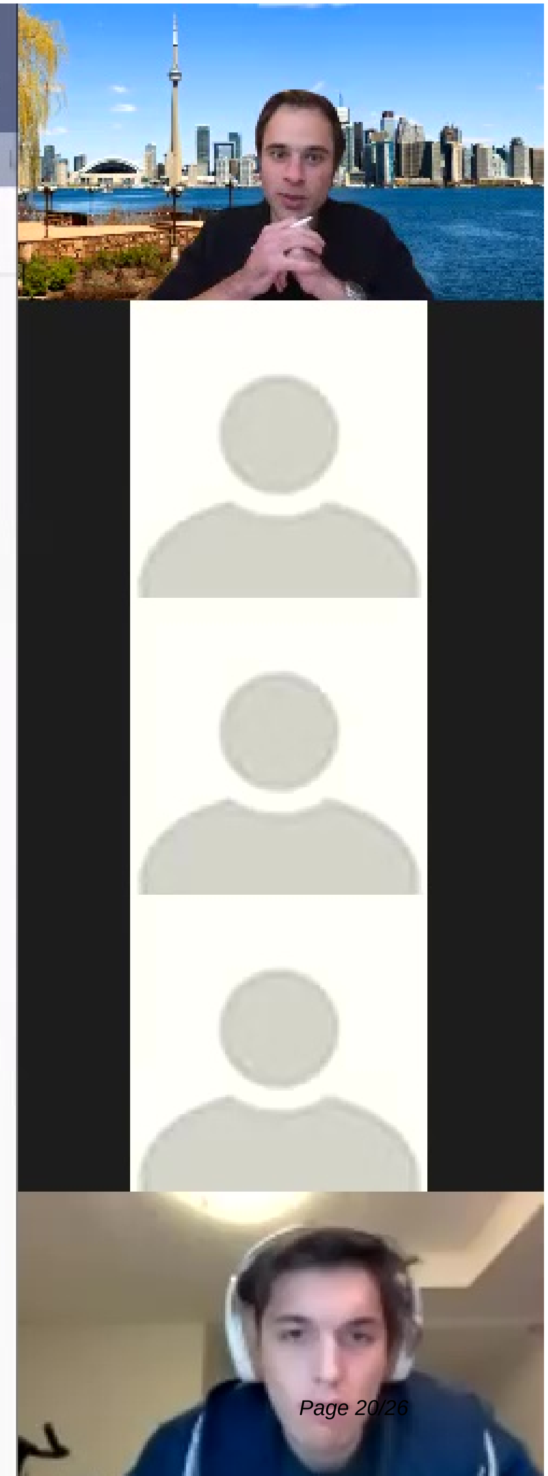


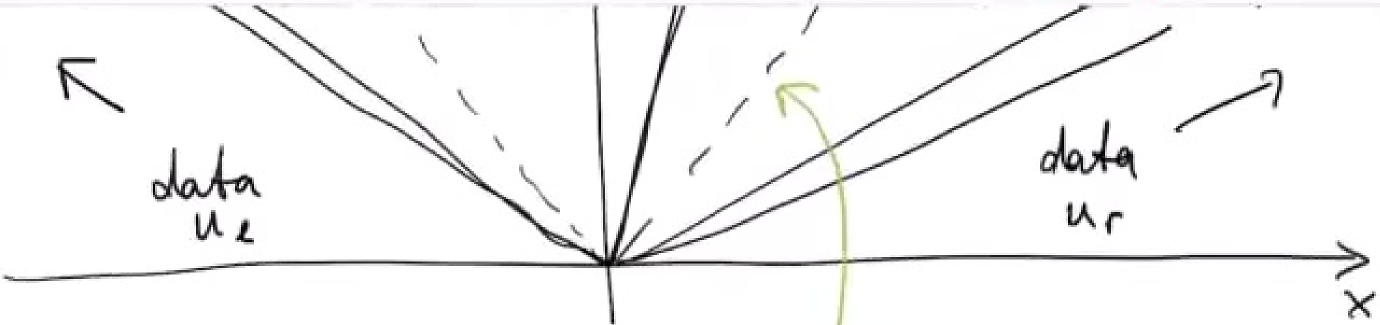
the curves $\{T_k\}$ in $\tilde{D}$.

Infinitesimal version (linearized Riemann problem)

$$u_r - u_\ell = \sum_{k=1}^{m} \varepsilon_k r_k(u_\ell)$$

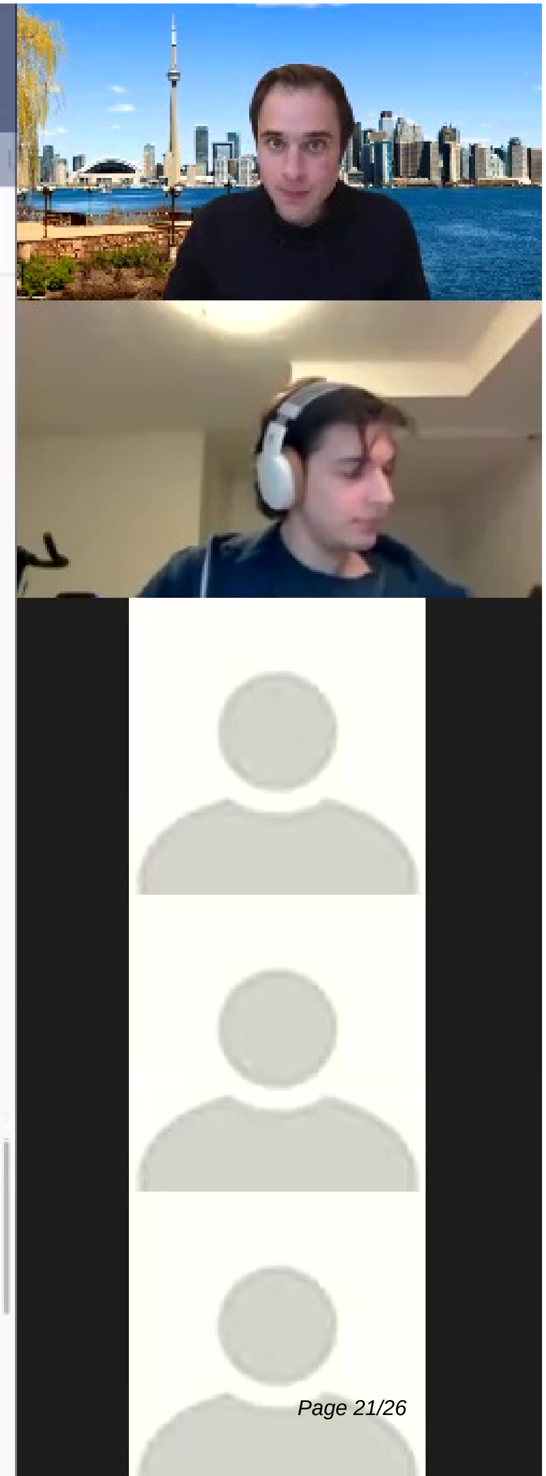






solution constant on lines through the origin $\frac{x}{t} = \text{const}$

$$|u_l - u_r| \sim \frac{2}{\gamma - 1}(c_l + c_r)$$



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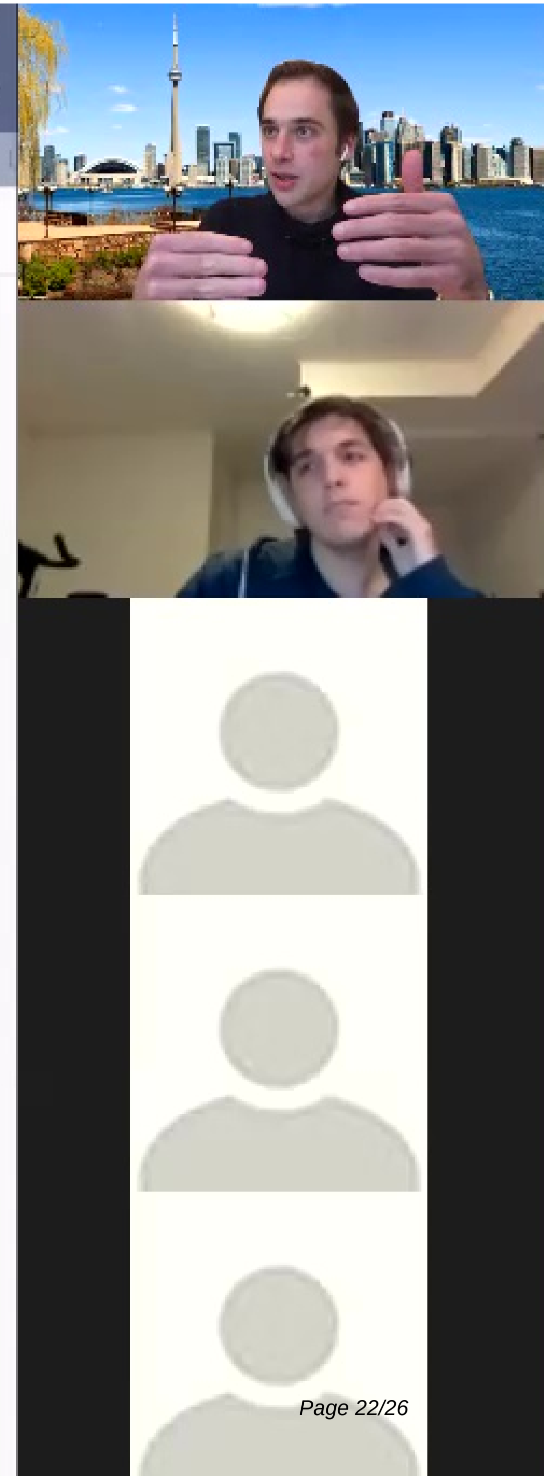
Chapter 5: The RP for the Euler equations

5.1 General setup

Consider



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Chapter 5: The RP for the Euler equations

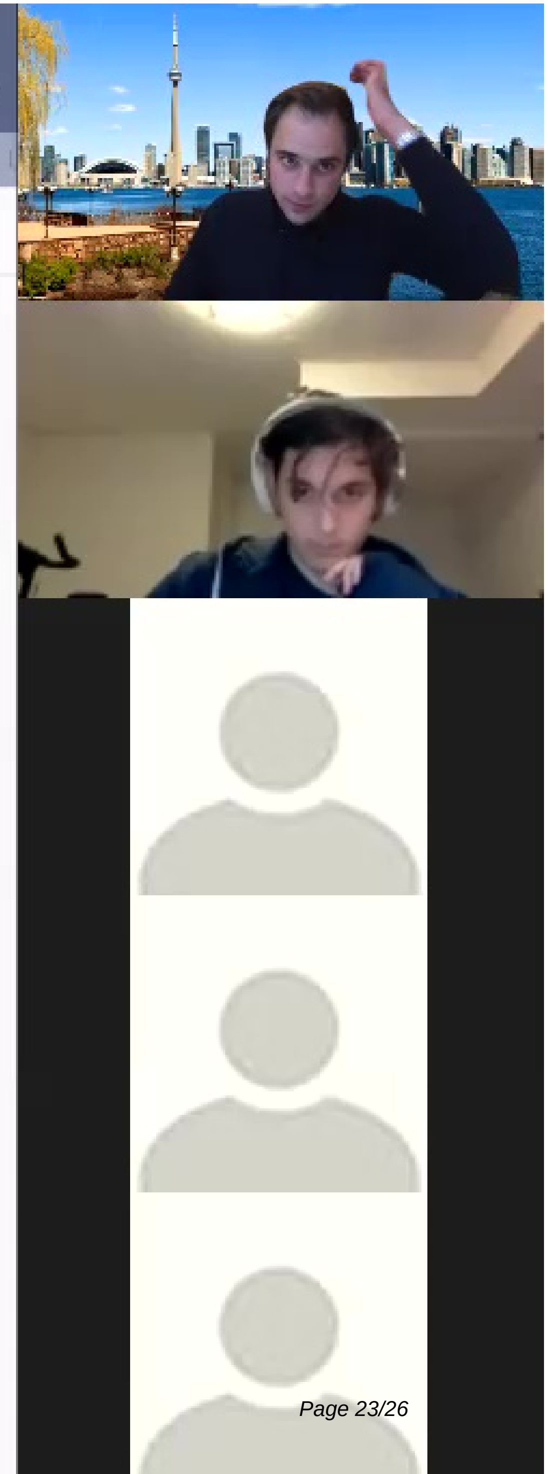
5.1 General setup

Consider Euler equs

$$u_t + f(u)_x = 0$$

$$\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}_x = 0$$

Ideal gas EOS: $p = (\gamma - 1)\, e_{int}$



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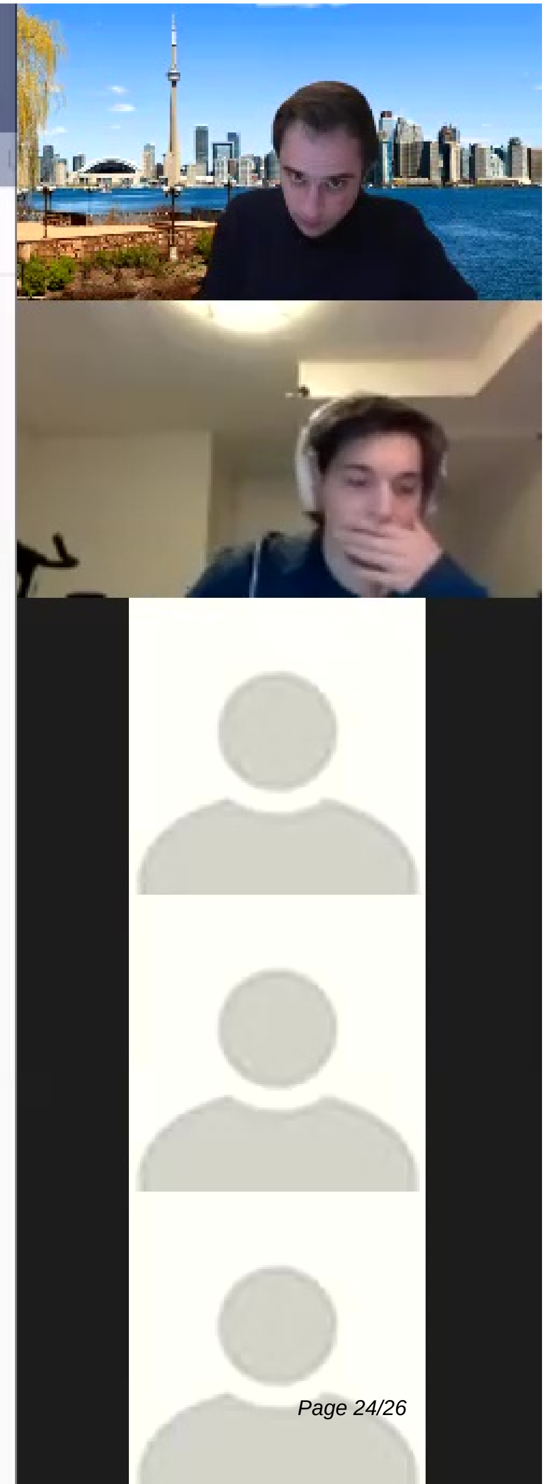
$$r_1 = \begin{pmatrix} 1 \\ u-c \\ H-uc \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 \\ u+c \\ H+uc \end{pmatrix}$$

Exercise:
- $(\lambda_1, r_1), (\lambda_3, r_3)$: genuinely non-linear
- (λ_2, r_2): linearly degenerate
- s: Riemann invariant

k-Riemann invariants:

1-RI: $s, \quad u + \frac{2c}{\gamma - 1}$

2-RI: $u, \; p$



- (λ_2, r_2): linearly degenerate

- s: Riemann invariant

k-Riemann invariants:

1-RI: $s, \quad u + \frac{2c}{\gamma - 1}$

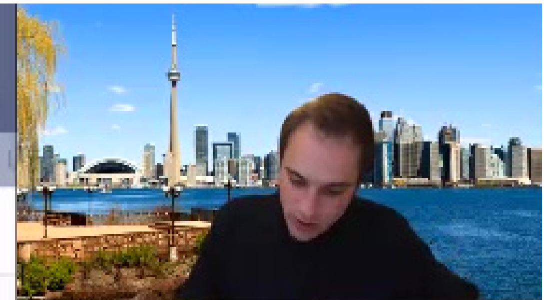
2-RI: $u, \quad p$

3-RI: $s, \quad u - \frac{2c}{\gamma - 1}$

⇝ General structure of the RP for the Euler equations

$\lambda_1 = u - c$ $\quad w_{*l}$ $\quad t \uparrow$ $\quad \lambda_2 = u$ $\quad w_{*r}$ $\quad \lambda_3 = u + c$

contact ...



9:41 AM Tue Jan 9

Lecture_14

Lecture_14 | Chap_4_Propertie... | Lecture_13 | Chap_5_Riemann... | Riemann - 1860 -...

⇝ General structure of the RP for the Euler equations